Mathematical Blending

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1 Introduction

Mathematics–"higher mathematics"–is a singular intellectual discipline. Although often classified with the sciences, it is not inductive. Rather, it is rigorously deductive, its inceptions found in Euclid's exposition. However, it is not a mechanical sterile exercise in manipulation of definitions and logic. Rather, mathematics is considered one of the apogees of human intellect, with the scrupulousness of science and the aesthetics of a fine art.[1] To remain, over the centuries, a vibrant human activity, it must encompass in its methodologies the modi of human cognition. As such, it is a worthy subject of cognitive science.

Authors George Lakoff and Rafael Núñez address the epistemology of mathematics with the perspective of current cognitive science in their book *Where Mathematics Comes From*. They introduce their book as "an early step in the development of a cognitive science of mathematics"(Lakoff and Núñez, 2000, p. 11). The reception to this book—praise and riposte—indicates the timeliness of cognitive-science perspectives.

Cognitive scientists Gilles Fauconnier and Mark Turner, especially, and others, have focused attention on conceptual integration, or "blending." Blending is a common, but sophisticated and subtle mode of human thought, somewhat, but not exactly, analogous to analogy, with its own set of constitutive principles, explicated for example, in Fauconnier/Turner's book *The Way We Think: Conceptual Blending and the Mind's Hidden Complexities*. To quote Fauconnier/Turner, "Building an integration network involves setting up mental spaces, locating shared structures, projecting backwards to inputs, recruiting new structure to the inputs or the blend, and running various operations in the blend itself" (Fauconnier and Turner, 2002, p. 44). Despite its sophistication, "[b]lending is child's play for us human beings, but we are children whose games run deep" (Fauconnier and Turner, 2002, p. 50).[2]

The logician Meir Buzaglo, in his book *The Logic of Concept Expansion*, intends to develop a logic of "concept expansion," especially in mathematics,

[1] "Mathematics seems to be the one place where you don't have to choose, where truth and beauty are always united" (Goldstein, 2005).

[2] To which one might append, in the present spirit: „Gott ist ein Kind, und als er zu spielen begann, trieb er Mathematik. Sie ist die göttlichste Spielerei unter den Menschen.“ ["God is a child; and when he began to play, he went in for mathematics. It is the most godly of humankind's games."] (Erath, 1954).

largely in refutation of [Friedrich Ludwig Gottlob] Frege (1848–1925), who maintained that mathematical concepts are logically fixed, and not to to be expanded. Concept expansions are thus a structured type of cognitive blend, comparable to the idea of closure of Lakoff and Núñez (2000, p. 21), and Buzaglo sees them as essential to scientific advances, so even a system of logic, if it is to reflect human cognition, must have scope for blends.

> Now it may be possible to place metaphors, analogies, and vague concepts outside the realm of logic, as Frege does, but it definitely undesirable to present a theory of logic in which this is the fate of the expansion of concepts of the sort presented here [e. g., zeroth powers], since it is impossible to imagine modern mathematics and physics without such expansions. Here is the general structure of the argument I present in this book. While Frege claims that the idea of expansions detracts from the principles of reference and sense, and therefore there cannot be a logic that includes this process, I claim that there can be a logic that includes non-arbitrary expansions, and that there are convincing reasons to believe that a certain type of expansion expresses human rationality. Therefore, instead of allowing some principles to place this phenomenon outside logic, the principles must be changed so as to include this process. These changes will eventuate in a different conception of logic that is not confined to a general study of the space of reference and truth *after* they have already been consolidated, but also includes an analysis of how this space is established (Buzaglo, 2002, p. 2–3) (emphasis in original).

Blending is a powerful force in mathematics. Indeed, a number of Fauconnier/Turner's exemplars are mathematical. Lakoff/Núñez state, "Blends, metaphorical and nonmetaphorical, occur throughout mathematics. Many of the most important ideas in mathematics are metaphorical conceptual blends"(Lakoff and Núñez, 2000, p. 48) (the role of metaphor in the epistemology of mathematics is a main motif of that book). For example, Núñez (2004) and forthcoming, continuing the theme of Lakoff and Núñez (2000), has analyzed particular mathematical blends in this cognitive-science framing, especially investigating potential and actual infinity, both as an end in itself, but also with the viewpoint that mathematics, as a totally human construct, is an arena to examine human creative cognition—a theme explored also here, especially in Section 9.

The formal structure of mathematics is a framing. It is a mystery to be explored that mathematics, in one sense a formal game based on a sparse foundation, does not become barren, but is ever more fecund. I posit, and wish to explore, the proposition that mathematics incorporates blending (and other cognitive processes) into its formal structure, as a manifestation of human creativity melding into the disciplinary culture, and that features of blending, in particular emergent structure, are vital for the fecundity. As Fauconnier/Turner note,

> [f]ormal systems are not the same thing as meaning systems, nor are they small translation modules that sit on top of meaning systems to encode work that is done independently by the meaning systems. Like the warrior and the armor, meaning systems and formal systems are inseparable. They co-evolve in the species, the culture, and the individual (Fauconnier and Turner, 2002, p. 11).

My purpose here is to explore, by cases and in general context, this phenomenon. I consider several basic cases, also explored by Fauconnier/Turner and also Lakoff/Núñez, both because they are elementary mathematics (by today's standards) and thus more broadly accessible, and also because they are more interesting, in that modern (roughly post 1900) mathematics has a fixed cognitive paradigm, whereas in the time these cases were under development (sometimes millennia), the paradigms were different, and the comparisons shed light on the phenomenon of blending. In particular, one can ask the question: if blending is child's play, why in fact can it sometimes be so problematic? Since mathematics is so structured, features of blending and other cognitive processes can be isolated, and thus mathematics is a good laboratory for probing them.

In fact, some of the basic mathematical blends proved quite problematic. Intellectual history is replete with blends that did not work out and were cast aside, for examples, Ptolemaic epicycles, inheritance of acquired characteristics, phrenology, phlogiston, ... Many successful intellectual blends initially met resistance and did not become firm for a considerable time, for example the heliocentric theory in opposition to the Ptolemaic epicycles. In these cases, often it was a matter of building up sufficient experimental and observational data to overcome resistance. However, in mathematics, a purely deductive discipline, it is not a case of developing evidence. A number of the blends we consider—negative numbers, irrational numbers, complex numbers—required centuries—even millennia—to secure. These blends seem obvious to us; indeed Buzaglo might argue they are inevitable:

> Let us say that we manage to send a spaceship to a planet of Alpha Centauri and we discover that its inhabitants use arithmetic. Does this give us reason to predict that if we return there a thousand years later we will find them using negative numbers, perhaps even complex numbers? And if we find that they use the power function, is is also probable that they will expand it to the zero? It seems that there is some basis for believing that we will find all these developments (Buzaglo, 2002, p. 15).

However, in the documented history, one can discern the dissonance. The blends were not exactly repudiated; rather they remained arrested. After examining the formal blends, I revisit, in Section 9, with the outlook of cognitive science, the phenomenon of arrested blends, and contrast them with the modern formal style.

Let me first, with Turner, bring the framing of mathematics back to the human scale.

> As long as ... mathematical conceptions are based in small ... stories at human scale, that is, fitting the kinds of scenes for which human cognition is evolved, mathematics can seem straightforward, even natural. The same is true of physics. If mathematics and physics stayed within these familiar story worlds, they might as disciplines have the cultural status of something like carpentry: very complicated and clever, and useful, too, but fitting human understanding. The problem comes when mathematical work runs up against structures that do not fit our basic stories. In that case, the way we think begins to fail to grasp the mathematical structures. *The mathematician is someone who is trained to use conceptual blending to achieve new blends that bring what is not at human scale, not natural for human stories, back into human scale, so it can be grasped* (Turner, 2005) (emphasis mine).[3]

Let us regard this a bit. Mathematics is an edifice. Blending is a means of construction, the mortaring, brick by brick. The mortar must be strong; it is the rigorous formalism of the discipline that permits one to surely make the next blend—lay the next brick. The physicist Eugene Wigner mulled this in a rather famous speech and article, "The unreasonable effectiveness of mathematics in the natural sciences," where he mused "The great mathematician fully, almost ruthlessly, exploits the domain of permissible reasoning

[3]The ellipses above are the word "geometric," which is the emphasis of Turner (2005). However, his observation is more universal, and I wish to present it in the broader context.

and skirts the impermissible. That his recklessness does not lead him into a morass of contradictions is a miracle in itself: certainly it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess." (Wigner, 1960). A well-placed brick opens up new architectural perspectives—emergent structures—towards the next advance of the construction. The modern foundation on which it stands is usually set theory, formalized in the late 1800s and early 1900s, along with a vocabulary and logic with which to handle it. It came to be understood during this development that some caution is required, otherwise contradictions can be effected (e. g., the set of all sets is a no-no). However, the pale is well-demarcated—one treads carefully around self-reference—and does not affect the main body of mathematics, or how mathematicians operate. There are several set-theoretic foundations available. The one assumed by more working mathematicians is called the [Ernst Friedrich Ferdinand] Zermelo (1871–1953)-[Adolf Abraham Halevi] Fraenkel (1891–1965) theory, named after its inventors, freely augmented with the axiom of choice. However, the particulars of the foundation are largely irrelevant to working mathematicians, and to this investigation, which is principally concerned with the methodology of working mathematicians. The mathematical edifice is erected by various operations on sets. These operations are the implements of working mathematicians.

A blend consists of a gathering—a joining of concepts—into a meaningful reassemblage. In mathematics, dual processes of expansion and reorganization are quite explicit set operations. The first is done via a variety of methods; the second is accomplished by a rather ubiquitous mechanism, namely quotient sets. For our examples, ordered pairs is a convenient first method. An ordered pair melds a single identity from two. It is simply a pairing of an element of one set with an element of a second. This construction is a common cognitive blend or metaphor (Lakoff and Núñez, 2000, e. g., p. 141+), for example married couples (in an ordered pair, there is a first-named and a second-named; there is also the underlying unordered pair, in which the orderering is immaterial). We are all familiar with ordered pairs (x, y) of real numbers, on horizontal and vertical axis, respectively, marking points in the Cartesian plane or on a map. For this reason, often the set of ordered pairs is called the Cartesian product. Quotient sets amount to taking collective nouns seriously. We humans collectivize all the time. Thus we fashion families, cohorts, companies, troops, brigades, lots (of goods), gardens, forests, ...[4] If one sees more the forest than the trees, one is thinking in terms of quotient sets. In fact, what drives the use of quotient sets is not the collective impulse, but rather the concept of "sameness." One ignores the irrelevant differences; things that are the same, in some way or another, are collected together under a collective identity and the collective becomes an object in itself. Fouconnier/Turner use the very apt word "fuse" (full context in Section 3 below). For example, there are a variety of architectural types of houses—rancher, colonial, etc. There are descriptions, but the types can also be designated by all the representatives. The quotient set is thus the collection of types. Of course, there can be questions of the identity of any individual or how fine grained the classification (although I have never seen a colonial rancher). However, in mathematics, definitions are precise and identities clear. Quotient sets are very efficient way of creating blends. A simple, but typical, example (see Section 3 below) is fractions. Thus 2/4 is different from 1/2 as symbols, but the difference is irrelevant to their meaning as quantities, and for that blend they are quotiented together. Virtually all of mathematics can be reduced to the human scale of a basic set and judicious use of set operations.

Most cognitive blends depend on an aggregation of earlier blends (recursion). However, when constructing a blend, a person does not generally reflect on what has gone before, which is already at that human's scale. Similarly, a working mathematician does not reflect on all the structure underlying a new blend. Thus, one may ponder, how does a mathematician know the integrity of the mathematical edifice is not being weakened. The answer is that the mathematician's blend is constructed with a certain collection of mechanisms; such as ordered pairs or quotient sets. The mathematician knows, as a matter of logic, that these mechanisms are safe. Blends in mathematics are quite structured; they are haikus, not free verse, and indeed, therein lies much of the beauty, as well as the power, of the subject.

My thesis here is largely orthogonal (as mathematicians say) to Lakoff and Núñez (2000). Except for a few historical observations, I am not concerned here with the philosophy or epistemology of mathematics. Much of Lakoff and Núñez (2000) concerns how humans perceive basic mathematical entities—numbers, geometric points on a line, ... Those authors devote a few pages explicitly discussing the meaning of the mathematical edifice—

[4] Some of these words serve as more than collective nouns; for example, a garden or forest can also mean an area or region. Here the import is the collective set of trees.

what they call the "formal reduction metaphor" (Lakoff and Núñez, 2000, p. 369+)—which is the context for the present discussion. Thus, for example, whether formal reduction is a "cognitive interpretation" or "foundations interpretation"(Lakoff and Núñez, 2000, p. 372) is here immaterial. Moreover, it is not the foundations, per se, that is the topic here. If a different foundation is used, the present discussion would be largely unchanged. Rather, it is the methodology—the use of constructions—that is the present focus.

