A Fresh Look at the Foundations of Mathematics: 
Gesture and the Psychological Reality of Conceptual Metaphor

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Abstract:
The study of speech-gesture-thought co-production serves multiple purposes, providing deep insight into many areas of investigation that go from psycholinguistics and cognitive linguistics to discourse analysis, and to the neuroscience of motor action and language. In this piece, building on my previous work (with George Lakoff) on the cognitive science of mathematics (Lakoff & Núñez, 1997, 2000; Núñez & Lakoff, 1998, 2005), I focus on the study of gesture production in order to address the question of the nature of mathematics and its foundations. I analyze (gestural) convergent evidence of the psychological reality of fundamental conceptual metaphors that we claim make infinitesimal calculus possible (in particular, what concerns limits and continuity). These conceptual metaphors, which are analyzed in detail in Where Mathematics Comes From (Lakoff & Núñez, 2000), emerge from fundamental mechanisms of everyday human imagination, language, and cognition, and structure the inferential organization of mathematical concepts and ideas. In this chapter, I show how the study of the gesture production of professional mathematicians turns out to be crucial in characterizing (in real-time) fundamental metaphorical contents that, while making the very mathematical ideas possible, are not captured by the standard well-accepted formalisms that are taken to “define” what mathematical concepts really are.

1. Introduction

All over the world people gesture when speaking. Despite its universality and ubiquity, the natural phenomenon of gesture is poorly understood. The reason for this is relatively simple: With a few rare exceptions\(^1\), the investigation of human gesture has been, for a long time, left out of the study of the human mind. It is not an exaggeration to say that gesture constitutes, unexcusably, a fundamental forgotten dimension of thought and language. One of the main problems has been that in western academic circles (especially in the United States), gesture has not been visible because it falls between the cracks of

\(^1\) See, for instance, the creative, but largely unknown, work of the French Jesuit anthropologist Marcel Jousse (1886-1961) (Jousse, 1974, 1975).
the well-established academic disciplines. In Chomskian linguistics, for instance—where language has been seen mainly in terms of abstract grammar, formalisms, and combinatorics—there was simply no room for “bodily production” such as gesture. In mainstream experimental psychology gesture was ignored, among other reasons, because, on the one hand, the field took language processing as the study of word-by-word sentence treatment (written or spoken) focusing on lexical items, phonology, and so on. On the other hand, being produced in a spontaneous manner, gesture was seen as very difficult to operationalize and hard to grasp experimentally. In mainstream cognitive science, which in its origins was strongly influenced by classic artificial intelligence, there was simply no room for gestures either. Cognitive science and artificial intelligence, being originally shaped by body-free abstract rule-driven digital computational paradigms, simply left no room for even considering the possibility that gesture may have mattered in the study of human cognition. In all these cases, gestures were completely ignored and left out of the picture that defined, for decades to come, what constituted genuine subject matters for the study of the mind. At best gestures were considered as a kind of epiphenomenon, secondary to other more important and better-defined phenomena.

But in the last twenty years or so, this scenario has changed in a radical way with the pioneering work of A. Kendon (1980, 1982, 2004), D. McNeill (1985, 1992, 2000, 2005), S. Goldin-Meadow (2003), and many others. Research in a large variety of areas, from child development, to neuropsychology, to linguistics, and to anthropology, has shown the intimate link between oral production, gestural production, and thought. Finding after finding has shown, for instance, that gestures are often produced in astonishing synchronicity with speech, that they develop in close relation with speech, and that brain injuries affecting speech production also affect gesture production. There are several sources of evidence supporting (1) the view that speech and gesture are in reality two facets of the same cognitive linguistic reality, and (2) an embodied approach for understanding language, conceptual systems, and high-level cognition. Some of these sources are the following:

2) Largely unconscious production: Gestures are less monitored than speech, and they are to a great extent unconscious. Speakers are often unaware that they are gesturing at all (McNeill, 1992).

3) Speech–gesture synchronicity: Gestures are very often co-produced with speech, in co-timing patterns that are specific to a given language (McNeill, 1992).

4) Gesture production with no visible interlocutor: Gestures can be produced without the presence of interlocutors; for example, people gesture while talking on the telephone, and in monologues; congenitally blind individuals gesture as well (Iverson & Goldin-Meadow, 1998).

5) Speech–gesture co-processing: Stutterers stutter in gesture, too, and impeding hand gestures interrupts speech production (Mayberry & Jaques, 2000).

