2. Meaning and Formalization

Formalization and cognitive (un)friendliness

- **Limits** of infinite series

“We describe the behavior of \( s_n \) by saying that the sum \( s_n \) approaches the limit 1 as \( n \) tends to infinity, and by writing

\[
1 = 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + \ldots
\]

(Courant and Robbins, 1978, p. 64)
Equations of curves in the Cartesian Plane

“The hyperbola approaches more and more nearly the two straight lines $qx \pm py = 0$ as we go out farther and farther from the origin, but it never actually reaches these lines. They are called the asymptotes of the hyperbola.”

(Courant and Robbins, 1978, p. 76)
and …

- Continuity

“… since the denominators of these fractions increase without limit, the values of $x$ for which the function $\sin(1/x)$ has the values 1, -1, 0, will cluster nearer and nearer to the point $x = 0$. Between any such point and the origin there will be still an infinite number of oscillations of the function.”

(Courant and Robbins, 1978, p. 283)
Pure Math: nothing moves

- Axiomatic approaches, existential and universal quantifiers, and the Least Upper bound axiom don’t tell us anything about
  - Approaching
  - Tending to
  - Oscillating
  - Going farther and farther
  - … and so on
Continuity

- A function $f$ is continuous at a number $a$ if the following three conditions are satisfied:
  
  1. $f$ is defined on an open interval containing $a$
  2. $\lim_{x \to a} f(x)$ exists, and
  3. $\lim_{x \to a} f(x) = f(a)$. 
Where \( \lim_{x \to a} f(x) \) is ...

Let a function \( f \) be defined on an open interval containing \( a \), except possibly at \( a \) itself, and let \( L \) be a real number. The statement

\[
\lim_{x \to a} f(x) = L
\]

means that \( \forall \, \varepsilon > 0, \exists \, \delta > 0, \)

such that if \( 0 < |x - a| < \delta, \)

then \( |f(x) - L| < \varepsilon. \)
So?
What moves? …

- $\varepsilon$-$\delta$ formalisms
- and other formal definitions don’t give the answer
- Weierstrass-Dedekind and the Arithmetization of Calculus (19th Century) don’t either.
Are these just “dead” expressions?

- the sum $s_n$ approaches ...
- the hyperbola approaches ...
- … as $n$ tends to infinity.
- … oscillations of the function ...
- … it never actually reaches ...

... the reading assignment
Metaphorical Fictive Motion

- Metaphorical Fictive Motion operates on a network of precise conceptual metaphors (e.g., Numbers Are Locations in Space)
- providing the inferential structure required to conceive mathematical functions as having motion and directionality
  - $\sin \frac{1}{x}$ oscillates more and more as $x$ approaches zero
  - $g(x)$ never goes beyond 1
  - If there exists a number $L$ with the property that $f(x)$ gets closer and closer to $L$ as $x$ gets larger and larger; $\lim_{x \to \infty} f(x) = L$
Metaphorical Fictive Motion

- Are these mere linguistic expressions (using meaningless and arbitrary notations)?
- NO. They are actual manifestations of the inherent semantics (meaning), seen through extremely fast, highly efficient, and effortless cognitive mechanisms underlying thought: speech-gesture co-production
The cognitive components of these mathematical ideas (e.g., continuity)
- Dynamic
- Source-path-goal based, etc.,

DON’T GET CAPTURED BY THE ABOVE FORMALISMS and are COGNITIVELY INCOMPATIBLE WITH THEM
- Static existential and universal quantifiers
- Preservation of closeness
- Gaplessness

Núñez & Lakoff, 1998
Núñez, Edwards, Matos, 1999
Núñez, 2006, 2007