2 Integers

For the integers, as counters, Turner (2005) starts with three mental spaces, whole (positive) numbers, discrete points on a line (like walking steps), and objects in a container. The number zero is constructed as a blend (counterpart of the empty container). In the modern formalism, zero is the first number constructed from sets, and the positive natural numbers (integers) are constructed recursively; e. g, in the construction of [John] von Neumann (1923), see also Stillwell (1989, p. 315).[5] Zero is the empty set, and each subsequent number is the set consisting of all the previous ones. The integers are, almost literally, built from nothing. Actually, the key word here is "built." The integers are not proclaimed into existence, but firmly rooted in set theory. However, there is more to the integers, namely arithmetic operations. The formal construction includes the concept of successor, which is identified with "+1." From this, all addition, and then multiplication, is defined, and subtraction and division tag along. However, the construction of the integers is not the best exemplification of my thesis, and we pass by it, except for the following observation.

Clearly a child's apprehension of the counting numbers (itself a blend) is not via this formalism.[6,7] The formal construction has formal advantages. One obtains the full set of natural numbers in one rigorous fell swoop,

[5]This book is a textbook, organized by the mathematical concepts. It is thus concise, but it does trace the intellectual developments and implications of mathematics. It does assume of the reader some mathematical background and sophistication.

[6]This author remembers trying to explain the von Neumann construction in a social setting to a person who asked, "How *do* you define 'two'?" [Bill Cosby (1965): "The teacher said, 'One and one is two,' and we were all going, 'Alright! Yeah! One and one is two! That's cool, man . . . ' 'Uh, what's a two?'"]. The lesson learned was: don't do it again.

[7]Lakoff and Núñez (2000, p. 141–142) discuss this construction in their context.

and indeed more (namely infinite ordinals). Concision and generality are virtues. Trained mathematicians think this way. However, the (quite sophisticated) blends of young humans that lead to the "intuitive" knowledge of basic counting, geometry, proportion, dynamics, etc., are not rooted in formalism. Indeed, it might be that "higher" mathematics is a different mental process than "intuitive" mathematics, actually involving, via training, a different part of the brain. Every professional mathematician has made a transition from "computing mathematics," often called "engineering mathematics," i. e., how to compute in calculus, linear algebra, and such—often the culling field is analysis, the formalism of limits—to a mathematical kenning, a different mode of thought, and thus become inhabitants, as those who live with mathematicians know, in another mental (and sometimes social) universe into which they retreat.

3 Fractions

Let us thus begin with the whole (or natural) numbers (= non-negative integers), and blend with another concept, namely proportion. Fauconnier/ Turner:

> if in one [mental] space we have whole numbers and in the other space we have proportions of objects, then in the blend we have all the proportions, all categorized as numbers. Those proportions that had whole-number counterparts are fused with those counterparts, so that, for example, 6:3, 12:6, and 500:250 are fused in the blend with 2. But now 3:4, 256:711, and 5:9, which had no whole-number counterparts, are now also numbers in the blend (Fauconnier and Turner, 2002, p. 242).

In fact, both phylogenically and ontogenically, this blend is not a single process. A child probably first learns 1:2 early on ("she got more than I did"), and then 1:n. The classical Greeks and others often used "unit fractions" or "Egyptian fractions," with numerators 1 (the Rhind papyrus contains such computations). Thus if n' denotes $1/n$, one would write 12/17 as $2' + 12' + 17' + 34' + 51' + 68'$ (Fowler, 1999, p. 235). The proportion 256:711, or even more 711:256 is a much later blend. There is considerable examination of the conceptualization and manipulation of such objects by early cultures. In particular, D. H. Fowler notes,

> Addition and multiplication of common fractions,
>
> $$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq} \quad \text{and} \quad \frac{m}{n} \times \frac{p}{q} = \frac{mp}{nq} \tag{1}$$
>
> are operations so basic to arithmetised mathematics that it may be difficult for us today to conceive of a mathematics in which they are unknown or unimportant, ... [but] I believe that this may indeed have been the case for early Greek mathematics (Fowler, 1999, p. 133),[8]

and in any event is not the modern formulation which "involves a comprehensive apparatus of equivalence relations,[9] purely formal tricks of the last hundred years" (Fowler, 1999, p. 111). However, these purely formal tricks are precisely the blends of mathematical thought that we wish to explore.

Fauconnier and Turner discuss the conceptual blend that results in rational numbers (= fractions) (Fauconnier and Turner, 2002, p. 242+). A key point is the "gaps" between the integers which are filled in (not completely, it turns out—another blend is required, see "completion" below) by fractions.[10] This blending emphasizes the *ordering* of the numbers—$0 < 1/3 < 1/2 < 3/4 < 1 < 3/2 \cdots$—and thus accentuates the role of spatial conceptualization for humans; indeed the role of "geometric narrative" is a major point of Turner (2005). "Emergent structure," discussed below, is an important aspect of blending—particularly so in mathematics. Fauconnier and Turner note, "The [fraction] blend has considerable emergent structure. It turns out that there is an 'addition' operation in the blend that will correctly preserve addition from the number input and ordering from the proportion input" (Fauconnier and Turner, 2002, p. 243).

[8]But perhaps not so unreservedly. Although ratios existed not as fractions, but as proportions, not to be arithmetically manipulated, some manipulation occurred. The Antikythera mechanism (Freeth et al., 2006, and references therein) is a Greek orrery dating to 100–50 BC, clearly indicating the Greeks had a sophisticated understanding of gearing, which requires multiplication of fractions, or something equivalent. The moon travels $13.368267\cdots$ times as fast as the sun through the Zodiac. The Greeks had a geometric computational method, anthyphairesis (Fowler, 1999, chap. 2), equivalent to modern continued fractions, that gives rational approximations, in some sense best possible, to quantities. The approximation $254/19 = 13.368421\cdots$ is used in the mechanism, effected by a train of three gears of gear ratios (number of gear teeth) 64/38, 48/24, 127/32 with required product 254/19.

[9]For "equivalence relations" one can read "quotient sets."

[10]Thus for Turner (2005), where integers are blended from steps, perhaps "baby steps."

The blend of "arithmetised mathematics" inverts the input and the emergent structure of Fauconnier and Turner (2002). Namely, addition and multiplication are essential components of the structure of the integers. Division, the inverse of the operation of multiplication, is thus also desirable. But universal division fails; e. g., 1 is not divisible by 2 in the integers. So we construct fractions and extend arithmetic so that division is always possible,[11] i. e., make more numbers. This is a key paradigm of modern mathematics: if some useful concept or structure (here division) is not always possible, we enlarge our domain of conversation so (a) the useful concept is possible and (b) incorporate the previous entities (here integers with their arithmetic) into our enlarged domain. The "purely formal tricks" are precisely the blending of our present focus. We return to this point below the presentation of some details in this case.

Modern notation closely parallels the mathematical construction.[12] A fraction is denoted m/n where m and n are integers. Reverting to set notation, we consider all ordered pairs (m, n) of integers, with $n \neq 0$. These come with no additional *a priori* structure; rather we define structure, taking care to make certain anything we define works well. The first care we take is to note that for what we want (rm, rn) should be the same thing as (m, n), for any non-zero integer r; this is commonly known as canceling a common factor from the numerator and denominator. Let us use the set notation $(rm, rn) \equiv (m, n)$; this is the "equivalence relation" alluded to by Fowler, and the quotient set is the collective of all of these into what we denote m/n. We then define addition and multiplication by the prescriptions

$$(m, n) + (p, q) = (mq + np, nq) \quad \text{and} \tag{2}$$

$$(m, n) \times (p, q) = (mp, nq) \tag{3}$$

(compare (1)). We emphasize the arithmetic operations are part of the construction, not preordained in any way. A technical point is that to make the operations "well-defined," it has to be verified that they are independent

[11] For completeness, we note we do not define division by zero. There is no ukase against this; mathematical blends are human activity, and one can define what one wants. But it turns out division by zero does not work well, does not lead to anything useful, and does not advance mathematics. Indeed, in other situations where a fraction blend is useful, one excludes non-zero "zero divisors" as denominators, because they behave sufficiently like a zero to effect the same untoward results.

[12] "Modern" relatively speaking—putting one integer over another came from India; the horizontal bar was an Arabic addition; the slanted bar is a later typographical convention.

of the representation, e. g.that $(rm, rn) + (sp, sq) \equiv (m, n) + (p, q)$; this is a routine verification. Finally, we embed our existing integers m into our construction by $m \to (m, 1)$, and verify that the old and new arithmetic operations are congruent. Note that $(m, 1) \times (1, m) = (m, m) \equiv (1, 1)$; so that we have created reciprocals, hence division. We also explicitly note the important point that the extension of the arithmetic structure—addition and multiplication—was defined totally in terms of the preexisting arithmetic on integers. That is, equations (2), (3) use only integer arithmetic. Thus, not only the objects—fractions—are defined in terms of earlier objects, but their structure—arithmetic—is also.

There is extra structure, which leads to a richness—an emergent structure. As mentioned by Fauconnier/Turner, the ordering $m < n$ of the integers can be extended; namely we define $(m, n) < (p, q)$ if $mq < np$ (again there are some verifications required to verify that everything is consistent). Thus the geometry is an emergent structure, however not of the arithmetic structure, but of the "auxiliary" ordering. With the ordering, one can "insert" the fractions between the integers in consistent manner so as to fill in the gaps. This is not preordained. There are contexts where the fraction concept makes sense and is useful, but where there is no ordering. For example one can begin with polynomials and construct fractions, e. g.

$$\frac{1 + 2x^2 + x^3}{2 - x^4}$$

to obtain "rational functions." In this case, m and n are polynomials ($1 + 2x^2 + x^3$ and $2 + x^4$ above). Addition and multiplication are defined via prescriptions (2), (3). In this case, there is no ordering, and no geometry.

The blend is completed by incorporating the newly constructed fractions into the family of "numbers." We compress the blend. That is, we elide the construction; we do not think of fractions as ordered pairs, but as single entities in their own right. We do not think of fractions as something created out of integers, but rather integers as certain fractions.

4 Blending

Fauconnier and Turner (2002) is rich with details on the governing principles and mechanisms of blending. Most, possibly all, are manifested in the formalities of mathematics. Here we mention only a few, particularly relevant to our discussion. The first is "emergent structure." "The blend develops emergent structure that is not in the inputs" (Fauconnier and Turner, 2002, p. 42). There are three mechanisms for the development of emergent structure: composition, completion, and elaboration. Composition: "Blending can compose elements from the input spaces to provide relations that do not exist in the separate inputs." Completion: "We rarely realize the extent of background knowledge that we bring into a blend unconsciously. Blends recruit great ranges of such background meaning." Elaboration: "We elaborate blends by treating them as simulations and running them imaginatively according to the principles that have been established for the blend" (Fauconnier and Turner, 2002, p. 48). Emergent structure is extremely important in mathematics. Indeed, it likely accounts for the fecundity of the discipline. The inputs of effective blends almost always come with additional structure (for example the ordering of the integers fed into the blend of fractions). Technically, i. e., logically, it is not the case that the emergent structure is not in the inputs; recall that logically everything can be reduced to basic set theory. However, on a human scale, the emergent structure may not be apparent until the blend is completed.

A second important component of blending is compression (Fauconnier and Turner, 2002, chap. 6, 7, 16). Compression operates towards the overall goal of achieving human scale. There are a number of mechanisms outlined in Fauconnier and Turner (2002); for us, compression mostly involves giving the blended objects their own identity. That is, after launching the blend, the inputs are relegated to behind the scenes. For example, above, fractions are given their own identity as full members in the family of numbers; the fact that they actually are ordered pairs of integers is ignored. Another clear example is negative numbers, below. In point of fact, for elementary "intuitive" mathematics, the mental process is often the reverse. Fractions and negative numbers were conceived as numbers, along with heuristic principles for their behavior, long before they were rigorously defined.

A third important component of blending is recursion.

> One crucial corollary of the overarching goal of blending to Achieve Human Scale is that a blended space from one network can often be used as an input to another blending network. Once blending delivers a new blend at human scale, that new blend is a potential instrument for achieving yet more compression to human scale (Fauconnier and Turner, 2002, p. 334).

Recursion is exceedingly important in mathematics. Perhaps for no other discipline are the areas of active investigation so far removed from the first principles. The analysis of Fauconnier and Turner (2002, p. 335–336), where is discussed that “the concept of number has seen many successive blends, where at each stage a blended concept of number serves as the input to a new integration network, whose blended space has yet a newer concept of number,” closely parallels some of our present discussion, although Fauconnier/Turner do not focus on the formal mathematics.

There is a fourth important component of blending that is important to mathematics. This aspect is implicit in Fauconnier and Turner (2002), but it is useful to make it explicit. It is a combination of emergent structure, especially completion, and multiple blends. It often happens that a blend has, perhaps as emergent structure, a structure that the inputs do not have. There is an example discussed below. This structure, and all the mathematical machinery developed around it, can be blended with the just-consummated blend. Often this is intentional on the part of the mathematician; such is the value of experience. However, both logically and usually psychologically, this blend is separate from the first. The first creates an object, such as fractions. The second incorporates this object into a new context. Although the initial blending is often done with malice aforethought towards the second blend, equally often the full emergent structure only becomes apparent after passage of time and exploration. I call this “found structure.”