6) Speech–gesture development: Gesture and speech development are closely linked (Bates & Dick, 2002; Goldin-Meadow, 2003; Iverson & Thelen, 1999).

7) Speech–gesture complementarity: Gesture can provide complementary (as well as overlapping) content to speech content. Speakers synthesize and subsequently cannot distinguish information taken from the two channels (Kendon, 2000).

8) Gestures and abstract metaphorical thinking: Linguistic metaphorical mappings are paralleled systematically in gesture (Cienki, 1998; Cornejo, Simonetti, Aldunate, Ceric, López, & Núñez, in preparation; McNeill, 1992; Núñez & Sweetser, 2001, 2006; Sweetser, 1998; see also contributions in this volume, which is especially dedicated to this subject).

As a result, the study of gesture is, in contemporary academia, a rich field that informs many areas of investigation that go from psycholinguistics and cognitive linguistics to discourse analysis, to the neuroscience of motor action and language, and even to applied areas such as computer software and video game development, and human-computer interaction. The scholarly work published in the relatively new journal *Gesture* gives an excellent picture of the variety of domains that benefit from the study of gesture. In this
chapter, I want to show yet another interesting application of the study of gesture, speech, and thought, especially gesture and metaphorical thought: the study of the inferential organization of abstract conceptual systems—in this case mathematics—through the analysis of gesture production.

Building on my previous work with George Lakoff on the cognitive nature of mathematics (Lakoff & Núñez, 1997, 2000; Núñez & Lakoff, 1998, 2005), I want to address the question of the (cognitive) foundations of this discipline, providing empirical evidence for the psychological reality of crucial conceptual metaphors that make mathematics what it is. In order to make my argument as clear as possible, I will focus on a particular case study, namely, limits and continuity in calculus. It will be important to keep in mind that within this framework gestures are not investigated as mere aids for teaching and learning mathematics, but rather— when produced by professional mathematicians — as manifesting important dimensions of the actual inferential organizational that constitute mathematical concepts. In other words, in this chapter I will show how gesture studies can inform deep questions about the nature of mathematics itself.

2. What is mathematics? And what makes it so special?

What is the nature of mathematics? What makes it possible? What is the cornerstone of such a fabulous objective and precise logical edifice? These are questions that have been treated extensively in the realm of the philosophy of mathematics, becoming, in the 20th century, specific subject matters for the rather technical fields of formal logic and metamathematics. Ever since, the foundations of mathematics have taken to be intra-mathematical (i.e., inside mathematics proper), as if the tools of formal logic alone are to provide the ultimate answers about the nature of mathematics. But can the foundations...
of mathematics be themselves, mathematical entities? Or do they lie outside of mathematics? And if yes, where do these entities come from? I’ll argue that they come from the human imagination and everyday cognition, and that therefore one must address the above questions taking into consideration advances made in the scientific study of the mind, especially what concerns the mechanisms that make abstraction possible.

Indeed, an essential characteristic of mathematics is that, despite being a highly technical domain, the very entities that constitute what Mathematics is are idealized mental abstractions. These entities cannot be perceived directly through the senses. Even, say, a point, which is the simplest entity in Euclidean geometry can’t be actually perceived. A point, as defined by Euclid is a dimensionless entity, an entity that has only location but no extension! No super-microscope will ever be able to allow us to actually perceive a point. A point, with its precision and clear identity, is an idealized abstract entity. The imaginary nature of mathematics becomes more evident when the entities in question are related to infinity where, because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself. Yet, infinity in mathematics is essential. It lies at the very core of many fundamental concepts such as limits, least upper bounds, topology, mathematical induction, infinite sets, points at infinity in projective geometry, to mention only a few.

Moreover, mathematics has a unique collection of features. It is (extremely) precise, objective, rigorous, generalizable, stable, and, of course, applicable to the real world. Any attempt to address the nature of mathematics must explain these features. So, the intriguing question that comes to mind then is: If mathematics is the product of human imagination and abstraction—human ideas, how can we explain the nature of mathematics with this collection of unique features? It is easy to see that such a question doesn’t represent a real problem for approaches inspired in Platonic philosophies (very influential in philosophy of mathematics), which rely on the existence of transcendental worlds of ideas beyond human existence. The world of ideals exists outside, and independently, of human beings. But this view doesn’t have any support based on scientific findings and doesn’t provide any link to current empirical work on human ideas and conceptual systems (although it is supported, as a matter of faith by many Platonist
scientists and mathematicians). The question doesn’t pose major problems to purely formalist philosophies either (also very influential), because in that worldview mathematics is seen as a manipulation of meaningless symbols. The question of the origin of the meaning of mathematical ideas doesn’t even emerge in the formalist arena. For those studying the human mind scientifically, however (e.g., cognitive scientists), the question of the nature of mathematics is indeed a real challenge, especially for those who endorse an embodied oriented approach to cognition. How can an embodied view of the mind give an account of an abstract, idealized, precise, sophisticated and powerful domain of ideas if direct bodily experience with the subject matter is not possible?

