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4

Inventing Arithmetic: Making Children's Intuition Work in School

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I begin this chapter with an apparent contradiction: School mathematics—especially beyond the primary grades—seems difficult and inaccessible to many people. Yet most young children entering school have a great deal of knowledge about quantities and their relations. What is more, children's knowledge of quantity relations seems to be organized in ways that are coherent with the formal structures of mathematics, including some fundamental laws of algebra. How can it be that children know so much about mathematics and yet have such a hard time learning it in school? My efforts to answer this question raise a series of fundamental questions about the nature of knowledge and cognitive competence, the relations between cognition and social processes, and the role of schooling in adaptive sociocognitive development.

KNOWING MATHEMATICS: A SOCIOCOGNITIVE INTERPRETATION

In the mid-1980s, an important article by a group of Brazilian developmental psychologists made a distinction between mathematical performances "in the streets" and "in the school" (Carraher, Carraher, & Schliemann, 1985). They documented stunning, flexible computational skill by child street vendors who could not manage school arithmetic. This group and other investigators subsequently documented extensive practical arithmetic among children (e.g., Saxe, 1988) and adults (Lave, 1988; Nunes, Schliemann, & Carraher, 1993; Scribner, 1984) and showed how such performances were embedded in the demands of everyday life. They also documented nearly error-free computation, often on large numbers and in complex situations, among *street math* performers. But the status of street math as "real mathematics" was not clear at first. Perhaps these arithmetic strategies were just tricks of the trade, convenient procedures passed on within a working culture but not requiring much mathematical understanding on the part of those who used them.

About the same time that the street math studies were becoming known, mathematics educators and students of mathematical cognition were combining forces to challenge common school math practices as also not being real mathematics (e.g., Charles & Silver, 1988; National Council of Teachers of Mathematics, 1989; Schoenfeld, 1985). These reformers and scholars wanted to recraft mathematics education to focus on underlying principles, rather than on computation, and on strategies for mathematical problem solving and communication, rather than on routine procedures. The gist of their agenda might be described as wanting to replace school math with *math math*, that is, child-accessible versions of the kind of thinking that mathematicians do.

The distinctions among *street math*, *school math*, and *math math* serve as a starting point for my consideration of what it means to know mathematics. I try to show that street math—a version of which can be observed in school children if we capture their everyday intuitions about number and quantity—is based on substantial conceptual knowledge of mathematical principles. Street math is, however, a different kind of *social* practice than math math. American school children sometimes engage in math math practices, but such practices are very rare among unschooled street math performers.

Street Math: Practical Performances Based on Intuitive Principles

The top of Fig. 4.1 gives an example of numerical reasoning by a 9- or 10-year-old selling coconuts on the streets of a northeastern Brazilian city. This child was not "playing store." He was engaged in real work, part of the informal economy that is vital in the life of very poor communities in much of the world, work that is essential to the welfare of his family. Errors in pricing cannot be tolerated, for the child would lose money for his family either by receiving too little for the coconuts sold (profit margins are very small in the informal economy) or by driving customers away to a competitor likely to be waiting just a few steps away.

In Recife, coconuts are customarily sold in bunches of three, so the children learn the price of a single coconut and of a bunch of three. Because of rampant inflation, these prices may change every few days. There is little opportunity to memorize the prices of bundles of coconuts other than the standard group of three. This means that the question asked by the interviewer, posing as a customer, is a true problem for the child.

The bottom of Fig. 4.1 shows an analysis of this apparently simple exchange, which reveals a surprisingly complex level of mathematical thinking. Underlying everything the child does is a mathematical principle of *additive composition*. Stated in ordinary language, this principle specifies that quantities and numbers are compositions of other quantities and numbers (e.g., 9 is composed of 4 and 5; 5 is composed of 2 and 3; 9 can also be combined with another 9 to yield 18 and so on for larger and larger numbers). This principle permits both decomposition of a target amount (10 coconuts are made up of 3 coconuts, plus 3 more, plus 3 more, plus 1 more) and composition of a new amount by adding known amounts (350 cruzeiros, plus 350 more, plus 350 more, plus 35 more).

How much is one coconut? ***35. I'd like ten.*** **How much is that? [Pause]** ***Three will be 105; with three more, that will be 210.*** **[Pause]** ***I need four more. That is . .*** **[Pause]** ***315. . . . I think it is 350.***

Coconuts	Cruzeiros
3 coconuts	*105 cruzeiros*
3 more coconuts	(105 more cruzeiros) *210 cruzeiros so far*
(6 coconuts so far)	
6 needs 4 more to reach 10	
(3 more coconuts)	(105 more cruzeiros) *315 cruzeiros so far*
(9 coconuts so fac)	
(9 needs 1 more to reach ten)	
(1 more coconut)	(35 cruzeiros)
(10 coconuts in all)	*350 cruzeiros in all*

FIG. 4.1. Numerical reasoning by a child coconut vendor. From Carraher, Carraher, and Schliemann (1985, p. 23).

The problem, however, was not presented to the child vendor as an exercise in rules of numerical composition, as it might be in school (e.g., "What combinations add up to 10?" or "What is 3 times 350 cruzeiros plus 35?"). This means that the child needed additional knowledge: about how two series of composed numbers relate to one another. The bottom part of Fig. 4.1 schematizes the relational knowledge that this child vendor possessed. The analysis reveals a form of *protoratio* reasoning (see Singer, Kohn, & Resnick, in press). The child vendor cannot (or at least does not) construct a relational equation such as 35::1 = 350::10 (cf. Vergnaud, 1983). But he solves a ratio-like problem by treating it as a problem in coordinating two additive compositions.

In this scale, you see, each two centimeters will be worth one meter. Four meters will be eight centimenters. But it is nine centimeters here, that means half a meter more. Nine centimeters is a wall four and a half meters.

blueprint size	wall size
2 cm	1m
8 cm	4m
(1 cm more)	
9 cm	
	1/2m more
9 cm	4-1/2m

FIG. 4.2. Protoratio reasoning by an uneducated construction worker. From Carraher (1986, p. 22).

The same basic strategy is used by minimally educated adults in Brazil working as construction foremen and thus needing to use scale information on architectural plans to decide on the length of sections of walls they are building. Fig. 4.2 gives an example. This man is working from a mental table of the kind shown in the bottom part of the figure. It is basically the same kind of *protoratio* table as the coconut example—two decomposed series of numbers are coordinated to yield a correct answer—although in this particular case, the foreman also expresses the ratio directly at one point.[1]

Protoratio reasoning is also the most common way that school children solve ratio problems, at least until a certain point in their education. A particularly clear example comes from a study by Ricco (1982), who presented to French school children in grades 2 to 5 (roughly 7- to 11-years-of-age) problems in which, given the price of a certain number of pens, they had to calculate the price of other multiple-pen purchases. The problems were presented in tabular form with some lines filled in and others left for the child to complete. For example, one problem was presented as shown in Table 4.1.