Let us now set the governing principles for formal blending. In its higher reaches, as noted by Turner (2005), mathematics is far removed from the ready human scale. To maintain its integrity, mathematics insists on “rigor,” firmly basing each advance in existing structure; thus in principle, able to be brought back to its very basic elements. Generally, there is a structure, with something desirable missing, such as the positive integers, with their arithmetic, but division is not always possible. The basic idea is to extend the domain of discussion to fill in what is missing. In the case above, “numbers”—things on which arithmetic can be done—were extended to include fractions. This is a conceptual blend. But the demand of the discipline require it be done “rigorously.” Thus there is a protocol. One blends (a) existing mathematical objects (e. g., integers) with (b) constructions from basic set theory (e. g., ordered pairs, equivalence relations) to the desired end. Thus in one sense, the strictly logical one, there is nothing new in the blend. That is, the new entities can be recursively reduced to the basic foundations of the subject. On the other hand, if the blend is to be fruitful, it must be something new; i. e., it must have emergent structure. There is a word in mathematics for this process—"generalization." Thus arithmetic was generalized from integers to fractions.

At this point, we return to the quotation of Rebecca Goldstein in footnote 1. Some generalizations are more potent than others. To a practitioner, a potent generalization is beautiful—it can actually effect physical excitement. The aesthetics come from everything fitting together just so correctly—it feels so right. The potency usually derives from the concomitant emergent structures. Fauconnier/Turner note, "crucial scientific leaps involve the discovery of powerful blends that can be run ever more to develop ever more useful emergent structures" (Fauconnier and Turner, 2002, p. 307). A hallmark of "higher" mathematics is that blending cascades. That's how mathematics gets high. It is also how mathematics leads to "structures that do not fit our basic stories" (Turner, 2005), taking it out of the realm of common human cognition, and hence seemingly esoteric. And it is why the formal structure of mathematics is important, so the scaffolding is firm—one knows one can always climb down to the foundation—and does not collapse like a house of cards at the slightest breeze.

There is a closely allied mathematical process called abstraction. Abstraction is more a case of framing. A mathematical concept is disembodied—removed from particular instances, characterized by a set of properties, often called axioms (although this is not the original meaning of the word), and usually given a name. For example, there is the concept of "group." Something is a group if it has a law of composition of pairs $(\alpha, \beta) \to \alpha \cdot \beta$ satisfying certain axioms, e. g.$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$ (the purpose of the group concept is to capture the notion of symmetry). An abstraction creates a context and is a framing. A generalization is a blending. However, sometimes it is realized the generalization is so useful and widely applicable that it is abstracted and a generalization leads to an abstraction. In fact the boundary between generalization and abstraction is fuzzy, and sometimes the words are used interchangeably, and indeed sometimes it is not useful to distinguish them. For our purposes, however, a generalization always involves a construction, and is seen to be a blend.

5 Functors, forgetful and otherwise

Would it be surprising that within the formalism of mathematics, generalization itself has been abstracted? In the mid-20th century, category theory, a framing of large areas of mathematics, was conceived. Many mathematicians hold that the substance of their discipline is the study of structures. Category theory brings this view into sharp focus. Indeed, some mathematicians have used category theory as a foundation alternative to set theory. For example, an (integral) domain is an object with with the structures of addition, subtraction and multiplication, obeying certain relations (e. g., the distributive law $a(b + c) = ab + ac$), but not necessarily division. The integers form a domain, as do the rational numbers, as do polynomials $p(x)$ with coefficients in some domain (e. g., polynomials in a variable x with integer coefficients, rational coefficients, . . .), and a host of other objects. Thus one has the category $\mathfrak{D}$ of domains, consisting of all domains. A field is a domain with division. The rational numbers form a field; the integers do not. Rational functions form a field; polynomials do not. One also has the category $\mathfrak{F}$ of fields.

To understand relations among different structures, on moves from one category to another via functors. One might, for some reason, ignore—forget about—division in a field, as if the ÷ button on the calculator were broken. That is, any field is a domain, and we view everything though polarizing glasses that blind us to division. Taking any field to its underlying domain is called a "forgetful functor." A forgetful functor is rather trivial and seemingly uninteresting. However, within the edifice of category theory, forgetful functors often have associated "adjoint functors," that go in the other direction, here from domains to fields, that to each domain associates a field that has, in the most efficient and natural manner, the structure that the forgetful functor forgets. It is not standard terminology, but one might call such a functor a "generalization functor," or a "blending functor," if you will. Unlike a forgetful functor, a generalization functor is highly non-trivial, and may or may not exist. Usually a generalization functor is effected by a construction of some kind—a generalization. Thus adjoint to the forgetful "ignore division" functor: $\mathfrak{D} \rightarrow \mathfrak{R}$ domains is the so-called "field of fractions" functor: $\mathfrak{R} \rightarrow \mathfrak{D}$, which is the construction that introduces fractions by exactly the ordered-pair construction preceding equations (2), (3), with arithmetic structure given by equations (2), (3). Note that it does not impose division on the original domain, but rather constructs a new object, a mathematical blend, that has the additional structure of division. If the original domain happens to support division, the blend is token; the new object is the given one, only with division made visible. The field of fractions of the integers is the rational numbers. The field of fractions of the rational numbers is itself, since the rational numbers already support division. The field of fractions of polynomials is rational functions. More generally (or more abstractly), the concept of generalization has been abstracted into the category-theory notion of adjoint functor to a forgetful functor. One may also consider a generalization functor as the mathematical formulation of the closure concept of Lakoff and Núñez (2000, p. 21) and of non-arbitrary concept expansion of Buzaglo (2002). For more on category theory: there are many books on the subject, and a web search will turn up a number of expository articles.

6 Negative numbers

We next turn to the fact that subtraction is not always defined on the non-negative numbers (either integers or fractions). That is, although $m - n$ is defined if $m \geq n$, not so if $m < n$. This inhibits our arithmetic. The solution is a blend, quite analogous to the construction of fractions, although the notation is handled differently. We consider ordered pairs (m, n) of non-negative numbers, which are supposed to represent $m - n$. There is redundancy and thus we set $(m, n) \equiv (p, q)$ if $m + q = n + p$ (again, equivalence relations and quotient sets). We define

$$(m, n) + (p, q) = (m + p, n + q) \qquad \text{and} \tag{4}$$

$$(m, n) \times (p, q) = (mp + nq, mq + np). \tag{5}$$

We embed the positive numbers in this new set by $m \to (m, 0)$. We introduce a new term, the "negative of a non-negative number m," denoted $-m$, by $(0, m)$. This notation eliminates the need for using ordered pairs (compression at work), and these new entities are welcomed into the family of numbers, as the negative numbers, with no further comment. Again, some routine verification establishes that this is a generalization, and that arithmetic is extended to the complete family.

We learn about negative numbers early on, and are so used to them, that we seldom consider the blend that led to them. It was not always so. As late as the 1600s, negative numbers were under some suspicion, and the value of

$(-1) \times (-1)$ was open for discussion. [Girolamo] Cardano (Cardani) (1545) called them "numeri ficti." [René] Descartes (1637), in accordance with the usage of his time, called negative roots of polynomials «faux». Again, both phylogenetically and ontogenetically, without precise definitions, clarity is elusive. Modern students learn at some point in early school that "negative × negative = positive." A quick web search will reveal that young inquiring minds look for a rationale. In response, educators offer all kinds of heuristic reasons, sometimes with the implicit or explicit message, "If you don't like this one, maybe this other one will work."[13] Moreover, for the student, in the end it really comes down to, "Because my teacher told me so."[14] Of course, more than heuristics is not possible if the basic definition is heuristic, but with a precise definition, a precise answer is possible. Since $-a$ is represented by any $(0, a)$, then by (3), the product $(-a) \times (-b)$ is represented by $(0, a) \times (0, b) = (ab, 0)$. This is not to maintain that the rigorous generalization should be taught in early grades, but rather to make the point that the cascade of blends that goes into modern mathematics requires precision to maintain its integrity. If there is a question about the value of $(-1) \times (-1)$,

[13]This is not altogether fortunate. Mathematics is not litigation, in the exercise of which one presents a variety of rationales, hoping the court will accept one, any more than science is done by voting. Rather the one correct well-grounded argument should serve. But if heuristics is the horse one has mounted, heuristics is the horse one must ride.

[14]Thus, W. H. Auden (1970, p. 92) on "Cultures, The Two":

> Of course, there is only one. Of course, the natural sciences are just as "humane" as letters. There are, however, two languages, the spoken verbal language of literature, and the written sign language of mathematics, which is the language of science. This puts the scientist at a great advantage, for, since like all of use, he has learned to read and write, he can understand a poem or a novel, whereas there are very few men of letters who can understand a scientific paper once they come to the mathematical parts.
>
> When I was a boy, we were taught the literary languages, like Latin and Greek, extremely well, but mathematics atrociously badly. Beginning with the multiplication table, we learned a series of operations by rote which, if remembered correctly, gave the "right" answer, but about any basic principles, like the concept of number, we were told nothing. Typical of the teaching methods then in vogue is this mnemonic which I had to learn.
>
> > Minus times Minus equals Plus:
> > The reason for this we need not discuss.

(Thus, incidentally, the common ascription of this couplet to Ogden Nash seems to be erroneous.)

it means that issue is not, at least at that moment, at a "human scale." The methodology of formal blending permits such issues to be backed off through the blendings, i. e., "reduced to a known problem."

Then of course, the blend is completed by compression. Negative numbers are not thought of as ordered pairs, but as entities in their own right, equal to the positive numbers.

The sequel: An object with the structure of the non-negative numbers—addition but not necessarily subtraction—is called a monoid. Monoids turn up reasonably often. There is a category of monoids. In the mid-1950s, [Alexandre] Grothendieck (1928–)[15] realized the generalization above—using ordered pairs in a certain monoid of sheaves to introduce negatives—was precisely what was need to formulate what is now called the Grothendieck-Riemann-Roch theorem.[16] He called the resulting negative objects „Klassen" (classes) and the resulting extension "K-groups." See below, end of this section. They are now known as Grothendieck groups, and have been one of the most fruitful concepts of late 20th-century mathematics. One may ask why such a simple-minded generalization as appending virtual negatives could be so fruitful. The answer is that K-theories come with an incredible amount of emergent and found structure, some surely yet to be unearthed. K-theories blend with other structures of mathematics with the precision of a well-made tenon in a mortise. [Michael] Atiyah and [Friedrich] Hirzebruch almost immediately blended Grothendieck's concept with some other geometry to create topological K-theory, a Grothendieck group of vector bundles on topological

[15]Grothendieck was (at this time of writing, he is alive, but long ago withdrew from mathematics) one of the giants in mid-twentieth-century mathematics, who transformed a number of areas of research. Much of his genius was his adeptness at generalization and abstraction, as noted in the following portrayal:

> He [Grothendieck] had an extremely powerful, almost other-worldly ability of abstraction that allowed him to see problems in a highly general context, and he used this ability with exquisite precision. Indeed, the trend toward increasing generality and abstraction, which can be seen across the whole field since the middle of the twentieth century, is due in no small part to Grothendieck's influence. At the same time, generality for its own sake, which can lead to sterile and uninteresting mathematics, is something he never engaged in (Jackson, 2004, p. 1038).

This article, and its continuation, are highly recommended for a perspective on the individual and his milieu.

[16]Grothendieck did not publish his results; the Grothendieck-Riemann-Roch theorem first appeared in Borel and Serre (1958).

spaces (bundles are themselves a blend of geometry and algebra), and in particular the found structure of "generalized cohomology theories"(Atiyah and Hirzebruch, 1961).[17] In topological K-theory, [J. Frank] Adams blended the Grothendieck group with constructions on bundles[18] (Adams, 1962). There are today a plethora of K-theories (topological, algebraic, ..., with various modifiers), under currently active investigation.

It should be noted that there is no story here of intuitive or heuristic formulation later reworked with "purely formal tricks." By the mid-twentieth century, the intellectual culture of mathematics was set (Adams particularly was known for his punctiliousness). But there are stories to tell. As noted by Fauconnier and Turner (2002), blending is a uniquely human activity, and for each blend, there is a greater or lesser story of invention. Good histories of mathematics tell these stories.

A categorical paragraph: A mathematical object with (commutative) addition and an identity element (zero) is called a monoid. If it also has negatives and hence subtraction, it is a group. With respect to the earlier note on category theory, we note there is a "forgetful functor–adjoint functor" system here. The forgetful functor forgets subtraction, thus regards groups only as monoids; the adjoint functor is the Grothendieck construction. Technically: for any monoid, one considers pairs (a, b), eventually denoted $a - b$. Two pairs (a, b), (a', b') are equivalent if $a + b' + t = a' + b + t$ for some t. The monoid elements a are included via $a \to (a, 0)$. Addition is defined by $(a, b) + (c, d) = (a + c, b + d)$. Elements $(0, b)$ (or $-b$) are called virtual elements. This enlarges a monoid to a group, the associated K-group. The Grothendieck construction on the monoid of positive numbers yieids the group of positive and negative numbers. Other examples are fiber bundles over a topological space (with Whitney sum as addition), and modules over a domain (with direct sum as addition). Both of these example are rich with emergent and found structure. Another possibility: consider knots—ordinary

[17]More correctly perhaps, it should be stated that, by this example, uncovered the concept—the abstraction—of generalized cohomology theory.