In *Where Mathematics Comes From*, George Lakoff and I give some preliminary answers to the question of the cognitive origin of mathematical ideas (Lakoff and Núñez, 2000). Building on findings in mathematical cognition, and using mainly methods from cognitive linguistics, a branch of cognitive science, we suggest that most of the idealized abstract technical entities in mathematics are created via human cognitive mechanisms that extend the structure of bodily experience while preserving inferential organization. Such mechanisms are, among others, *conceptual metaphors* (Lakoff & Johnson, 1980; Sweetser, 1990; Lakoff, 1993; Lakoff & Núñez, 1997; Núñez, 2000; Núñez & Lakoff, 1998, 2005), *conceptual blends* (Fauconnier & Turner, 1998, 2002; Núñez, 2005), *conceptual metonymy* (Lakoff & Johnson, 1980), *fictive motion* and *dynamic schemas* (Talmy, 1988, 2003). Lakoff and I analyzed many areas in mathematics, from set theory to infinitesimal calculus, to transfinite arithmetic, to Boolean algebra, and showed how, via everyday human embodied mechanisms such as conceptual metaphor and conceptual blending, the inferential patterns drawn from direct bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains⁴.

An important component of the argument Lakoff and I provide rests on the role conceptual metaphor plays in structuring the inferential organization of mathematical concepts and ideas. Because conceptual metaphor theory, originally based mainly on

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⁴ The details of how conceptual metaphor and conceptual blending work go beyond the scope of this piece. For a general introduction to these concepts see Lakoff & Núñez (2000, chapters 1-3), and the references given therein.
purely linguistic grounds, has made important claims about human cognition, abstraction, and mental phenomena, some psychologists have raised and addressed the question of empirical evidence to support the psychological reality of conceptual metaphor (Gibbs, 1994, in press; Núñez, in press). How do we know, for instance, that some of the metaphors we observe in linguistic expressions are not mere “dead metaphors,” expressions that were metaphorical in the past but which have become “lexicalized” in nowadays language? How do we know that these metaphors are the actual result of real-time cognitive activity? And how can we find out the answers to such questions? As we’ll see, one answer is provided by the study of gesture-speech co-production.

We are now in a position to turn into our case study.

3. Limits and continuity

In the spirit of how the semantics of everyday human language is studied, let us see how mathematical ideas are actually presented, described, characterized, and even formally defined in mathematics books, academic journals, and college textbooks. A careful analysis of technical books and articles in mathematics (through the study of specific wordings, technical notations, figures, diagrams, etc.) provides very good insights into the question of how the inferential organization of human everyday ideas has been used to create mathematical concepts. So, let’s take a look, for instance, at the classic book *What is Mathematics* by R. Courant & H. Robbins (1978), and see what it says about limits in infinite series, and continuity of functions.

a) Limits of infinite series

In characterizing limits of infinite series, Courant & Robbins write:

“We describe the behavior of \( s_n \) by saying that the sum \( s_n \) approaches the limit 1 as \( n \) tends to infinity, and by writing

\[
1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots
\]“ (p. 64, my emphasis)

We can see that, strictly speaking, this statement refers to a sequence of discrete and motion-less partial sums of \( s_n \) (real numbers), corresponding to increasing discrete and motion-less values taken by \( n \) in the expression \( 1/2^n \) where \( n \) is a natural number. If we examine this statement closely we can see that it describes some facts about numbers, and
about the result of discrete operations with numbers, but that there is no motion whatsoever involved. No entity is actually approaching or tending to anything. So why then did Courant and Robbins (or mathematicians in general, for that matter) use dynamic language to express static properties of static entities? And what does it mean to say that the “sum \( s_n \) approaches,” when in fact a sum is simply a fixed number, a result of an operation of addition?

b) Continuity
Later in the book, the authors analyze cases of continuity and discontinuity of trigonometric functions in the real plane. Referring to the function \( f(x) = \sin \frac{1}{x} \) (whose graph is shown in Figure 1) they say: “… since the denominators of these fractions increase without limit, the values of \( x \) for which the function \( \sin(\frac{1}{x}) \) has the values 1, -1, 0, will cluster nearer and nearer to the point \( x = 0 \). Between any such point and the origin there will be still an infinite number of oscillations of the function” (p. 283, our emphasis).