From third grade on, the most common method used by Ricco's subjects was to compute a constant difference that was added to successive rows in the Price Paid column, taking into account the "jumps" in the Number of Pens Bought rows: for example, "Five pens cost 20 francs . . . because there is 4 francs difference between 12 and 16. Six pens . . . 24; I add 4. Eight will cost 28 francs. Oh, no! Seven is missing. Seven . . . 28 francs, 8 pens . . . 32 francs. . . ."

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[1] There is no developmental connection claimed between child performance of street math and adult performance. These studies have not followed individuals through different life-stages.

TABLE 4.1
Type of Table Used by Ricco to Assess Ratio Reasoning

Number of Pens Bought	*Price Paid*
1	?
2	?
3	12
4	16
5	?
6	?
8	?
10	?
15	?
16	?
18	?
71	?
72	?
75	?

School Math: Errors and Alienation

In the Brazilian research, children and adults whose everyday work required mathematical performances of the kind just described carried them out without errors. Yet many of the same individuals, when given written arithmetic probems in school form, with identical numbers in them, could not do the arithmetic at all (Nunes et al., 1993). The contrast between nearly perfect performance on street math and highly errorful performance on school math highlights an almost total disjunction of everyday and school competence that, at first glance, calls into question the entire venture of school mathematics.

Such total disjunctions are more rare in more fully schooled societies, probably because there is less compelling economic reason to practice street math and because the school forms of mathematical knowledge dominate our lives so much. Most Americans, for example, eventually learn to do written school arithmetic without undue error. But they attain nothing like the sense of nearly perfect command that the Brazilian street mathematicians—child and adult—display. What is more, the learning process for school arithmetic is not a smooth one. Most children make many errors along the way, errors that have come to be called *buggy algorithms* (Brown & VanLehn, 1982; VanLehn, 1990) because they are systematic transformations of correct algorithms, made, apparently, because children are *not* applying schemas of additive composition (Resnick & Omanson, 1987).

Math Math: Beyond Utility

Yet there is plenty of evidence that American and European school children know the principles of additive composition and can use them to think about numbers and arithmetic (Resnick & Greeno, 1990). To learn this, we have had to get

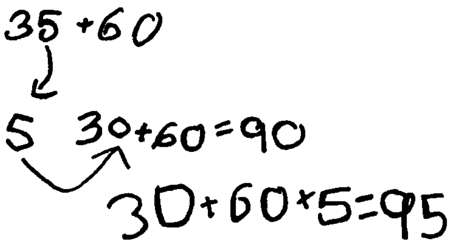


FIG. 4.3. Additive composition reasoning by a first grader.

ahead of school math, interviewing children about procedures and concepts they have not yet encountered in school. Fig. 4.3 gives an example of a first grader, Pitt, who has not yet been taught two-digit addition, explaining in words and numerical symbols how he mentally solved the problem 30 plus 65. Pitt's writing appears in Figure 4.3. His accompanying verbal description was, "I would take away the 5 from the 35. Then I'd add the 60 and the 30, which equals 90. Then I'd bring back the 5 and put it on the 90, and it equals 95."

What Pitt did was to temporarily remove a component from one of the numbers, allowing him to use a known "number fact" (60 + 30 = 90). He then added back the removed component at the end. Thus, he knew that it is permissible to change a number in the course of calculation, as long as a compensating change is made at another point to preserve the total quantity. The constraints and permissions that guided Pitt's invented procedure derive from the mathematical principles of additive composition of number.

A feature of Pitt's performance that is rarely encountered among unschooled street math performers is that he was clearly reasoning about *numbers* as such and not about quantities of material, such as coconuts or pens. This is one of the features of math math. Pitt demonstrated a command of and a taste for math math in many other interviews. For example, his explanation of why 2 × 3 and 3 × 2 yield the same answer (Fig. 4.4) does not reach the status of a proof, but he was searching for justifications for why certain procedures work, explaining the relations among numbers. In short, he was exploring the properties of numbers as a domain of interest in and of itself, beyond any immediate practical utility.

What's two times three? *Six.* How did you get that? *Well, two threes . . . one three is three; one more equals six.* Okay, what's three times two? *Six.* Anything interesting about that? *They each equal six, and they're different numbers. . . . I'll tell you why that happens. . . . Two has more ways; well, it has more adds . . . like two has more twos, but it's a lower number. Three has less threes, but it's a higher number. . . .* Alright, when you multiply three times two, how many adds are there? *Three . . . and in the other one there's two. But the two--that's two threes--but the other one is three twos, 'cause twos are littler than threes, but two has more . . . more adds, and then the three has less adds but it's a higher number.*

FIG. 4.4. A child's informal reasoning about commutativity of multiplication.

THE SCHOOL MATH PROBLEM: COGNITIVE OR MOTIVATIONAL?

Let us return now to my opening contradiction: Why is school math so difficult when street math, apparently based on intuitive appreciation of fundamental mathematical principles, is available to children? The by-now traditional answers to this question come in separate categories that we can roughly label *cognitive* and *motivational*. Although I begin with these categories, I do not mean to accept the traditional separation between them but rather want to show that cognitive and motivational factors interact in ways that produce an unproductive *situation* for mathematical performance in school.

The Cognitive Disconnect. On the cognitive side, there is evidence that the forms of school math teaching do not connect well with the street math that children have acquired at home. Arithmetic (nearly the only math that is in the American elementary school curriculum) is taught as if children come to school knowing nothing at all about it—none of the counting principles that Gelman and Gallistel (1978), for example, have documented for preschool children; none of the language of quantity that Sera, Troyer, and Smith (1988) and others have established; none of the intuitive schemas that even the most "deprived" children have for understanding additive composition of material in the physical world (Resnick, 1992).

Instead, arithmetic (and later algebra) is taught as a series of rules for manipulating the written numerical symbols. Learning the particular procedural rules that have been handed down over the years, mainly as ways to insure reliability in written calculation, and applying them correctly have become the major goals of instruction. There is nothing wrong with the rules per se. In fact, every one of the algorithms taught in elementary school reflects a systematic and elegant application of some subset of the basic principles of additive composition (see Resnick & Omanson, 1987, for an analysis of some of these). But only minimal efforts to help children understand the algorithms as principled derivations are usually made. Furthermore, because math is viewed by educators as an exact subject, there is little invitation to children to talk—in their necessarily imprecise and nonmathematical language—about what they are practicing, except to recite the rules they are expected to learn (Resnick, 1988). All in all, little is done either to use children's street math knowledge or to cultivate math math practice in the classroom.