[18]These are now called Adams operations and they are a good example of generalizations as found structure in a succession of multiple blends. Their pedigree can be traced to simple particular polynomials (Newton polynomials), taken over as found structure to the blend of vector spaces; this algebraic structure is taken over as found structure to the geometric blend of bundles, and then taken over again as found structure to the algebraic blend of Grothendieck group. At each stage, there is a "reduction to human scale," at least for human mathematicians.

knots in strings (but with the ends—leading and trailing—of the string held so the knot cannot be untied). Addition of two knots is concatenation, i. e., taking a the leading string end of one knot and melding it with the trailing string end of the other knot. The identity element is the "unknot," a string with no knot. This defines a monoid. It is possible to construct the associated K-group (thus with "negative" knots). However, there does not seem to be any useful emergent structure in this case. As another demonstrative example, consider non-zero integers with multiplication as the monoid operation, and thus 1 as the identity. The associated Grothendieck group is the rational numbers.

7 Completion (irrational numbers)

As the story is told, the Pythagoreans discovered that the length of the diagonal of the unit square, namely $\sqrt{2}$, cannot be written m/n for integral m and n. Exactly what the argument was is not known. The Pythagoreans were intrigued with the dichotomy of evenness/oddness of integers and the dichotomy can form the basis of an argument (somewhat subtle in that it is a proof by contradiction); namely a fraction even/even is not in lowest terms. This argument is found in Aristotle's *Prior Analytics* (Stillwell, 1989, p. 9), and is the one usually presented today. On the other hand, geometry and arithmetic were kept well-separated, and the diagonal of the unit square is a geometric object. The original argument could well have been geometric (Fowler 1999, chap. 2; Stillwell 1989, §3.4). In any event, classical Greek mathematics insisted on a separation of the geometric from the arithmetic; $\sqrt{2}$ did not exist as a number, although it was clearly a length. The separation was maintained through the succeeding centuries, and what entities were entitled to the name "number" was an issue.

In their Chapter 3, Fauconnier and Turner (2002) relate the riddle of the Buddhist monk who ascends a mountain along a path from dawn to dusk one day, and descends along the path from dawn to dusk a succeeding day. Is there a place on the path that the monk occupies at the same time of day on the two journeys? They solve the riddle by having two clones of the monk make the up and down journeys on the same day—where and when the clones meet is the desired point. Fauconnier and Turner (2002) analyze the blending involved in solving the riddle. But for us—suppose the clones meet at π pm? If π is not a number, then at no quantifiable time do they cross paths—do the clones meet?

Of course, this is all simply too esoteric; one should rather count the number of angels that can fit on the head of a pin. Clearly the two monk clones meet. However, with the invention of the calculus, it became necessary to closely inspect the concept of a limit, and thus of number, to be used in infinitesimal analysis[19] (Boyer 1959). In 1821, [Augustin-Louis] Cauchy (1789–1857) wrote *Cours d'analyse de l'ècole royale polytechnique: Analyse algébrique*, an early major treatise attempting to carefully and expansively develop the emerging subject of infinitesimal analysis. Cauchy did not give a precise definition of real number, assuming rather the geometric concept of the line sufficed. However, modulo this omission, Cauchy did give a careful exposition of limits. A sequence a_1, a_2, a_3, $\ldots$ of numbers has a limit a (converges to a) if the "remainders" $a_n - a$ become vanishing small as $n \to \infty$.[20] Cauchy developed a necessary condition for convergence (which was not completely original with him) that does not involve the limit a. It has a geometric rendering in terms of pixels. Suppose one sets a pixel size $\epsilon > 0$—think of it as a tolerance. Suppose one narrows down the tolerance by decreasing ϵ; one now has a subpixel, and by iterating, a nested set of pixels. The decimal notation for numbers is a realization of this conceptualization. Thus 1.4... encompasses all numbers from 1.35 to 1.45, a pixel of size .1. Similarly 1.41... implies a pixel size of .01. Each finite expansion is rational (with denominator a power of 10), and each additional digit decreases the pixel size ϵ by a factor of 10. As $\epsilon \to 0$, the nested pixels contract down, but to what? One might say, as was said, that such a nested sequence of pixels converges *internally*. If there is an actual number a to which it converges, we can say it converges *externally*. Suppose our mathematical blends have not gone beyond rational numbers. The expansion .3333... (all '3's) contracts down to, and hence converges externally to, the the rational 1/3. The square of the expansion 1.414213562373095048 is pretty close to 2, and the squares of appropriate longer expansions come closer and closer to 2. But there is no rational number with square equal to 2, so there is no number available for the expansion to contract down to. The pixels contract down to a hole; although they converge internally, they do not converge externally.

[19] Which led to the δ-ϵ formulation, the bane of calculus students everywhere. More relevant to our narrative, it was the realization that precision was necessary that led to the paradigm of modern mathematics.

[20] More precisely, for every $\epsilon > 0$ (the measure of smallness) there exists N (the measure of going to ∞), depending on ϵ such that $|a_n - a| < \epsilon$ if $n > N$.

Nowadays, seduced by the potency of decimal notation, we effortlessly create the blend of an infinite expansion, and think of it as a number—there are no holes, and internal convergence implies external convergence. However, in the time of Cauchy, no one consummated that mathematical blend. Rather, it took another 31 years and a completely different blend to pull it off. In 1858, [Richard] Dedekind (1831-1916) conceived what are now called Dedekind cuts [„Schnitt“]. Note that the pixel concept relies on the notion of distance between numbers, which is metered against ϵ. Dedekind did not use distance in his blend, but rather a different input from geometry, namely the ordering—left and right. He divided the rational numbers into two non-empty sets A_1 and A_2 such that every $a_1 \in A_1$ was less than every $a_2 \in A_2$. For example A_1 might consist of all negative rationals and all non-negative rationals a_1 such that $a_1^2 < 2$, and A_2 the complement. Such a cut *defines* a (real) number. The number is rational if there is a largest member of A_1 or a smallest member of A_2 (namely, that largest or smallest member) and irrational otherwise. This bails out our Buddhist monk: let A_1 be the times for which the ascending clone has never been above the descending clone, and A_2 the later times. In 1872, Dedekind published *Stetigkeit und irrationale Zahlen.*[21] For our present purposes, this is a remarkable publication. He details the modern mathematical blending process, in its fullness. Lakoff and Núñez (2000) devote their chapter 13 to “Dedekind’s Metaphors,” and note, ”What we see in these pages from Dedekind is one of the most important moments in the history of modern mathematics” (Lakoff and Núñez 2000, p. 305). It is almost worth reproducing Dedekind’s full tract, since the blending process is so clearly articulated. Rather, we recommend the document to the reader, and here quote significant segments, with bullets to emphasize the issues. The excerpts below are from Dedekind (1872).

• He lays out the deficiencies of the existing situation and the need for a blend.

Die Betrachtungen, welche den Gegenstand dieser en Schrift bilden, stammen aus dem Herbst des Jahres 1858. Ich befand mich damals als Professor am ei-	My attention was first directed toward the considerations which form the subject of this pamphlet in the autumn of 1858. As professor in the

[21]Modern terminology is Vollständigkeit (completeness) instead of Stetigkeit (continuity).

dgenössischen Polytechnicum zu Zürich zum ersten Male in der Lage, die Elemente der Differentialrechnung vortragen zu müssen, und fühlte dabei empfindlicher als jemals frührer den Mangel einer wirklich wissenschaftlichen Begründung der Arithmetik. Bei dem Begriffe der Annäherung einer veränderlichen Größe an einen festen Grenzwerth und namentlich bei dem Beweise des Saßes, da jede Gre, welche beständig, aber nicht über alle Grenzen wächst, sich gewiß einem Grenzwerth nähern muß, nahm ich meine Zuflucht zu geometrischen Evidenzen. Auch jeßt halte ich solches Heranziehen geometrischer Anschauung bei dem ersten Unterrichte in der Differentialrechnung vom didaktischen Standpuncte aus für außerordentlich nüßlich, ja unentbehrlich, wenn man nich gar zu viel Zeit verlieren will. Aber daß diese Art der Einführung in die Differentialrechnung keinen Anspruch auf Wissenschaftlichkeit machen kann, wird wohl Niemand leugnen. Für mich war damals dies Gefühl der Unbefriedigung ein so überwältigendes, daß ich den festen Entschluß faßte, so lange nachzudenken, bis ich eine rein arithmetische und völlig strenge Begründung der Principien der Infinitesimalanalysis gefunden haben würde. Man sagt so häufig, die Differentialrechnung beschäftige sich mit den stetigen Größen, und doch wird nirgends eine Erklärung von dieser Stetigkeit gegeben, und auch die strengsten Darstellungen der Differentialrechnung gründen ihre Beweise nicht auf die Stetig-

Polytechnic School in Zurich I found myself for the first time obliged to lecture upon the elements of the differential calculus and felt more keenly than ever before the lack of a really scientific foundation for arithmetic. In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theorem that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis. The statement is so frequently made that the differential calculus deals with continuous magnitude, and yet an explanation of this continuity is nowhere given; even the most rigorous expositions of the differential calculus do not base their proofs keit, sondern sie appelliren entweder mit mehr oder weniger Bewußtsein an geometrische, oder durch die Geometrie veranlaßte Vorstellungen, oder aber sie stüßen sich auf solche Säße, welche selbst nie rein arithmetisch bewiesen sind. Zu diesen gehöhrt z. B. der oben erwähnte Saß, und eine genauere Untersuchung überzeugte mich, daß dieser oder auch jeder mit ihm äequivalente Saß gewissermaßen als ein hinreichendes Fundament für die Infinitesimalanalysis angesehen werden kann. Es kam nur noch darauf an, seinen eigentlichen Ursprung in den Elementen der Arithmetik zu entdecken und hiermit zugleich eine wirckliche Definition von dem Wesen der Stetigkeit zu gewinnen. Dies gelang mir am 24. November 1858...

upon continuity but, with more or less consciousness of the fact, they either appeal to geometric notions or those suggested by geometry, or depend upon theorems which are never established in a purely arithmetic manner. Among these, for example, belongs the above-mentioned theorem, and a more careful investigation convinced me that this theorem, or any one equivalent to it, can be regarded in some way as a sufficient basis for infinitesimal analysis. It then only remained to discover its true origin in the elements of arithmetic and thus at the same time to secure a real definition of the essence of continuity. I succeeded Nov. 24, 1858...

• One clearly sees the geometry coming into the blend.

Von der größten Wichtigheit ist nun aber die Thatsache, da es in der Geraden L unendlich viele Puncte giebt, welche keiner rationalen Zahl entsprechen. Entspricht nämlich der Punct p der rationalen Zahl a, so ist bekanntlich die Länge op commensurabel mit der bei der Construction benußten unabänderlichen Längeneinheit, d.h. es giebt eine dritte Länge, eine sogenanntes gemeinschafliches Maß, von welcher diese beiden Längen ganze Vielfache sind. Aber schon die alten Griechen haben gewußt und bewiesen daßes Längen giebt, welche mit einer gegebenen Längeneinheit incommensu-

Of the greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number. If the point p corresponds to the rational number a, then, as is well known, the length op is commensurable with the invariable unit of measure used in the construction, i. e., there exists a third length, a so-called common measure, of which these two lengths are integral multiples. But the ancient Greeks already knew and had demonstrated that there are lengths incommensurable with a given unit of length, rabel sind, z. B. die Diagonale des Quadrats, dessen Seite die Längeneinheit ist. Trägt man eine solche Länge von dem Puncte o aus auf der Geraden ab, so erhält man einen Endpunct, welcher keiner rationalen Zahl entspricht. Da sich ferner leicht beweisen läßt, da es unendlich viele Längen giebt, welche mit der Längeneinheit incommensurabel sind, so können wir behaupten: Die Gerade L ist unendlich viel reicher an Punct-Individuen, als das Gebiet R der rationalen Zahlen an Zahl-Individuen. Will man nun, was doch der Wunsch ist, alle Erscheinungen in der Geraden auch arithmetisch verfolgen, so reichen dazu die rationalen Zahlen nicht aus, und es wird daher unumgänglich nothwendig, das Instrument R, welches durch die Schöpfung der rationalen Zahlen construirt war, wesentlich zu verfeinern durch eine Schöpfung von neuen Zahlen der Art, daß das Gebiet der Zahlen dieselbe Stetigkeit gewinnt, wie die gerade Linie...

e. g., the diagonal of the square whose side is the unit of length. If we lay off such a length from the point o upon the line we obtain an endpoint which corresponds to no rational number. Since further it can be easily shown that there are infinitely many lengths which are incommensurable with the unit of length, we may affirm: The straight line L is infinitely richer in point-individuals than the domain R of rational numbers in number-individuals. If now, as is our desire, we try to follow up arithmetically all phenomena in the straight line, the domain of rational numbers is insufficient and it becomes absolutely necessary that the instrument R constructed by the creation of the rational numbers be essentially improved by the creation of new numbers such that the domain of numbers shall gain the same completeness, or as we may say at once, the same continuity, as the straight line...

Im vorigen Paragraphen ist darauf aufmerksam gemacht, daß jeder Punct p der Geraden eine Zerlegung derselben in zwei Stücke von der Art hervorbringt, daß jeder Punct des einen Stückes links von jedem Puncte des anderen liegt. Ich finde nun das Wesen der Stetigkeit in der Umkehrung, also in dem folgenden Princip: „Zerfallen alle Puncte der Geraden in zwei Classen von der Art, daß jeder Punct der ersten Classe links von jedem Puncte der zweiten Classe

In the preceding section attention was called to the fact that every point p of the straight line produces a separation of the same into two portions such that every point of one portion lies to the left of every point of the other. I find the essence of continuity in the converse, i. e., in the following principle: "If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second liegt, so existirt ein und nur ein Punct, welcher diese Eintheilung aller Puncte in zwei Classen, die Zerschneidung der Geraden in zwei Stücke hervorbringt.

class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions."

• The essence of a generalization is explicit. In particular, the blend must be totally in terms of the inputs.

Daß solche Anknüpfungen an nicht arithmetische Vorstellungen die nächste Veranlassung zur Erweiterung des Zahlbegriffes gegeben haben, mag in Allgemeinen zugegeben werden (doch ist dies bei der Einführung der complexen Zahlen entschieden nicht der Fall gewesen); aber hierin liegt ganz gewiß kein Grund, diese fremdartigen Betrachtungen selbst in die Arithmetik, in die Wissenschaft von den Zahlen aufzunehmen. Sowie die negativen und gebrochenen rationalen Zahlen durch eine freie Schöpfung hergestellt, und wie die Geseße der Rechnungen mit diesen Zahlen auf die Geseße der Rechnungen mit ganzen positiven Zahlen zurückgeführ werden müssen und können, ebenso man dahin zu streben, da auch die irrationalen Zahlen durch die rationalen Zahlen allein vollständig definirt werden. Nur das Wie? bleibt die Frage.

That such comparisons with non-arithmetic notions have furnished the immediate occasion for the extension of the number-concept may, in a general way, be granted (though this was certainly not the case in the introduction of complex numbers); but this surely is no sufficient ground for introducing these foreign notions into arithmetic, the science of numbers. Just as negative and fractional rational numbers are formed by a new creation, and as the laws of operating with these numbers must and can be reduced to the laws of operating with positive integers, so we must endeavor completely to define irrational numbers by means of the rational numbers alone. The question only remains how to do this.

• Quite clearly, the blend incorporates the ordering of the rationals—ordering enters the definition of cut. Dedekind explicitly carries the ordering through the blend.

• Dedekind explicitly incorporates the rationals (input) into the reals (blend), as those cuts (A_1, A_2) where either A_1 has a greatest member or A_2 has a least member. Thus it is appropriate to extend the name "number" to the new entities.

• Moreover, he shows the blend is complete. That is, if a cut is made in the newly defined real numbers, one obtains a real number again, not again something else new (the blend has effectively filled in all the holes).

• The emergent structure (in this case, arithmetic operations) is made explicit.

Um irgend eine Rechnung mit zwei reellen Zahlen α, β auf die Rechnungen mit rationalen Zahlen zurückzufhren, kommt es nur darauf an, aus den Schnitten $(A1, A2)$ und $(B1, B2)$, welche durch die Zahlen α und β im Systeme R hervorgebracht werden, den Schnitt $(C1, C2)$ zu definiren, welcher dem Rechnungsresultate γ entsprechen soll… [M]an gelangt auf diese Weise zu wircklichen Beweisen von Säßen (wie z. B. $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), welche meines Wissens bisher nie bewiesen sind.

To reduce any operation with two real numbers α, β to operations with rational numbers, it is only necessary from the cuts $(A1, A2)$, $(B1, B2)$ produced by the numbers α and β in the system R to define the cut $(C1, C2)$ which is to correspond to the result of the operation, γ …[I]n this way we arrive at real proofs of theorems (as, e. g., $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before.

• Then finally, Dedekind discusses the found structure. He shows that limits are well-defined, and thus his blend of reals is a solid foundation for infinitesimal analysis.

Altogether an impressive display of mastery of a mathematical blend, suitable for our appreciation.

As noted above, the distance $|a - b|$ between two numbers does not enter at all into Dedekind's account. Distance extends through the blend to irrationals, and is thus an emergent structure. However, others were following up Cauchy's development, and not long after Dedekind's publication, [Georg Ferdinand Ludwig Philipp] Cantor (1845–1918), [Heinrich Eduard] Heine (1821–1881) and others realized that *defining* a real number by its Cauchy sequences was an effective blend (in fact, some have conjectured that knowledge of these investigations is what led Dedekind to publish—hence his explicit mention of 1858). For a brief discussion of the history, see

Boyer and Merzbach (1989, §25.6, 25.7).[22] Cauchy did not discuss pixels in his *Cours d'analyse*. Rather he considered sequences of numbers. He noted that in order for a sequence a_1, a_2, a_3, ... of rationals to have a limit, it had to have the property that for any $\epsilon > 0$, there is an N, depending on ϵ, such that for all m, $n > N$, $|a_m - a_n| < \epsilon$. This is an analytic way of expressing internal convergence. Note that the distance $|a_m - a_n|$ between terms of the sequence is integral to the concept. A sequence with this property is nowadays called a Cauchy sequence.[23] A sequence of decimal expansions with increasing numbers of digits is a Cauchy sequence. One then insists, as a blend, that any internally converging sequence converges externally, i. e., a number is just a Cauchy sequence. In particular, with our decimal notation, Cauchy is often the way we think of irrationals. When we calculate $\sqrt{2}$ on our hand calculators to obtain 2.141214, we perceive the result as a range of numbers in which there is a "correct" result, and that with more precision, the range is narrowed. It is a simple blend, that with a calculator of infinite precision, a precise answer would result.

The point is that there are two completely different blends that we use to obtain all real numbers, both with geometric significance. With Dedekind cuts, the ordering is input, the idea of a line as a continuum embraces the geometry—as Dedekind noted, if two lines cross, they must have a point, not a hole, in common—and distance is an emergent structure. With Cauchy sequences, the distance is input, the idea of nested pixels embraces the geometry—again, no holes—and the ordering is an emergent structure. Sometimes the Dedekind blend suits, and sometimes the Cauchy blend. In the end, both work, and indeed both blends produce the same set of real numbers. We can use whichever is most appropriate in any situation.

Again compression occurs. Irrational numbers—infinite decimal expansions—are numbers, on a equal footing with rational numbers. Rational numbers are special real numbers (in fact, precisely those whose decimal expansions eventually become periodic, with the same sequence of digits repeating forever).

However, although the two constructions get us to the same result in the case of real numbers, they are not equivalent. Sometimes one or the other structure, distance or ordering, is absent, and only one construction is

[22]See also the mention of Méray below in Section 9.

[23]Indeed, Augustin-Louis has been freed from noun phrases; the sequence is said to be Cauchy.

available to us. There is a rich abstract theory of orderings (with words such as partially ordered sets, lattices, Boolean algebras), for which Dedekind's construction is ideal. If distance is the issue, Cauchy sequences are the input to the standard mathematical blend, now abstracted and called the completion. Thus in this terminology, the reals are the completion of the rationals with the usual distance. The blend is powerful, and turns up in more exotic situations. An example: let $b > 1$ be an integer, and consider base b expansions of integers ($b = 10$ for our usual decimal expansions). Define the b-adic size of a nonzero positive integer as b^{-n}, where n is the number of final 0's in its expansion in base b. Thus 10 has 10-adic size 1/10, as does 5280, 1500 has 10-adic size 1/100, 25,000 has 10-adic size 1/1000 (0 has size zero). This size extends easily to rationals—the size of a rational is the size of the numerator divided by the size of the denominator. Define the distance between two numbers as the size of their difference, so-called b-adic distance. Thus two integers are close if they have lots of identical final digits. The sequence 1, 10, 100, 1000, ... converges to 0. The structure to define Cauchy sequences is all there. There are sequences that converge internally, but not externally. That is there are holes—b-adic holes. The analogue of an infinite decimal expansion is an expansion extending infinitely far to the left.[24] Thus $\cdots 11111111.$ is an internally converging sequence—possibly a hole.[25] The new entities one obtains in the completion are not at all like the real numbers we think of. It is counterproductive to try to interpret this blend in terms of real numbers. Rather one obtains what are called b-adic numbers. For each b, there is an infinitesimal analysis called (no surprise) b-adic analysis, distinctive to b and very different from the usual analysis—thus b-adic limits, b-adic derivatives, b-adic integrals, etc. The full power of the analysis occurs for b a prime (otherwise, complicating zero divisors occur). Although our usual geometric visualization fails, it is possible to visualize all this (in terms of (mathematical) trees); see Holly (2001) for an elementary exposition, and references to more advanced material. But there is no ordering; there is no sense of left and right, greater and lesser, and Dedekind cuts cannot be defined.[26]

[24]It is also common notation to reverse the order, so that the infinite expansion runs to the right.

[25]Possibly, but in fact not; $\cdots 11111111.$ is the 10-adic expansion of $-1/9$ (multiply by 10 (shift left one place), add 1, obtain the same number).

[26]Thus, to follow up a thread of Lakoff/Núñez, who as modern cognitive scientists, note, "The detailed nature of our bodies, our brains, and our everyday functioning in

It should be explicitly noted what the arithmetization of analysis accomplished. It united number and geometry, which had been kept separate since the classical Greeks. A geometric line no longer stood as a fundamental entity. A line was an emergent structure. Dedekind explicitly felt he was capturing the essence of a line in his cuts, and a line is nothing more, nothing less, than the set of real numbers, with the concomitant ordering. Moreover, the real numbers, and hence the line, "as [Hermann] Hankel [(1839–1873)] foresaw, ... real numbers are to be viewed as 'intellectual structures,' rather than intuitively given magnitudes inherited from Euclid's geometry" (Boyer and Merzbach 1989, p. 628), i. e., the result of a human activity, a blend. From Lakoff/Núñez:

> Not only is arithmetic freed from the bonds of the Cartesian plane, but the very notion of space itself becomes reconceptualized—and mathematically redefined!—in terms of numbers and sets. In addition, the real-number line becomes reduced to the arithmetic of rational numbers, which had already been reduced to the arithmetic of natural numbers plus the use of classes. This sets the state for the Foundations movement in modern mathematics, which employs the Numbers are Sets metaphor to reduce geometry, the real line, the rational numbers, and even the natural numbers to sets. All mathematics is discretized—even reason itself, in the form of symbolic logic (Lakoff and Núñez 2000, p. 305).

It is also worth considering the reduction to human scale. With a geometric intuition, the geometric line is the foundation of human scale. However, for

the world structures human concepts and human reason. This includes mathematical concepts and mathematical reasons" (Lakoff and Núñez 2000, p. 5). They note what they call the Numbers are Points on a Line Metaphor, which goes into what they, and we, call Dedekind's blend (Lakoff and Núñez 2000, p. 295). That is, only by our geometric sense do we complete the rationals to the reals. Indeed Núñez (2005) investigates fictive motion in mathematics to demonstrate that at some level our conceptualization of number is inextricably bound up with our kinesthetic perception of physical space, a premise also supported by recent neurological investigations (Hubbard et al. 2005). For an idle moment, grant that it is conceivable that some alien race has a different intuition. Mathematical trees are often used to represent decision (or more generally, so-called branching) processes. Suppose the alien race, perhaps disembodied beings, does not have our human sense of space, but rather, suppose its collective cognition developed emphasizing decision making, and developed trees as a dominant cognitive abstraction. This race might have gotten to integers and rationals (see the quote of Buzaglo, page 5), but then diverged from us. Perhaps, if we make contact with them, we will have to communicate in b-adics.

mathematics, the foundation is set theory. The arithmetization of analysis effects the final step in that reduction. Mathematics is not common intuition.

And a categorical note: there is a (actually several) "forgetful functor–adjoint functor" situation here again. For example, one category consists of ordered spaces (possibly with holes), and the other of such spaces with no holes. The adjoint construction of course, is the Dedekind construction.

8 Imaginary numbers

The name of Dedekind's blend is "real numbers." The name is perhaps itself an emergent structure; it has nothing to do with the reason for the blend. Rather it was introduced by Descartes in 1637 along with, and in opposition to "imaginary" numbers, in other words, the familiar in opposition to the strange. Imaginary numbers were introduced in the 1500s, but as bastards. That is, they were there, but were largely disowned, even by their creators. They did not gain full credence until about 1800.

The formula for the roots r of a quadratic equation $ax^2 + bx + c = 0$, namely $r = (-b \pm \sqrt{b^2 - 4ac})/2a$, was known in antiquity. If $b^2 - 4ac < 0$, the solutions involve the square root of a negative number. In fact, this caused no difficulty, since in fact no solutions was quite consistent with whatever application led to the equation. Rather, cubics caused the crisis. 'Umar al-Khayyami (Omar Khayyam), around 1100, found a methodology for solving some cubic equations as intersections of conic sections and lines. The Cardano algebraic formulae[27] for roots of a cubic, "the first clear advance in mathematics since the Greeks"(Stillwell 1989, p. 54) was developed in the sixteenth century.