The Motivational Disconnect. The failure of much of our present teaching to make a cognitive connection between children's own math-related knowledge and the school's version of math feeds a view held by many children that what they know does not count as mathematics. This devaluing of their own knowledge is especially exaggerated among children from families that are traditionally alienated from schools, ones in which parents did not fare well in school and do not expect—however much they may desire—their children to do well, either. In the eyes of these children, math is what is taught in school. Because they do not encounter much at home that resembles school math, they are inclined to discount the street math that surrounds them and that they are learning as not real math. What they do know, often rich and robust, does not count as school knowledge, and so they do not try to use it to make sense of what they encounter in school.

This point is illustrated by the responses we encountered in a set of informal interviews with African-American children about their perceptions of mathematics. Interested in the ways in which home activities might relate to school math learning, we asked several fifth graders to tell us whether they did math at home and what that math was like. All said they did math at home, but when asked for examples, all but one described only their own school homework or the homework of family members enrolled in adult education programs. One girl broke out of the mold, describing how she used math to "prove" that her mother could afford the new dress she wanted, by calculating the cost of clothing items the whole family needed from the Sears catalog and then comparing the total cost of those items with the amount budgeted for family clothing. The specifics of the arithmetic are not particularly interesting in this case. What is important is that this child understood the very practical and desire-motivated activity of pricing clothing to be math, whereas the others did not. For them, only the school-like activity counted as math.

RECONNECTING: FROM STREET MATH TO MATH MATH

The cognitive and motivational disjunctions between what children bring from home and what the school offers can be overcome. Victoria Bill is an elementary school teacher who studied the body of research I have just summarized and then worked to invent a new way of teaching math that would actively use children's street math intuition. The children with whom Bill worked were from an inner-city school. Almost all were African-American, and about 70% were eligible for free or subsidized lunch. In Bill's first year of work, the median first-grade class percentile rankings on the school district's standardized tests moved from around the 30th percentile to above the 80th, with the lowest child scoring at the 66th percentile (Resnick, Bill, & Lesgold, 1992). Similar gains in test performance were made with children in several different grades that Bill taught in succeeding years using her new approach.

These test-based successes—we might call them school math successes—were obtained despite a program that spent little time on the activities typical in school math programs. Bill's method can be characterized as setting up sequences of situations in which street math can be done by children and then, via a carefully structured form of classroom discourse, pushing them toward math math. There is almost no school math intervening.

Principles for Teaching Math Math

Everyday Problems. Figure 4.5 shows some typical work of second graders. The page is copied from a child's notebook dated in January of the school year. The work results from a typical day's lesson, lasting perhaps 40 minutes. The lesson was built around a single problem that the children analyzed under the teacher's direction and then developed solutions to in small groups. The theme of the problem for January 24, 1990, like most problems used in Bill's teaching, came from the children, who are encouraged to bring problems to school that they have encountered or made up. Bill does not necessarily use the problems as given. She reworks them somewhat (keeping them recognizable in such content as children's names, barrettes and other materials being dealt with, and the like) in order to represent a particular mathematical structure. This way of developing problems illustrates one of the principles on which the program is based:

> Encourage everyday problem finding.

The goal is to undo the disconnection between home or street learning and school learning: to bring street math to school. By using problems that the children bring to school as the basis for a formal lesson, Bill teaches the children implicitly that what they do and think about at home *is* math. The same message is carried in other program practices, such as miniprojects that children are expected to do with parents or other family members. For the youngest children, miniprojects might include finding matching sets of four objects in the household or recording the numbers and types of items removed from a grocery bag after a shopping trip.

Accelerated Introduction of Concepts. The problem shown in Fig. 4.5 is a compound problem involving both multiplication and subtraction. This problem is substantially more complex (because it involves multiplication and because it involves two separate arithmetic steps) than children would normally encounter in a typical second-grade textbook. This means that most children will require substantial *scaffolding* in order to solve the problem. Scaffolding is provided through guided class discussion, which is also the teacher's opportunity to evoke and shape mathematical thinking and mathematical explanation by the children.

Monique told her friend TaRae that she would give her 95 barrettes. Monique had 4 bags of barrettes and each bag had 9 barrettes. Does Monique have enough barrettes?

The class first developed an estimated answer. Then they were asked, "How many more does she need?" The solutions below were generated by different class groups.

Group 1 first solved for the number of barrettes by repeated addition. Then they decomposed *4 x 9* into *2 x 9* plus *2 x 9*. Then they set up a missing addend problem, *36 + __ = 95*, which they solved by a combination of estimation and correction.

Group 2 set up a subtraction equation and then developed a solution that used a negative partial result.

Group 4 began with the total number of barrettes needed and subtracted out the successive bags of 9.

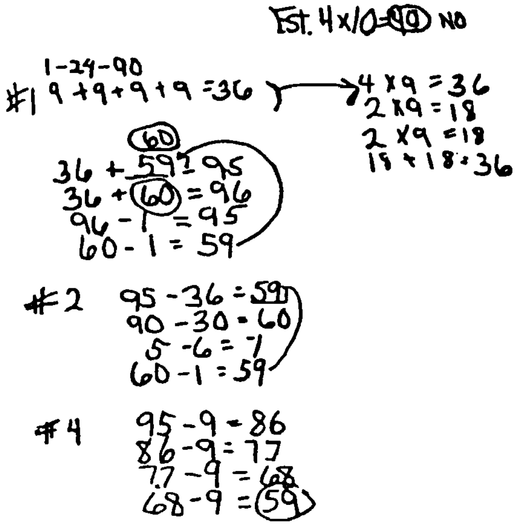


FIG. 4.5. A second-grade problem and several solutions.

Children also scaffold each other's thinking as they work in independent small groups to develop numerical solutions. The choice of a problem as complex as this one for children at this age is guided by another principle:

Introduce key mathematical structures as quickly as possible.

According to the research alluded to here, young children come to school already knowing a lot about the additive composition of physical material in the world (Resnick & Greeno, 1990). But this knowledge about physical material is not about numbers. In her first grade teaching, Bill tries to help children see how what they know about physical material also applies to numbers, which, at this age, they appreciate mainly via counting. For children, additive composition is not broken down into separate arithmetic rules such as addition, subtraction, and multiplication. To help them develop the connection between their additive composition knowledge and what can be done with numbers, it is important to help them glimpse, even before they can master, the additive composition structure at work across several arithmetic operations and in large as well as small numbers. For this reason, Bill introduces concepts, always embedded in a familiar problem, at a very fast pace. Commutativity, distributivity, and additive inverse form part of the conceptual curriculum beginning in first grade. The concepts are revisited over time to allow full mastery to develop.