A cubic equation always has at least one real root. The case of three real roots, the *casus irreducibilis*, involves the square root of a negative number; Cardano avoided this case. Cardano is in fact willing to write the roots of the quadratic $x^2 - 10x + 40 = 0$ as $5 \pm \sqrt{-15}$ (but: "*& hu-*

[27]Evidently at least partially found in the early 1500s by [Scipioni] del Ferro (1465–1526), later independently by [Niccolò Fontana] Tartaglia (1500–1557), teased from Tartaglia by [Girolamo] Cardano (1501–1576) and published in Cardano (Cardani) (1545). The plural is used advisedly. Following [Muhammad ibn Musa] Al-Khwarizmi and other Arabic mathematicians, the argument was geometric, involving breaking a cube into eight solids and comparing volumes. All quantities were geometric, and thus had to be positive. Accordingly, the justification for $x^3 + 15x = 38$ was different than that for $x^3 = 15x + 38$. However, the final result was algebraic, involving square and cube roots.

cusque progreditur arithmetica subitilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile" (Cardano (Cardani) 1545, p. 287) ("And to this extent the exactness of arithmetic emerges, which at this extent I assert, precisely as it is exact, so is it useless"), but for example, a root of the cubic $x^3 - 15x - 4 = 0$ is $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$, which was beyond him. A few decades later, [Rafael] Bombelli had the "absurd idea" («*un'idea assurda*») ((Bombelli 1572)) to treat imaginary numbers as if they could be manipulated arithmetically—added and multiplied—and handled the *casus irreducibilis*.[28] Thus Bombelli is generally credited with unmasking imaginary numbers. However, imaginary numbers were by no means put within a human scale. There are a number of excellent histories of complex numbers. As discussed in the next section, they remained enigmatic for centuries. Over these centuries, they found their way into computations and theory. Finally, the mystery (and imaginariness) of imaginary numbers was disposed of by [Caspar] Wessel in 1799 (but published in Norwegian, and unknown for a century), [John-Robert] Argand in 1806, and most influentially by [Johann Carl Friedrich] Gauss in 1831 (although Gauss noted the idea in his diary in 1797), by blending them into geometry, as points in a plane. Thus Gauss:

...sicuti omnis quantitas realis per partem rectae utrinque infinitae ab initio arbitrario sumendam, et secundum segmentum arbitrarium pro unitate acceptum aestimandam exprimi, adeoque per punctum alterum repraesentari potest, ita ut puncta ab altera initii plaga quantitates positivas, ab altera negativas repraesentent: ita quaevis quantitas complexa repraesentari poterit per aliquod punctum in plano infinito, in quo recta determinata ad quantitates reales refertur, scilicet quantitas complexa $x + iy$ per punctum, cuius abscissa $= x$, ordinata (ab altera lineae abscissarum plaga	...just as every real quantity can be represented in such a way as a point on a two-sided infinite straight line by means of an arbitrary point selected as the origin and an arbitrary favored segment designated as unit length marked off, so that the points on one side of he origin represent positive and the points on the other side represent negative quantities, so can every complex quantity be represented by a point in an infinite plane, in which a specified line serves to represent the real quantities, that is to say, the complex quantity $x + iy$ by a point, whose abscissa equals x and whose ordinate (on which one side of the abscissa-line is understood as positive and the

[28] And in particular $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4$, which is indeed a root.

positive, ab altera negative sumta) $= y \ldots$	other negative) $= y \ldots$
Hoc nodo metaphysica quantitatum, quas imaginarieas dicimus, insigniter illustratur (Gauss 1832, section 38).	Thusly the metaphysical knot of quantities that we declare imaginary is significantly clarified.

Gauss continues by interpreting the algebra of imaginary numbers with geometry, and concluding with the words quoted in the next section. The geometric blend is certainly alive and well today, because it catalyzes the emergent structure of analysis—calculus—of a complex variable; indeed the branch of study is called complex analysis. This subject had begun in the previous century. By the middle of the 18th century, [Leonhard] Euler (1707–1783) and [Jean le Rond] d'Alembert (1717–1783) had a largely complete theory of elementary complex analysis, in terms of infinite series and algebraic curves.[29] With the blend completed in the 1800s, compression occurred immediately, emergent structures spilled out, and within a relatively few years, the foundations of complex analysis were in place, and well-understood. Complex analysis is rich in found structure. Gauss' student [Bernhard] Riemann (1826–1866), the most influential mathematician of his time, laid the groundwork for connecting complex analysis with advanced geometry (hence Riemann curvature), number theory (hence Riemann zeta function—which Euler had defined, but not in its entire complex scope), differential equations (hence Riemann-Hilbert problem) mathematical physics, ..., and Riemann surfaces, which are present in all of the above. Indeed, more mathematical concepts and results may have Riemann's name attached to them than for any other mathematician. Together with the found structure, recursion was well underway, and still reverberates.

[29] For one well-known example, one has Euler's equation $e^{\sqrt{-1}\nu} = \cos\nu + \sqrt{-1}\sin\nu$. How Euler came by this formula is a good example of the manipulation of the time—ingenious formal extension of real manipulations. Thus Euler expands $(\cos z + \sqrt{-1}\sin z)^n$ and equates real and imaginary parts to obtain formulae for $\sin^n z$ and $\cos^n z$. He lets $\nu = z/n$ and with some algebraic manipulation, obtains

$$\cos\nu = \frac{1}{2}\left[\left(1 + \frac{\nu\sqrt{-1}}{n}\right)^n + \left(1 - \frac{\nu\sqrt{-1}}{n}\right)^n\right],$$

at least as $n \to \infty$, and similarly for $\sin\nu$ (Euler also could manipulate limits, although not formally defined). He then extends the known real result that $(1 + z/n)^n \to e^z$ to imaginary $z = \nu\sqrt{-1}$ to obtain (now standard) expressions for sin and cos in terms of the exponential, which leads immediately to his result (Euler 1748, §132+).

In 1835, [William Rowan] Hamilton wrote such numbers as ordered pairs of real numbers, with rules for addition and multiplication, thus eliminating the need for geometry—his intent.[30] Lakoff and Núñez write:

> From a formal perspective, much about complex numbers seems arbitrary. From a purely algebraic point of view, i [$= \sqrt{-1}$] arises as a solution to the equation $x^2 + 1 = 0$. There is nothing geometric about this—no complex *plane* at all. Yet in the complex plane, the i-axis is 90° from the x-axis. Why? Complex numbers have a weird rule of multiplication [Hamilton's rule]:
>
> $$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i.$$
>
> Why? Is this an arbitrary invention of mathematicians? (Lakoff and Núñez 2000, p. 423) (emphasis in original)

Answers: Complex numbers are a non-arbitrary expansion in the sense of Buzaglo (quotes on pages 3, 5, 37). As noted above, it is emergent structure, beginning with Euler's formula and progressing to complex analysis, that leads to the complex plane with orthogonal real and imaginary axes, not arbitrariness. As to the second question: No. The opposite. It is no weirder than formula (2) (or (1)) above for addition on fractions. The multiplication rule comes from expanding $(a + bx) \cdot (c + dx)$ and then setting $x^2 = -1$ and relabeling x as i. In other words, the multiplication rule is the simplest generalization of the usual rule of arithmetic. Set constructions, especially quotient-set constructions, permit one effectively to do exactly that: create a new symbol and declare its arithmetic. There is no other interpretation—geometric or otherwise—needed. "Nothing remains of the mystic flavor that was so long attached to the imaginary numbers" (Weyl 1949, p. 32). Thus

[30]However, Buzaglo, with echos of Lakoff and Núñez (2000), believes the situation is more complicated:

> It is not exactly obvious that we can eliminate the fictional aura of the complex numbers by declaring that the square root of -1 is a point on a plane or identifying it with the ordered pair $(0, 1)$. After all, the mathematicians who use points on a plane as an interpretation of the complex numbers do not actually mean that these numbers really *are* these points, nor does anyone believe that they are merely ordered pairs. These identifications involve conceptual difficulties associated with our basic understanding of what mathematics is (Buzaglo 2002, p. 12) (emphasis in original).

for example, in 1843 Hamilton defined quaternions, with elements i, j, k, satisfying $i^2 = j^2 = k^2 = ijk = -1$, followed by octonians (or Cayley numbers), Clifford algebra, ..., with similar, but more complicated defining symbols and relations. An extremely useful construction is to start with the field of rational numbers and append roots of polynomials (thus append i as the root of $x^2 + 1$, or $\sqrt{2}$ as the root of $x^2 - 2$); this leads to, for example, the lush subject of algebraic number theory. Here is an instance where generalization has given way to something else. With mathematical formalities, cognitive effort is not required to append elements.[31]

9 Arrested blends, blended frames

With hindsight, many of the blends of mathematics seem inevitable. Each seemingly follows so logically and recusively from its predecessors. One early and continuing popular viewpoint is that there is a mathematical existence independent of human experience, and that "doing mathematics" means discovering verities of that existence. This is the Platonic philosophy, and the inevitability is one argument advanced for it. It is argued that it is possible only to imagine that any intelligent alien species would have developed the

[31]Here is one application. Recall those three problems from classic Greek geometry: doubling the cube, trisecting an arbitrary given angle, and squaring the circle, by means of straightedge and compass constructions. Doubling the cube means; starting with a line segment representing the edge of a cube, construct another segment which is the length of a cube of twice the volume. This amounts, via analytic geometry, to constructing a segment of length the root of $x^3 - 2 = 0$. Starting from a line segment, which we set of length 1, one can construct a segment of length any rational number (a standard exercise in grade-school geometry). One shows (using analytic-geometry equations of lines and circles) that any length that can be constructed is the real root of a quadratic equation with previously-constructed lengths as coefficients. One appends these roots to the field of computations. Thus one can construct segments of lengths $\sqrt{5/2}$ (root of $2x^2 - 5 = 0$) and $\sqrt{2\sqrt{10}+1} - 1$ (root of $x^2 + 2x - 4\sqrt{5/2} = 0$), but with algebra, one shows not the root of $x^3 - 2 = 0$; the $\sqrt[3]{2}$ never appears in any of the extended fields of computation. This is early 19th-century mathematics. Hence one cannot double the cube with straightedge and compass. Trisecting the generic angle also gives rise to a cubic equation, and is impossible. Squaring the circle gives rise to the equation $x^2 = \pi$; this is quadratic, but one can never get to π by such constructions, so this is also impossible.

A second application: the same type of algebra—appending roots of equations—establishes that the *casus irreducibilis* of the cubic equation, although it has only real solutions, cannot be solved by any expression involving only real roots.

same mathematics. Given the wisdom of current cognitive science, Platonism is a hard position to maintain, and indeed, it is roundly routed in Lakoff and Núñez (2000).

It is not the case that the advance of mathematics is along a decided path. A good example is explicated in Lakoff and Núñez (2000), who devote their Chap. 11 to the cognitive story of infinitesimals. In standard analysis, the "infinitesimals" dx and dy of calculus, as in $dy = 5x^2\,dx$, are convenient notational artifacts. However, Lakoff/Núñez trace back to [Gottfried Wilhelm] Leibniz the practice of regarding infintesimals as honest mathematical objects, which can be honestly manipulated. Infinitesimals are infinitely small, taking no room, and are not part of Dedekind's line. Each number comes accompanied by a tiny retinue of infinitesimals. As Lakoff/Núñez write, calculus becomes arithmetic. The pesky δ and ϵ in the arithmetization of analysis can be bypassed. But obversely, the geometric blend of Dedekind is lost. These infinitesimals are the basic of the famous objections of [Bishop George] Berkeley to calculus. As discussed below, infinitesimals have an intuitive appeal, and they are convenient mental hooks in engineering mathematics. As noted in the next section, their existence or not was a topic of debate after the creation of calculus. One can imagine that with different principals, the history would have been different. There are good reasons why δ and ϵ prevailed, but not ironclad ones.

However, one need not be a Platonist to believe that an alien intelligence would have developed counting—hence integers—and that its cognitive process would include some form of blending, thus leading at least to some basic mathematical blends. Buzaglo (2002) maintains there is a logical structure and constitutive principles of what he calls "forced expansion of concepts." Thus, he notes that "[a]lthough mathematicians may talk about the sine of a complex number, they do not try to define the sine function to apply to the moon" (Buzaglo 2002, Preface).

> The concept of square root was expanded to include the negative numbers; the concept of power, originally defined only for the natural numbers, was expanded to include zero, fractions, and real and complex numbers; the logarithm function, which was originally defined only for positive numbers, was expanded to the negative numbers; in general nearly every mathematical function has been expanded in a non-arbitrary [hence forced] way. But this is not only true of mathematics; in physics as well there are expansions of concepts that were originally defined only for a restricted range. The expansion of the

> concept of temperature to black holes, the notion of instantaneous velocity, the idea of imaginary time, and perhaps even the idea of determining the age of the universe are a few examples of this process. Metaphors and analogies can also be considered expansions of concepts beyond the sphere in which they were first used. Moreover, philosophy has always been suspected of expanding concepts beyond their legitimate range of applicability. It seem that every area that contains concepts also contains expansion of concepts.
>
> Various incidental remarks about expansion of concepts that have taken place throughout out the development of modern mathematics were made by Leibniz, Pascal, Bernoulli, and Gauss (Buzaglo 2002, p. 1).

However, "forced" does not connote "effortless." Rather, the clear historical evidence is that some of the cognitive blends we take for granted were very problematic at the time. We have in mind, each in its own time, negative numbers, irrationals, and imaginary numbers. Conversely, these cases, considered from a cognitive-science viewpoint, shed light on the process of blending. In each case, the ultimate blend that we take for granted took considerable time—centuries—to come together, certainly longer than individuals' lifetimes. When a blend is there for the taking, so to speak, but not seized, I call it an arrested blend, and it perhaps worth while to ponder this phenomenon.