Use Street Math Knowledge. In solving the problem shown in Fig. 4.5, after some discussion designed to insure that most of the children understand the *story* structure (not necessarily the numbers yet), children work in table groups to figure out solution approaches. The class first began with an estimation: *9 is close to 10, and 4 tens are 40, 40 is far from the 95 barrettes Monique needs to give to TaRae, so, "No, there are not enough barrettes."* Bill encourages estimation as one of several forms of *number sense* that she treats as integral goals of her teaching. Most days' lessons include at least one occasion in which children develop an estimated solution to a problem or part of a problem.

Group 1 wanted to "prove" that their estimation was correct, and they did so in two ways. First, they added 9 four times to get 36. Then, they proved that 36 was correct in another, more sophisticated way. They added 9 and 9 to make 18, building on their very accessible knowledge of doubles. They repeated the 9 plus 9 to yield another 18. And then they added the two 18s! This solution builds directly and obviously on the children's knowledge of additive composition. They are combining numbers to form partial sums and then combining those to form yet larger sums. This reliance on informally acquired, intuitive knowledge about doubling and accumulation illustrates another principle:

Draw children's informal knowledge, developed outside school, into the classroom.

In some ways, this is the central precept of Bill's program. It is the foundation principle, drawn from the research discussed earlier, showing that children have good intuitional foundations for mathematical thinking and reasoning about numbers. The usual approach to teaching, in which children must master the "standard" or "correct" procedures for arithmetic, not only fails to use the cognitive resources children bring to school but also creates a motivational problem, because children come to believe that their home-acquired knowledge does not count in school.

Equations at an Early Age. The algebraic notation used to record children's thinking illustrates yet another principle:

> Use formal notations (identity sentences and equations) as a public record of discussions and conclusions.

Early use of algebraic notation is the aspect of Bill's teaching practice that people usually find most surprising. It seems to run counter to the emphasis many developmentalists and mathematics educators place on manipulatives and avoidance of meaningless symbolic manipulation. In Bill's classroom, however, equations are introduced as a shared public record, in mathematically exact language, of ideas that have already been discussed and that therefore have meaning for the children. Public records that can be referred to by everyone are essential in discussion-based learning. Without them, talk is ephemeral, and there is no intellectual accountability for ideas. The use of standard mathematical notation as a shorthand record introduces mathematical precision but permits children to speak in their ordinary language during classroom discussions. The use of formal mathematical notation comes easily to children under these circumstances and is a source of pride to them. It is also, of course, a painless way to introduce a powerful tool for later mathematical use. Equation notations support reasoning about purely mathematical relationships that cannot be observed directly in physical material (Resnick, Cauzinille-Marmeche, & Mathieu, 1987).

The next step in Bill's lessons is to develop specific numerical solutions. In this case, the class was seeking an answer to a question that was not made explicit in the problem statement but was implied in the story structure of the problem: *How many more barrettes does Monique need?* The nonspecification of the *how many* question mimics the way in which problems come up in the real world. The situation typically makes clear what questions should be asked and answered; it provides a mutually understood *common ground* (Clark & Brennan, 1991) for activity. If Monique has to provide 95 barrettes for TaRae and she knows she has only 36, she obviously needs to get enough more to reach 95.

To work on this solution, children return to their table groups to develop their ideas. They know that, although they are expected to reach a mathematically correct answer, there are several ways to reach it. Group 1 interpreted the problem as a "missing addend" addition problem: In effect, how many are needed to get from 36 to 95? (The underlined number was initially a blank space, the class's conventional way of showing the "unknown" in an equation.) They then estimated 60 (a circled number is, by classroom convention, an estimate). But this, added to 36, produced one more than the necessary number of barrettes. Knowing that subtracting 1 from 96 would produce 95, the children also subtracted 1 from the estimated 60, to produce an exact answer of 95. This solution uses intuitive knowledge of *equivalence classes of differences*, one of the algebraic concepts that is rooted in the general principle of additive composition of number.

Group 2 set up the problem as a subtraction problem, 95 minus 36 equals *unknown*, a conversion based on a conceptual understanding of the inverse relationship between addition and subtraction. Next, they worked on the *tens place*, subtracting 30 from 90 to yield 60. Beginning there runs contrary to the computational practice normally taught in school, which begins at the right: thus, with the *ones place*. But it is perfectly sensible mathematically and is actually a more convenient way to proceed when doing arithmetic mentally. These children's next step is an even more startling departure from standard school arithmetic practice. Moving to the ones place, Group 2 proposed subtracting 6 from 5 to yield a *negative* 1. The standard school algorithm teaches children that "You can't take a large number from a small one," so you have to "borrow" (or "regroup"), moving *10* from the tens place over to the ones place. Some of the Group 2 children had been exposed to the concept of negative numbers while working with a computer program and, as far as we know, tried out the idea in the context of subtraction for the first time in the course of this lesson.

Build Confidence in Street Math as Knowledge that Counts. As is clear from these examples, correct answers are sought in Bill's classroom but not just a single correct solution path. Indeed, variety and invention are encouraged. Bill always asks for another way to solve the problem, and the groups are likely to favor new solutions over ones they have encountered before. No standard algorithms are taught. The strongest reason for this practice is another of the program's principles:

> Develop children's trust in their own knowledge.

Children need to believe that what they know is real math and that it counts in school. The only effective way to foster such a belief is to let them use what they know in school and to celebrate what they can do with it. That approach bridges the motivation gap that I discussed earlier. We have often been asked if it would not be just as effective to teach children one or another of the standard computational procedures—which are, after all, both efficient and mathematically correct—and then work with them to analyze how mathematical principles are exemplified in the procedure learned. We believe that such a practice would be perfectly reasonable for adults and perhaps for children who are extremely confident about their ability to cope well with school. But for children who have any doubt about themselves as competent school learners—and this includes most children who come from less privileged homes—the implicit message is that the procedure taught is the one that counts as real, school-sanctioned knowledge, which means that other procedures based on their home-acquired knowledge do not.

Talk is Central. The written material reproduced in Fig. 4.5 is the record, as made by one child in the class, of an extended discussion lasting about 40 minutes and orchestrated by the teacher. The discussion is a reflection of a final principle of Bill's program:

> Talk about mathematics; don't just do arithmetic.

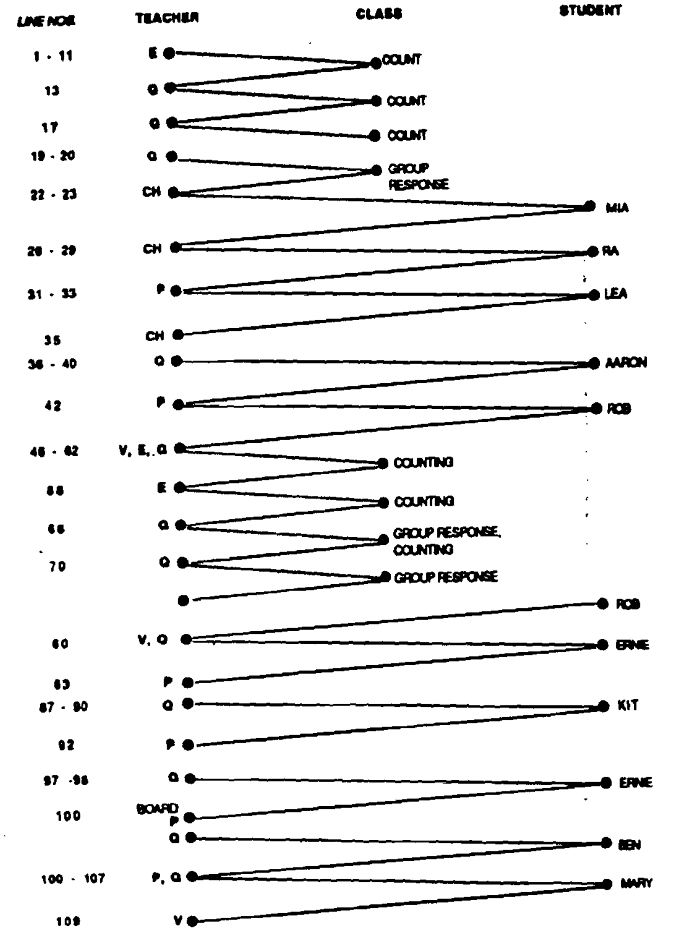
In traditional elementary school practice, math class is devoid of the normal chatter of children, the talk by which they learn most naturally. The focus in school math on performing procedures drives out talk. So does a false but widely held belief that mathematical talk must always be precise. It is true that the end result of mathematical thinking and talking is a set of precise statements. But conversation among mathematicians about their subject is full of the hesitations, restarts, and partially formulated ideas that characterize any conversation in which people are grappling with new ideas and trying to build explanations.

The Structure of Classroom Discourse

Communal Concept Building. To bring talk about mathematical ideas into the classroom, Bill has developed a special form of classroom discourse. She leads discussions designed to engage children in a form of shared problem solving in which the class as a whole, building on contributions from individuals and small working groups, constructs solutions and justifications to a problem. Like many other classroom discussions (see, e.g., Griffin & Humphrey, 1978; Mehan, 1979; Poole, 1990; Sinclair & Coulthard, 1975), Bill's classroom discourse is substantially controlled by the teacher. The teacher generally alternates turns with the children, whose contributions are much shorter than hers; thus, she controls the topic and direction of classroom talk. Unlike the standard school recitation form of discourse that consists of a series of unrelated questions and answers, however, Bill incorporates the children's contributions into relatively extended lines of reasoning that may last across several teacher–student exchanges.

Figure 4.6 schematizes the flow of conversation in a typical lesson. This diagram distinguishes teacher talk, student talk, and full class talk (almost always choral counting). Topically related turns are connected by lines. One can see in this diagram that very few turns are unconnected. Almost every statement is part of a chain that runs over several exchanges.

Revoicing. A look inside this general pattern uncovers another outstanding feature of the discourse: the teacher's frequent repeating or *revoicing* (Bill, Leer, Reams, & Resnick, 1992) of children's contributions. Each teacher contribution in a discussion can be coded as a repetition (P), revoicing (V), question (Q), challenge (CH), or explanation (E). Repetitions are counted when the teacher restates exactly what a child said; adds some term that functions deictically, such as *here*, *now*, or *again*; or writes what a child has said on the blackboard. These repetitions seem to serve several functions. In the general flow of conversation, they function as acceptances of a child's response to a teacher request. In addition, because children often speak in soft or indistinct voices in classroom settings, repeating their words is often necessary if their contributions are to become public: available to all members of the class and not just to the teacher.



Note. E = Explanation; Q = Question; CH = Challenge; P = Repetition; V = Revoicing

FIG. 4.6. Pattern of conversation in Victoria Bill's math classroom.

Revoicing carries the process of incorporating children's responses even further. Revoicing refers to occasions when the teacher rephrases or otherwise modifies the children's contributions. We regard the following kinds of rephrasings as revoicing:

- providing a label for a quantity after a child utters only a number
- rephrasing from dialect to standard English
- repeating only a part of what the child said
- making a mathematical restatement (e.g., changing a child's statement of repeated addition into a mathematically equivalent multiplication statement)

In a representative 10 min of classroom talk that we coded for one of Bill's lessons, 84% of all children's contributions were incorporated into the discussion. Of these, more than half were repeated by the teacher, and almost a third were revoiced. In three of the revoicing incidents, the teacher provided a label for a quantity stated by the child as only a number. These are potentially crucial discourse moves, because they serve to keep the class's attention focused on the semantics of the story problem, not just on the numerical arithmetic. One revoicing was a restatement in standard English ("Ones that don't have the icing") of a statement given by a child in vernacular dialect ("Ones that don't have *no* icing"). This kind of revoicing also serves to maintain the focus on meaningful communication while the teacher standardizes the language used.

Revoicings can be considered a form of conversational scaffolding (cf. Wood, Bruner, & Ross, 1976). Such scaffolding permits children to participate in a conversation that is beyond their current individual cognitive capacities. Other forms of conversational scaffolding are common in ordinary conversation, for example, the occasions on which a person provides the word or phrase for which a conversational partner seems to be searching. Repetition and revoicing are particularly prominent forms in teacher-controlled classroom discourse.

Revoicings sometimes serve as tools in the teacher's effort to maintain continuity in the conceptual content of the conversation. In these cases, Bill reformulates a child's contribution so that she can use it to forward her own pedagogical goal. For example, in a lesson focused on multiplication, the children are asked to figure out how many children are in the class, and one child proposes, "You could count them *by groups*." (He is referring to the table seating groups in the room.) Accepting his idea, Bill also transforms it, saying, "Oh. Oh, he says *by fours* . . . . So this is a group of four . . . ." This transformation allows her to pursue her pedagogical goal of showing the relationship between repeated addition and multiplication.

Linking Representations. Another special feature of Bill's teaching is her particular strategy for linking different representations—verbal, concrete, graphic, and formal—of quantities and their relations. Figure 4.7 schematizes 3 min of a discussion in which Bill led the class through several different ways of determining how many cupcakes there were on a tray. In these few minutes, five different representations were linked: a written equation; the children's oral language; the teacher's oral language; the actual tray of cupcakes; and a transparency graphic showing the cupcake tray schematically as a 3 × 7 array of squares.