There are several questions one should pose concerning the phenomenon. If blending is such a natural cognitive action, why should one be arrested? What are the phenomenological hallmarks of an arrested blend? What is the neurology of an arrested blend? And what are the specifics for our present cases?

There are any number of reasons a blend might be arrested. Here I proffer one, and discuss some of the relevant history in light of it. Many blends require some kind of reframing—perhaps a blending of frames. But this can be difficult, since frames have their own governing laws, and in particular, should be consistent. Moreover, individuals (and disciplines) become invested in frames, intellectually and sometimes emotionally. "Fighting the last war" is a well-known issue in military circles. Similarly for politicians and other policy makers, as well as, for example lovers. It is certainly the case in various sciences, and seems related to the idea of paradigm shift famously developed by Thomas Kuhn (1962), in that a paradigm is a framing writ large, and that some blends are so revolutionary that an individual cannot carry through the reframing, which thus must rely on a new generation who adopt the new framing before becoming invested in the old. This raises the questions: what is the cognitive response of an arrested blend, and how might this inform us about the blending process? One of course is denial; one simply refuses to entertain anything to do with the possibility of a blend. A second is cognitive dissonance; one tries to deal with two (or more), evidently incompatible, frames. One then is in a unstable situation, shifting back and forth from framing to framing, sometimes suspended between. One would expect this to be reflected in the language used to express the state of affairs, and an analysis of such could shed light on the cognitive process of creating blends. Euphemisms, vague phrasings, contradictions would be expected. Indeed, one can imagine visceral responses of individuals in some instances, reflecting hormonal aspects.

Fauconnier and Turner (2002, chap. 13) discuss a quite similar (perhaps identical) concept, they call category metamorphosis. They note:

> category metamorphosis can change fundamentally the structure of the category.... We have already seen in Chapter 11 how the category *number* changed to include zero and fractions, and, in the case of fractions, how complicated was the blending that produced the new version of that category. After the fact, it looks as if new elements have simply been added to the old ones, because we still use the same words for them. But in fact, in the metamorphosis of the category, the entire structure and organizing principles have been dramatically altered. It is an illusion that the old input is simply transferred wholesale as a subset of the new category (Fauconnier and Turner 2002, p. 270).

I posit that in some cognitive measure, there is a cost associated with blending frames or morphing categories, and that that cost can be too much at a particular time and cognitive state.

Blends such as irrationals and imaginary numbers were arrested for centuries—many generations—and thus provide case studies. Here we quickly survey the historical record. One sees in these cases that despite the best efforts to avoid such blends, their components kept forcing themselves into the arena. One can see cognitive bobbing and weaving to avoid being cornered.

The ancient historical record is well known. The irrationality of $\sqrt{2}$ caused a separation of the symbolic and the geometric. The diagonal of the unit square was permitted in geometry, but did not exist symbolically. This dichotomy was preserved until the arithmetization of geometry in the 1800s. For example, as noted, Cardano phrased his arguments in geometric terms, although he was deriving symbolic formulae for symbolic situations. [Felix] Klein writes

Zum ersten Male sollen die imaginären Zahlen 1545 bei *Cardano* allerdings mehr beiläufig, bei der Auflösung der kubischen Gleichung aufgetreten sein. Für die weitere Entwicklung können wir weider die gleiche Bemerkung machen wie bei den negativen Zahlen, *daß sich nämlich die imaginären Zahlen ohne und selbst gegen den Willen der einzelnen Mathematikers beim Rechnen immer wieder von selbst einstellten und erst ganz allmählich in dem Maße, in dem sie sich als nützlich erwiesen, weitere Verbreitung fanden.* Freilich war den Mathematikern dabei recht wenig wohl zumute, die imaginären Zahlen behielten lange einen etwas *mystischen* Anstrich, so wie sie ihn haute noch für jeden Schüler haben, der zum ersten Male von jenem merkwürdigen $i = \sqrt{-1}$ hört (Klein 1911, p. 136).

Imaginary numbers are said to have been used first, incidentally, to be sure, by *Cardano* in 1545, in his (partial) solution of the cubic equation. As for the subsequent development, we can make the same statement as for the case of negative numbers, *that imaginary numbers made their own way into computations without and even against the will of individual mathematicians, and obtained wider circulation at first quite gradually to the extent that they proved themselves expedient.* Certainly mathematicians thus unsurprisingly perceived imaginary numbers for a long time as manifesting a somewhat *mystical* air, as today for every student who for the first time hears of the curious $i = \sqrt{-1}$ (emphasis in original).

Brahmagupta in India in the 7th century, used negative numbers for debts. However, Arabic mathematicians, such as Omar Khayyām, rejected them. [Blaise] Pascal (1623–1662) rejected negative numbers because he believed a large number subtracted from a small one yielded zero. [Antoine] Arnauld (1612–1694) (and others) rejected negative numbers because they violated standard rules: for example, if $a < b$, then a/b cannot equal b/a; this if $-1 < 1$, how can it be that $-1/1 = 1/(-1)$?[32] Leibniz concurred, but main-

[32]Evidently such is still a viable issue. See for example [Alberto] Martinez (2005), in which is constructed a system satisfying $(-1) \times (-1) = -1$.

tained that none the less, they work for calculations (Leibniz 1712))=(Gerhart 1962, p. 387–389). Obversely, [John] Wallis (1616–1703) accepted negative numbers, but thought them larger than infinity. Cardano accepted them as roots of equations, but not as numbers. [Thomas] Harriot (c. 1560–1621) held the opposite. Collective cognitive dissonance run rampant.

[Michael] Stifel (1487–1567) published a major work, *Arithmetica Integra*, contemporaneously with Cardano's *Ars Magna*. In it, he wrote:

Merito disputatur de numeris irrationalibus, an ueri sint numeri, an ficti. Quia enim in Geometricis figuris probandis, ubi nos rationales numeri destituunt, irrationales succedut, probantque præcise ea, quæ rationales numeri probare non potetant, certe ex demonstrationi bus quas nobis exhibent: mouemur & cogimur fateri, eos uere esse, uidelicet ex effectibus eorum, quos sentimus esse reales, certos, atque constantes.	It is rightly discussed whether irrational numbers are genuine or counterfeit numbers. Since, in testing geometrical shapes, when rational numbers fail us, irrational numbers succeed and prove precisely those things which rational numbers can not prove by means of valid demonstrations, we are moved and compelled to acknowledge that they are genuine, as one may see by their effects, which we realize are real, certain, and secure.
At alia mouent nos ad diuersam assertionem, ut uidelicet cogamur negare, numeros irrationales esse numeros. Scilicet, ubi eos tentauerimus numerationi subijcere, atque numeris rationalibus proportionari, inuenimus eos fugere perpetuo, ita ut nullus eorum in seipso præcise apprehendi possit: id quod in resolutioibus eoru sentimus, ut inferius forte suo loco ostenda. Non aute potest dici numerus uerus, qui talis est ut præcisione careat, & ad numeros ueros nullam cognitam habeat proportionem. Sicut igitur infinitus numeros, no est numerus; sic irrationalis numerus non est uerus, que lateat sub quadam infinitatis nebula. Sitque non minus incerta	But other considerations move us to the opposite conclusion, so that one may clearly deny that irrational numbers exist as numbers. Without doubt, when we attempt to reveal their numeration and their relationship with rational numbers, they perpetually elude us, and thus are not precisely apprehendable in themselves; It it not possible to designate as a genuine number which is of such a kind that is without precision, and which does not have a known relationship with genuine numbers. Therefore, just as an infinitesimal number is not a number, so an irrational number is not genuine, but lies hidden in a certain infinitesimal obscurity. And there exists

proportio numeri irrationalis ad rationalem numerum, quam infiniti ad finitum (Stifel (Stiffelius) 1544, p. 103).

not less uncertainty apropros irrational numbers to rational numbers, as infinitesimal to finite.

Stifel's objection seems to be that it is not possible to precisely locate irrational numbers among the rationals on the number line, or equivalently (in modern terms, given Dedekind's blend) one cannot compute the decimal expansion. He goes on to note that one can make some determination. For example, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ all lie between 2 and 3, as do $\sqrt[3]{9}, \ldots, \sqrt[3]{26}$, etc., but does not make any additional refinement.[33] At one point, he declared that the circumference of a a mathematical circle (hence π), as neither a rational nor a root, had no number—no place on the line. Bombelli only timorously carried out his analysis of the cubic *casus irreducibilis.* He stated his calculations "appear based on sophistic considerations" (« *sembrava poggiare su considerazioni sofistiche* »).

The descriptors "real" and "imaginary" were introduced introduced in [René] Descartes' *La G'eométrie*:

...tant les vrayes racines que les fausses[34] ne sont pas tousiours réelles; mais quelquefois seulement imaginaires; c'est â dire qu'on peut bien toujiours en imaginer autant que aiy dit en châsque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde â celles qu'on imagine. comme encore qu'on en puisse imaginer trois en celle cy, $x^3 - 6xx + 13x - 10 = 0$, il n'y en a toutefois qu'une réelle, qui est 2, & pour les deux autres, quois qu'on les augmente,

...neither the true roots nor the false are always real; sometimes they are, however, imaginary; that is to say, whereas we can always imagine as many roots for each equation as I indicated, there is still not always a quantity which corresponds to each root so imagined. Thus, while we may imagine the equation $x^3 - 6x^2 + 13x - 10 = 0$ as having three roots, yet there is just one real root, which is 2, and

[33] Stifel is inadvertently onto something quite modern here. A hallmark of modern mathematics is precision. If the description of a mathematical term is not precise—in this case, sufficiently precise to locate a number as greater or lesser than any stated rational—then the description is not a definition, and the object is not mathematical. His error is that such irrational numbers can be precisely located. For example, our modern decimal notation intrinsically includes the ordering.

[34] True (« vrai ») roots are the positive ones; false (« faux ») roots the negative ones—telling terminology of the time.

ou diminué, ou multiplié en la façon que ie viens d'éxpliquer, on ne sçauroit les rendre autres qu'imaginaires (Descartes 1637).[35]

the other two, however, increased, decreased, or multiplied as just explained, never are other than imaginary.[36]

Terminology matters of course. It reflects, and sometimes guides, the framing. The French word «imaginaire» has a number of nuanced meanings, and one could parse Descartes' phrasing asking exactly how he meant the term (and exactly what "reality" the root $\sqrt{2}$ of $x^2 = 2$ has, or even 2 itself). In the bobbing and weaving, a variety of other locutions were used, up to and including "impossible."

The following two quotes discuss a perfect sound principle, namely that some (or all) solutions to the mathematical formulation of a physical or other situation may not be relevant to the application. Rather, they are included to illustrate the nomenclature (and as later history shows, they are wide of the ultimate mark, in that non-real solutions often are very relevant to applications). Thus [Isaac] Newton: "But it is just that the Roots of Equations should be often impossible, lest they should exhibit the cases of Problems that are impossible as if they were possible" (Newton 1719). Also Euler (1707–1783):

Endlich muss noch das Bedenken behoben werden, dass die Lehre von den unmöglichen Zahlen als nutzlose Grille angesehen werden könne. Dieses Bedenken ist unbegründet. Die Lehre von den unmöglichen Zahlen ist in der Tat von größter Wichtigkeit, da oft Aufgaben vorkommen, von denen man nicht sofort wissen kann, ob sie Mögliches oder

We must finally abandon the premise that the concept of impossible numbers might be viewed as an idle whim. This opinion is groundless. The premise of impossible numbers is in fact of greatest importance, since problems often arise in which one cannot know immediately whether what is asked for

[35]The connotation of the word «imaginaire»is more subtle than that of the English "imaginary," having more the feeling of "imaged," and Descartes' intention is open to interpretation. «Imaginaire» is certainly softer than earlier terms, "impossible," etc. I am indebted to Per Aage Brandt for discussions on this point. It is certainly the case that the connotation of "imaginary" became the effective one in mathematics.

[36]"increased, decreased, ... as just explained": that is, while negative roots can be shifted to be positive (and *vice versa*) by the substitution $x \to x - a$ for suitable real a, imaginary roots remain imaginary.

Unmögliches verlangen. Wann dann ihre Auflösung zu solchen unmöglichen Zahlen führt, hat man ein sicheres Zeichen dafür, dass die Aufgabe Unmögliches verlangt (Euler 1770, I, 1, par. 151).

is possible or impossible. Whenever their solution leads to such impossible numbers, one has a sure sign that the problem asks for something impossible.

As Gauss commented on all this:

Difficultates, quibus theoria quantitatum imaginariarum involuta putatur, ad magnam partem a denominationibus parum idoneis originem traxerunt (quum adeo quidam usi sint nomine absono quantitatum impossibilium). Si, a conceptibus, quos offerunt varietates duarum dimensionum, (quales in maxima puritate conspiciuntur in intuitionibus spatii) profecti, quantitates positivas directas, negativas inversas, imaginarias laterales nuncupavissemus, pro tricis simplicitas, pro caligine claritas successisset (Gauss 1832, section 38).