In the course of the conversation, the equation $5 + 5 + 5 = 15$ was produced. But this happened slowly, with each element in the equation written down only after child language and teacher language were carefully linked to the cupcakes and to the transparency graphic. In the opening line, Rob suggested that one could start the process of finding the total number of cupcakes by "do[ing] five plus five plus five." At this point, Rob had stated all of the left side of the equation. But Bill did not proceed directly to writing what Rob had said. Instead, she linked his words to their physical referent, the uniced cupcakes, saying "So when he's thinking of five plus five plus five, he's thinking of . . . ." As she said this, Bill pointed to the 3 rows of uniced cupcakes. She thus revoiced Rob's contribution and simultaneously attributed a particular referent (the 3 rows of cupcakes) to it. She also insisted on a label, "not iced," for the particular cupcakes, thus linguistically specifying the referent of Rob's statement.

Next, Bill analyzed the uniced cupcakes to show more precisely to what the 5s refer. First, she specified a row of 5, pointing carefully to one row. The class counted. The choral counting used frequently in this lesson served to reiterate the quantitative meaning of the 5s that Rob had spoken; they are the cardinals arrived at by counting each of the rows. After the children counted one row of cupcakes, the exercise was repeated, but this time the squares in the transparency schematic were counted. This shift highlights the fact that the schematic stands for the cupcake tray and can be used as if it were the tray itself. In other words, the second count refers to the same row of five cupcakes.

In just about one minute of talk, one row of 5 was selected for attention. Teacher and child language was linked to both displays. And only after all of this was done was a single 5 written into the developing equation.

THE DEVELOPMENTAL ROOTS OF MATH MATH

Bill's teaching approach is based on a developmental theory of how mathematical knowledge develops through a sequence of levels characterized by changes in representational content (see Resnick & Greeno, 1990). This theory is distinguished from Piaget's theory of mathematical development in two important ways. First, what develops over time is mainly the *kinds* of conceptual entities that are recognized and reasoned about rather than logical structures. In fact, fundamental logical or relational structures are maintained across the levels of mathematical development. Second, the different representational forms develop

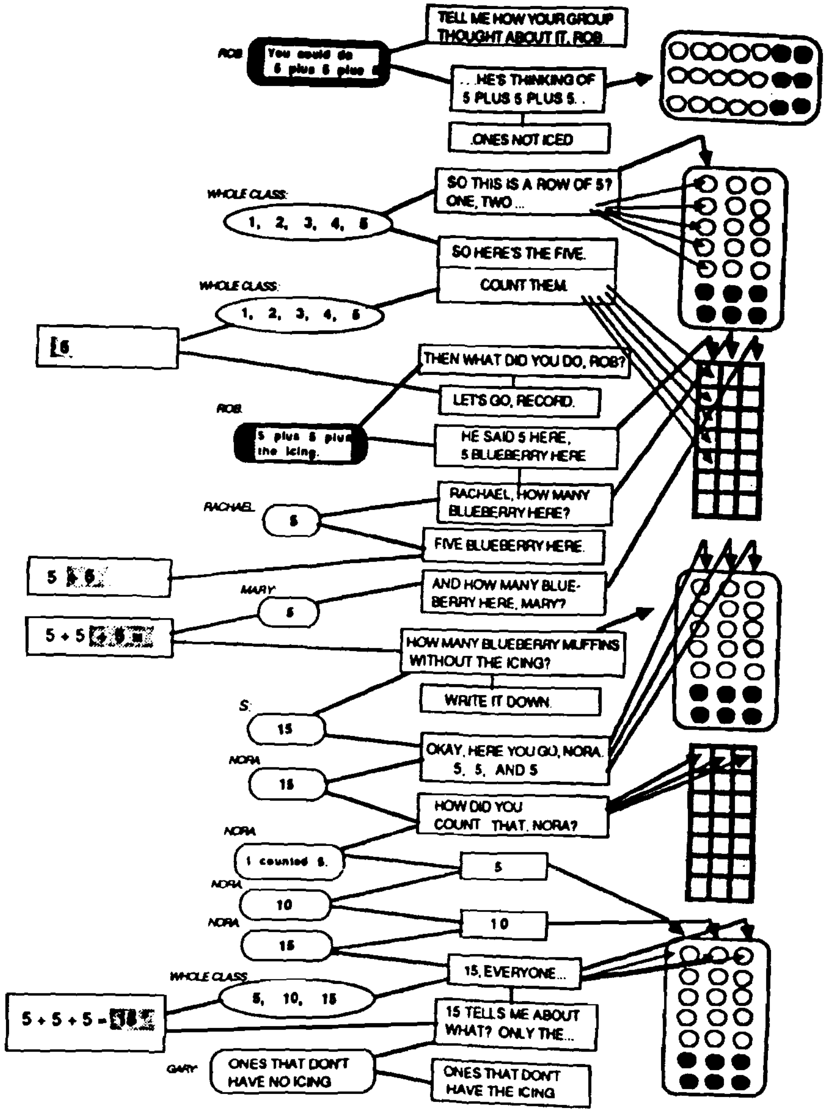


FIG. 4.7. Linking representations in three minutes of classroom talk.

unevenly for different numbers and operations, and the levels, therefore, can coexist in the same child.

Four Levels of Mathematical Thinking. Our ideas about the importance of representational structures in mathematical development are shared by a number of theorists (e.g., Fuson, 1988; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Most of those accounts, however, have focused heavily on the representations underlying counting and the transitions from counting objects to reasoning about cardinalities of sets. Our analysis is compatible with other representational development theories but extends them to include both a form of reasoning about quantity that precedes the ability to count individual objects or sets and an advanced form of reasoning in which operators and relations are represented as conceptual entities. The four kinds of mathematical thinking that Resnick and Greeno identified are summarized in Table 4.2.

In the *mathematics of protoquantities*, reasoning is about amounts of physical material. Comparisons of amounts are made and inferences can be drawn about the effects of various changes and transformations on the amounts; but no numerical quantification is involved. The language of protoquantitative mathematical thinking is a language of descriptive and comparative terms applied directly to the physical objects or amounts: a big doll, many eggs, more milk. In the mathematics of protoquantities, operations are actions that can be performed directly on physical objects or material: increasing and decreasing, combining and separating, comparing, ordering, pairing. The simplest form of protoquantitative reasoning is direct perceptual comparison of objects or sets of different sizes. More advanced protoquantitative reasoning works on a mental representation of amounts of material and allows one to reason about the results of imagined increases and decreases in amounts of material. Thus, protoquantitative reasoners can say that there *will be* more apples after mother gives each child some additional ones, or that some mice *must have been* removed if there are now less than before, without being able to look simultaneously at the objects in their before and after states. Similarly, mental combining and separating operations permit children to reason protoquantitatively about the relations between parts and whole: for example, more fruits in the bowl than either apples or oranges (Fuson, Lyons, Pergament, Hall, & Kwon, 1988; Markman & Siebert, 1976; McGarrigle, described in Donaldson, 1978).

In the *mathematics of quantities*, reasoning is about numerically quantified amounts of material. Numbers are used as measures: 4 dolls, 3 feet of board, 7 pounds of potatoes. In the mathematics of quantities, numbers function as adjectives; they describe a property (the measured amount) of a physical quantity. The numbers take their meaning from the physical material they refer to and describe. Terms from formal mathematics, such as *add* or *divide*, may be used, but their reference is to action on physical material. Operations in the mathematics of quantities are actions on measured amounts of material.

TABLE 4.2
Four Types of Mathematical Thinking

Mathematics of:	*Objects of Reasoning*	*Linguistic Terms*	*Operations*
protoquantities	physical material	much, many, more, less, big, small, etc.	increase, decrease, combine, separate, compare, order
quantities	measured physical material	*n* objects, *n* inches, *n* pounds, etc. Add, take away, divide	increase and decrease quantified sets by specific numbers of objects; increase and decrease measured amounts of material by measured amounts combine and partition quantified sets or measured amounts divide a set or measured amount into equal shares
numbers	specific numbers	*n* more than, *n* times, plus *n*, minus *n*, times *n*, *n* plus *m*, *n* divided by *m*	actions of adding, subtracting, multiplying, dividing, applied to specific numbers
operators	numbers in general, operations, variables	addition, subtraction, multiplication, division, difference, equivalence, times greater than, times less than, 1/nth of	commute, associate, distribute, compose, decompose

By contrast, in the *mathematics of numbers*, numbers function not as adjectives, describing something else, but as nouns. That is, numbers are conceptual entities that can be manipulated and acted on. One can add 3 and 4 (not 3 apples and 4 apples) or multiply 3 (not 3 books) by another number. In the mathematics of numbers, numbers themselves have properties rather than being properties of physical material. The properties of numbers are defined in terms of other numbers. Numbers have magnitudes in relation to one another: For example, 12 is *8 more than* 4; it is *3 times* 4; it is 1/3 of 36. Numbers are also compositions of other numbers: thus, 12 is 8 + 4, 7 + 5, 6 + 6, and so forth. Operations in the mathematics of number are actions taken on numbers, resulting in changes in

those numbers. Thus, 12 can be changed to 4 by subtracting 8 or, alternatively, by dividing by 3; it can be changed to 36 by multiplying by 3 or by adding 24. The numbers being compared, composed, and changed in these examples are purely conceptual entities. Their meaning derives entirely from their relations to one another and their place within a system of numbers. Physical material need not be imagined.

The final type of mathematical thinking that we identify is the *mathematics of operators*, in which not only numbers but also operations on numbers are conceptual entities that can be reasoned about. In the mathematics of numbers, operations are like transitive verbs. They describe actions that can be performed on numbers. But they are not themselves objects with properties, objects on which actions can be taken. Similarly, in the mathematics of numbers, one can describe relations between numbers, but the relations are essentially adjectives describing properties of the numbers. They are not themselves noun-like objects with properties that can be reasoned about. In the mathematics of operators, operations do behave like nouns. They can be reasoned about and not just applied. For example, it can be argued that the operation of addition by combining is always commutative, no matter what pair of numbers is composed. In the mathematics of operators, relations between numbers also are objects to reason about. A *difference of* 3, for example, can be understood as a property of the pair 11 and 8. Differences can be compared, so that one can recognize that [11 − 8] is less than [24 − 20], or even operated on so that [11 − 8] can be subtracted from [24 − 20]. This kind of reasoning about relations as mental objects is what it takes to understand functions.

Coexistence of the Levels. There is no reason to suppose that people pass all at once from a protoquantitative way of thinking to the mathematics of quantities or from the mathematics of quantities to the mathematics of numbers. There might be some developmental limits—of the kind developed by Case (1985) and Fischer (1980), for example—on how many chunks of information can be thought about at once, and this would be a brake of sorts on the pace at which new objects and properties could be incorporated into the mental system. But most evidence suggests that mathematical mental objects are built up specific bit by specific bit rather than emerging in discrete stages. So, for example, children may be doing the mathematics of quantity on dimensions such as manyness and length while still reasoning only protoquantitatively about weight or speed. Furthermore, they may convert small integers (e.g., 1, 2, 3, and 4) into mental entities and perform the mathematics of number on them while still using higher numbers only as descriptors of amounts of physical material. Similarly, operators may be transformed from verbs to nouns, one at a time. Children can, for example, reason generally about the commutativity of addition well before they can reason about multiplicative functions. Furthermore, as each new kind of number—for example, positive integers, negative integers, fractions—is encountered, it is likely that learning will entail passage through the successive layers of the mathematics of protoquantities, quantities, numbers, and operators. Thus, at any given moment, a child can be functioning at several different layers of mathematical thought.

Even more important, as more advanced forms of mathematical reasoning are developed, the earlier ones are not discarded but remain part of the individual's total knowledge system. People able to think abstractly about numbers and operators as conceptual entities can also use numbers to refer to specific measured quantities. This capacity to move back and forth among the mathematics of quantities, numbers, and operators is crucial in enabling people to relate their abstract mathematical knowledge to practical situations. When scientists use numbers, for example (cf. Schwartz, 1988), they are almost always referring to specific, measured quantities (the mathematics of quantities); their reasoning about functional relations (the mathematics of operators) refers back to these physical quantities. Engineers often reason protoquantitatively about physical systems (e.g., deKleer & Brown, 1985; Forbus, 1985), using the conclusions reached about how quantities should change or relate to one another to constrain and check the results of more formal calculations.

Coherence Across Levels. One reason that people can move back and forth easily across the four levels of mathematical thinking is that the basic logical relations are the same at all levels. This can be seen most clearly if we express the four levels as sets of equations, as in Fig. 4.8, which shows the evolution of the mathematical concepts of commutativity and associativity.

At the level of protoquantitative reasoning, before they can reliably quantify physical material, children know that a whole quantity (W) can be cut into two or more parts and that the parts can be recombined to make the whole (Equation 1). They also know that the order in which the parts are combined does not matter in reconstituting the original amount (Equations 2 and 3). Equation 2 is a protoquantitative version of the commutativity of addition property; Equation 3 is a protoquantitative version of the associativity of addition property.