Difficulties, one has believed, that surround the theory of imaginary magnitudes, is based in large part to that not-so-appropriate designation (it has even been discordantly called an impossible quantity). If, at the beginning, one proffered a manifold of two dimensions (which presents the intuition of space with greater clarity), the positive magnitudes would have been called direct, the negative inverse, and the imaginary lateral, there would be simplicity instead of confusion, clarity instead of darkness.

The problem was that negative, irrational, and imaginary numbers made things work, and would not go away. Thus the cognitive dissonance, and the bobbing and weaving. The terminology stuck because it reflected a real issue: did these things exist in some sphere or did they not? They were not at a human scale. Thus how should the human mind comprehend them? They could be bandied, but they could not be blended. Thus Leibniz, in a note describing quadrature methods, comments concerning partial fractions (working out the case $\int dx/(x^4-1)$ in detail, factoring $x^4-1=(x+1)(x-1)(x+\sqrt{-1})(x-\sqrt{-1})$),

Verum enim vero tenacior est varietatis suae pulcherrimae Natura rerum, aeternarum varietatum parens, vel potius Divina Mens, quam ut omnia sub unum genus compingi pa-

In fact Nature, the mother of limitless variations, or rather, Divine Reason, certainly adheres so closely to her own grand diversity than to permit all things to be confined within a single

tiatur. Itaque elegans et mirabile effugium reperit in illo Analyseos miraculo, idealis mundi monstro, pene inter Ens et non-Ens Amphibio,[37] quo radicem imaginariam apellamus (Leibniz 1702)=(Gerhart 1962, p. 356).	cast. Accordingly, she finds an elegant and extraordinary circumvention in that wonder of analysis, that Platonic entity, almost living a double life between being and not-being, that we call an imaginary radical.[38]

[Hermann] Weyl, in his book *Philosophy of Mathematics and Natural Science*, relates:

> For instance, Huyghens declares in 1674 (see Leibniz, *Mathematische Schriften*, II, p. 15) with reference to a complex formula [i. e., one involving imaginary numbers]: "Il y a quelque chose de caché là-dedans, qui nous est incompréhensible." ["There is something concealed here that is incomprehensible to us."] Even Cauchy, in 1821, still has a somewhat obscure idea as to the manipulation of complex [= imaginary] quantities. But negative quantities had produced almost as many headaches at an earlier time. Referring to the rule "minus times minus is plus," Clavius says in 1612: "debilitas humani ingenii accusdanda (videtur), quod capere non potest, quo pacto id verum esse possit." ["We see the the weakness of human intelligence, which is not capable of understanding how this can be true."] Descartes, in accordance with contemporary usage, still designates the negative roots of an algebraic equation as false roots. . . (Weyl 1949, p. 32).

The first edition of the Encyclopaedia Britannica (1768-1771) states: "[T]he square root of $-a^2$ cannot be assigned, and is what we call an impossible or imaginary quantity." A later major text by [Agustus] De Morgan (1806–1871) stated:

> We have shown the symbol $\sqrt{-a}$ to be void of meaning, or rather self-contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility. It depends upon the fact, which must be verified by experience, that the common

[37]A catchy standing translation of this phrase renders this word "amphibian"; I am indebted to Greg Meagher for his expertise in refreshing the translation, and in particular for noting that the natural-science meaning of "amphibio" was a later acquisition; its literal meaning in the Greek as used first by Democritus is living a double life.

[38]Or more prosaically: not all roots of some polynomials are real, but imaginary numbers salvage the situation.

> rules of algebra may be applied to these expressions without leading to any false results. An appeal to experience of this nature appears to be contrary to the first principles laid down at the beginning of this work. We cannot deny that it is so in reality, but it must be recollected that this is but a small and isolated part of an immense subject, to all other branches of which these principles apply in their fullest extent. There have not been wanting some to assert that these symbols may be used as rationally as any others, and that the results derived from them are as conclusive as any reasoning could make them. I leave the student to discuss this question as soon as he has acquired sufficient knowledge to understand the various arguments.... (De Morgan 1831, p. 151)

A decade later, a few years after Gauss' blend was published, De Morgan had reconsidered: "The motto which I should adopt against a course which seems to me calculated to stop the progress of discovery would be contained in a word and a symbol—remember $\sqrt{-1}$" (De Morgan 1842).

The fifty or so years in the 1800s leading to the real-number blend is interesting to explore with the perspective of cognitive science. A early as 1816–1817, [Bernard Placidus Johann Nepomuk] Bolzano (1781–1841) had explicitly raised the issue of firming up the foundation of analysis, what is called the "arithmetization of analysis" (and, it might be noted, formulated Cauchy sequences). In one sense the time was ripe. Terminology tip-toed around the blend. A sequence (of rationals) with a (rational) limit converged; as noted a Cauchy sequence "converged internally." In 1869, [Hugues Charles Robert] Méray (1835-1911), closely examined Cauchy's criterion. He wrote:

Il nous faut un terme spécial pour exprimer la propriété remarquable de cette différence dont il ságit : je dirai qe la variable progressive v est *convergente*, qu'elle ait ou non une limite numériquement assignable... On conçoit donc qu'il soit avantageux, dans le cas où il n'y a point de limite, de conserver le langage abrégé propre à celui où il en existe une; et, pour exprimer le convergence de la variable, on dira simplement : *elle a une limite (fictive)* (Méray 1870, p. 284).	We need a special term to express the remarkable property that concerns this difference : I will say a sequence v is *convergent*, whether or not it has a numerically assignable limit... One conceives therefore it will be advantageous, in case there is not a limit point, to conserve the established concise language if it has a limit; and to express the convergence of the series, one says simply : *it has a (fictitious) limit* (emphasis in original).[39]

[39]Thus Méray's « convergente » is today's "Cauchy," or as I labeled it, "internally conver-

Some historians credit Méray with the first published blend of irrational numbers—he had the mathematics right—and others claim his use of the expression « *limite (fictive)* » confirms that he missed the basic point, and credit Cantor and Heine with pinning down the details of the blend (hence "the Cantor-Heine construction") based on Cauchy's pixels. In either case, we note the linguistic tiptoe. It should be noted that the details of a blend based on Cauchy sequences is more technically complicated than when based on Dedekind cuts, and one needs some of the "formal tricks" of set theory, such as equivalence relations, which had not yet been developed in the early 1800s. In fact, to a large extent, it was the quest towards the arithmetization of analysis that led to the development of set theory. One could maintain that, as a blend or metaphor, set theory follows recursively from the arithmetization of analysis. For example, without the acceptance of the reality of infinite sets, it is not possible to establish the consistency of real numbers, i. e., that that blend is firmly mortared.

Given the framing posited above, namely that arrested blends occur when the cognition is not prepared to blend larger frames, or to morph appropriate categories, one should ask, what arrested negatives, irrationals, imaginaries, etc.? On the one hand, negatives, etc., permit computations and conceptual formulations impossible absent them, and indeed they force themselves onto the scene. Then what is the antagonistic? The only common theme in these cases seems to be an arresting reluctance to expand the concept of the term "number."[40] That is, that numbers are not a construct of human cognition, but rather have an external identity. This is closed related to—perhaps causative of—Platonism. If numbers have an external identity, we humans are not entitled to uninhibitedly expand the concept. There was some kind of barrier to each recursive expansion that required significant cognitive—psychic?—effort to overcome. Nowadays, the opposite view prevails: so long as one uses set-theoretic methods such as ordered pairs and quotient sets, one totally uninhibitedly expands concepts. There is no cognitive cost.

There are other instances of arrested blends in mathematics. Non-Euclidean geometry is one, and its history is well-known (and is closely tied with complex analysis, it should be noted). A second is the issue of the in-

gent." A « limite (fictive) » is what an internally convergent sequence that is not externally convergent has.

[40](Fauconnier and Turner 2002, p. 335–336) elucidate how our modern understanding of "number" is a succession of recursions. Lakoff and Núñez (2000, p. 89) discuss what they call the Number-Are-Things metaphor.

finitesimal, the infinitely many of the infinitely small (which—Stifel would recognize—itself is closely tied with the blend of irrational numbers). Thus for example, one can imagine polygons of ever increasing number of sides inscribed in a circle. Is the circle a polygon of an infinite number of infinitely small sides? Galileo (1638) so proposed. Thus, he argued, the tangent at a point a the circle (required for computing the velocity of a circularly-moving particle) it just the extension of the infinitely small segment. If not, then some kind of limit concept is required to get at the velocity. There was a vigorous discussion back and forth over a couple of centuries, famously following the creation of calculus (as every first-year calculus student knows, computing tangents is the job of differential calculus). In his tract, Weyl relates:

> The limiting process was victorious. For the *limit* is an indispensable concept, who importance is not affected by the acceptance or rejection of the infinitely small. But once the limit concept has been grasped, it is seen to render the infinitely small superfluous. . . .
>
> Incidentally, as far as I can see, the 18th century remained far behind the Greeks with regard to the clarity of the conception of the infinitely small. More than one writer of this enlightened era complains of the 'incomprehensibilities of mathematics,' and vague and incomprehensible indeed is their notion of the infinitesimal. As a matter of fact, it is not impossible to build up a consistent 'non-Archimedian' theory of quantities in which the axiom of Eudoxus (usually named after Archimedes) does not hold.[41] But as is pointed out just above, such a theory fails to accomplish anything for analysis. Newton and Leibniz seemed to have the correct view, which they formulated more or less clearly, that the infinitesimal calculus is concerned with the approach to zero by a limiting process. But they lack the ultimate insight that the limiting process serves not only to determine the value of the limit but also to establish its existence.[42] For that reason Leibniz is still quite unclear as to the summation of infinite series. Only slowly does the theory of limits gain a foothold. In 1784 D'Alembert declares emphatically in the *Encyclopédie*, "La théorie de la limite est la base de la vraie métaphysique du calcul différentiel. Il ne s'agit point, comme

[41] The previously-mentioned non-standard analysis, for example.

[42] For this discussion, this is an important observation: one can use internal convergence to fabricate external convergence. Weyl is also observing that the "limit blend" was not completed—indeed was arrested—until the 19th century.

> on le dit ordinairement, des quantités infinitement petites; il s'agit uniquement des limites des quantités finies." ["The theory of the limit is the foundation of the conceptual truth of differential calculus. It does not concern, as is ordinarily stated, infinitely small quantities; it is solely concerned with limits of finite quantities."] It was left to Cauchy, at the beginning of the 19th century, to carry these ideas out consistently. In particular, he discovers the correct criteria for the convergence of infinite series, the condition under which a number is generated as the limiting value through an infinite process. The proof of the criterion, however, requires the fixation of the number concept which was later accomplished by the principle of the Dedekind cut (Weyl 1949, p. 44–45).

The question of infinitesimal or not was integrally connected with the issue of irrational numbers. Most non-root irrationals turn up as limits of some kind or other, and as noted, the Cauchy definition of real numbers is nothing but asserting all internally convergences are externally convergent. In some sense the infinitesimal blend remained arrested. The idea was never abandoned completely—Weyl himself worked on it. as noted above, non-standard analysis, a modern manifestation of the concept, was developed in the mid-twentieth century, and has a vigorous, if small, community of practitioners. But this was essentially a resurrection, and it requires sophisticated formal logic and set theory to make it work. In the late 1800s, as Dedekind's and similar blends of the real numbers took force, limits were put on firm foundation, thus grounding calculus, the issue of infinitesimals faded into the background. It thus becomes an interesting case study of the cognitive process by which an incipient blend fades away.[43]

10 Coda

Modern mathematics—a formal edifice—is a relatively recent phenomenon—a bit over a century old. Lakoff and Núñez (2000, p. 89) call it the Numbers are Sets Metaphor. To us, it is not a metaphor on par with other metaphors

[43] This is not to say that the word or the general concept has faded away. Calculations with "infinitesimals" are done all the time in engineering mathematics. Here the term means "to first order smallness," and the use of such in computations causes no trouble so long as the user is in control of the technique and the applications are sufficiently differentiable.

discussed in Lakoff and Núñez (2000), but rather the footing of the modern mathematical paradigm. And as noted, it is not the particulars of the set theory formulation that matters, but rather the techniques the paradigm makes available to a mathematician to create blends, frames, etc., i. e., the cognitive space of the mathematical domain. There is a cost, in time and effort, for a person to position him/herself in this space, but once securely there, the mechanics of the blending process are, well, mechanical, and the practitioner can focus his/her cognitive exertion on the conceptual aspect of the blend.

That is, the formality of mathematics is cognitively liberating. The twentieth century, continuing into the twenty-first, was—is—a time of explosive development in mathematics—one of the golden ages of the discipline. Legendary conjectures were resolved, most famously perhaps, Fermat's Last Theorem in the 1990s (Wiles 1995). Others with some hoar include the four color theorem, solved in the 1970s—perhaps the first significant example of a computer-aided proof—Thomas Hale's proof of the 1611 Kepler conjecture (that stacking cannonball, oranges, etc., in the usual way is in fact the most efficient use of space)—another computer-aided proof—and just recently, the 1904 so-called Poincaré conjecture. However, it is a precept in mathematics that it is not the problem solved that is the advance, but the mathematics developed in the process—the blends, the framings. Usually, significant advances require cascading blends upon blends. It is as true in this continuing golden age of mathematics as in earlier eras. The premise of the present work is that mathematics has been able to thrive and advance as it has only because it incorporates in its cognitive structure, its own forms of basic cognition, such as blending.

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