In the mathematics of quantities, relations are between measured or counted amounts of material. However, all of the relationships among protoquantitative parts and wholes are maintained. As a result, children can now reason using quantified equations, such as Equations 4 through 6. Equations 5 and 6 constitute versions of the commutativity and associativity properties within the mathematics of quantities. Note that they maintain the identical form to Equations 2 and 3.

In the mathematics of numbers, the same schemas organize knowledge about relations among numbers. This is evident in Equations 7 through 9, which exactly parallel the structure of Equations 1 through 3 and 4 through 6. Equations 8 and 9 constitute versions of commutativity and associativity in the mathematics of numbers.

Finally, in the mathematics of operators, attention switches from actions on particular numbers to more general relations between numbers. Commutativity and associativity are *always* true for addition, no matter what the numbers. Thus, Equations 10 and 11 express in formal mathematical terms the same relationships that were expressed in protoquantitative Equations 2 and 3.

Mathematics of Protoquantities

(1) P1 + P2 + P3 = W
(2) P1 + P2 = P2 + P1
(3) (P1 + P2) + P3 = P1 + (P2 + P3)

Mathematics of Quantities

(4) 3 apples + 5 apples + 4 apples = 12 apples
(5) 3 apples + 5 apples = 5 apples + 3 apples
(6) (3 apples + 5 apples) + 4 apples = 3 apples + (5 apples + 4 apples)

Mathematics of Numbers

(7) 3 + 5 + 4 = 12
(8) 4 + 7 = 7 + 4
(9) (3 + 5) + 4 = 3 + (5 + 4)

Mathematics of Operators

(10) n + m = m + n
(11) (n + m) + p = n + (m + p)

FIG. 4.8. Four levels of reasoning about commutativity and associativity.

Emergent Math Math. So where, exactly, is math math? At first blush, it might seem to emerge with the mathematics of operators, where reasoning is about general patterns of relations among quantities and not about any specific quantities or even specific numbers. But Victoria Bill's young students were engaging in an emergent form of math math: noticing relations and justifying procedures. Their talk was about quantities and numbers, sometimes even protoquantities, but it concerned the same fundamental relations as would advanced students' or mathematicians' talk about operators. With this observation, we are in a position to reconsider the role of schooling in children's sociocognitive development.

IN CONCLUSION: BACK TO SCHOOL?

My analysis has been intended to make it clear that even young children have the cognitive wherewithal to engage in forms of thinking that contain many of the features of true mathematical reasoning. They can, under the right evoking and supporting conditions, practice intellectually honest forms of math math even in elementary school. They can also perform analogs of street math that depend on and help to develop fundamental mathematical concepts.

Yet, in today's schools, neither street math nor math math is usually the order of the day. Instead, children spend their time practicing forms of school math that many have trouble learning—for reasons of both cognitive and motivational disconnect—and that are increasingly hard to defend as useful tools for the lives they will lead outside school. This is clear from studies of modern work places, which suggest that the mathematics most people actually use looks more like Brazilian street math—highly accurate, tuned to specific situations rather than abstractions, dependent on tools and instruments—than like the ritualized, repetitive arithmetic drills of the schools.

Suppose we were to do away with all or most of this school math drill. What, if anything, would we substitute and to what ends? To answer this question sensibly, we need to return to my opening questions about the nature of knowledge and cognitive competence. In mathematics, as in other fields, *knowledge*, or even *conceptual understanding*, does not adequately define cognitive competence. Cognitive competence is also a matter of knowing how to *use* concepts and knowledge in ways that are socially and culturally adaptive. I have shown that, in some fundamental respects, the street math practiced by Brazilian children and adults is based on conceptual structures—such as additive composition—that are mathematically sound. Yet these individuals fail at school math and would almost certainly have difficulty participating in the kind of discourse that would qualify as math math. The only way, then, to make sense of the very different forms of mathematical competence discussed in this chapter is to think of them as different forms of mathematical *practice*, each valued in different communities, each with its own preferred ways of using knowledge.

In mathematical practice, as in other domains, cognitive competence and social competence are ineluctably intertwined. To be good at street math is to be good at participating in a particular set of activities that involve both certain kinds of knowledge and specific ways of interacting with other people. A good practitioner of street math knows not only how to calculate the price of coconuts or the length of building walls but also how to convince others (e.g., customers) that the calculation is an appropriate one. Similarly, a good practitioner of math math is someone who knows how to engage in particular kinds of discussions about the nature of numbers, or about operators and relations, that are sanctioned as "mathematical" in nature. These discussions are likely to focus on justifications, demonstrations, and, eventually, proofs that follow certain accepted rules for both logical relations and forms of presentation. School math, too, is a form of social practice. It has its particular rules of participation, ones that, by and large, tend to discourage both the practical and familiar activity of street math and the more intellectual, valued-for-its-own-sake, activity of math math discussions.

There is no way to acquire social practices except to participate in them. Children learn street math by participating in street math activities. They can learn math math competencies in the same way: by participating in the social practices that define math math. In any given social practice, beginners are not, of course, very expert. To participate, they need scaffolding of various kinds. Older siblings may help beginning street math participants in their price calculations, for example. Or, a teacher's revoicing may allow a young child to participate in an early form of math math discussion. The important point about participation is that it is just that: actual participation, not *preparation* for participation. This means that learning and cognitive development may be productively viewed as particular forms of *socialization*—specifically, socialization into practices that involve the use of concepts and forms of reasoning that we habitually study as cognitive achievements.

This, finally, brings us back to school. Where in the world might children be socialized into math math practice? We might send older children out into apprenticeships or internships to learn modern forms of street math or use simulations of workplaces in school. But math math cannot be learned in those environments. Math math is a social practice of its own, one that values discussion, rational justification, and reflection, all in pursuit of understanding for its own sake. To socialize children into math math requires deliberately creating places where math math is practiced in a form in which children can participate. That is what Victoria Bill has done in her classroom. Bill's young students were engaging in an emergent form of math math: noticing relations, constructing formal representations of these relations, justifying procedures.

In this era in which information is freely available from so many other sources, providing deliberately planned environments for cognitive socialization is arguably the central purpose of the school. And among the kinds of environments that schools can uniquely provide are those designed to socialize children into a culture of rational inquiry and reasoned debate. Using school in this way will require exchanging the social practices of school math for those of math math. There are several models, in addition to the one described here, of what such reorganized school practice might look like for mathematics. Leading mathematics educators such as Alan Schoenfeld (1992) and Magdalene Lampert (1986) have described how math math discourse can be organized in the classroom and how mathematical concepts can emerge within that discourse. Similar cases can be made for changed practice in other subject matters. These are fundamental changes, not just fine-tuning of the curriculum. The challenge of bringing about such change, in socially and politically responsible ways, is great. It is a challenge worthy of our best efforts.

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