![Figure 1. The graph of the function \( f(x) = \sin \frac{1}{x} \).](image)

Once again, if, as pure formal mathematics says, a function is a mapping between elements of a set (coordinate values on the \( x \)-axis) with one and only one of the elements of another set (coordinate values on the \( y \)-axis), all what we have is a static correspondence between points on the \( x \)-axis with points on the \( y \)-axis. How then can the authors (or mathematicians in general) speak of “oscillations of the function,” let alone an infinite number of them?
These two simple examples illustrate how ideas and concepts are described, defined, illustrated, and analyzed in mathematics books. You can pick your favorite mathematics books and you will see similar patterns. In both examples, static numerical structures are involved, such as partial sums, and mappings between coordinates on one axis with coordinates on another. Strictly speaking, however, absolutely no motion or dynamic entities are involved in the formal definitions of these terms. So, if no entities are really moving, why do mathematicians speak of “approaching,” “tending to,” “going farther and father,” and “oscillating”? Where is this motion coming from? What does dynamism mean in these cases? What role is it playing (if any) in these statements about mathematics facts? Do formal definitions in mathematics provide any help in answering these questions? We will first look at pure mathematics to see what it says about these questions.

Looking at pure mathematics

How does pure mathematics characterize what continuity is? What is, mathematically, continuity taken to be?

Mathematics textbooks define continuity for functions as follows:

• A function $f$ is continuous at a number $a$ if the following three conditions are satisfied:

1. $f$ is defined on an open interval containing $a$,

2. $\lim_{x \to a} f(x)$ exists, and

3. $\lim_{x \to a} f(x) = f(a)$.

Where by $\lim_{x \to a} f(x)$ what is meant is the following:

Let a function $f$ be defined on an open interval containing $a$, except possibly at $a$ itself, and let $L$ be a real number. The statement

$$\lim_{x \to a} f(x) = L$$

means that $\forall \varepsilon > 0, \exists \delta > 0,$

such that if $0 < |x - a| < \delta,$

then $|f(x) - L| < \varepsilon.$
As we can see, pure formal mathematics defines continuity in terms of limits (\(\lim_{x \to a}\)), and limits in terms of static universal and existential quantifiers quantifying over static numbers (e.g., \(\forall \varepsilon > 0, \exists \delta > 0\)), and the satisfaction of certain conditions which are described in terms of motion-less arithmetic difference (e.g., \(|f(x) - L|\)) and static smaller-than relations (e.g., \(0 < |x - a| < \delta\)). That’s it.

The moral here is that these formal definitions—especially concocted by mathematicians in the late 19th century to avoid vagueness (for a discussion see Núñez & Lakoff, 1998), which are meant to be precise and rigorous and taken to be what continuity ultimately really is, don’t tell us anything about our questions. Nowhere in those formalisms we can see why a sum “approaches” a number, or why a number can “tend to” infinity, or why a function “oscillates” between values (let alone doing it infinitely many times, as in the function \(f(x) = \sin(1/x)\)).

But this shouldn’t be a surprise. What Lakoff and I had suggested using techniques from cognitive linguistics (Lakoff & Núñez, 2000) showed what some well-known contemporary mathematicians had already pointed out in more general terms (Hersh, 1997; Thurston, 2006):

• The structure of human mathematical ideas, and its inferential organization, is richer and more detailed than the inferential structure provided by formal definitions and axiomatic methods. Formal definitions and axioms neither fully formalize nor generalize human concepts.

We can see this with a relatively simple example (taken from Lakoff & Núñez, 2000). Consider the function \(f(x) = x \sin(1/x)\) whose graph is depicted in Figure 2.

\[
f(x) = \begin{cases} 
  x \sin(1/x) & \text{for } x \neq 0 \\
  0 & \text{for } x = 0 
\end{cases}
\]
According to the $\epsilon$ - $\delta$ definition of continuity given above, this function is continuous at every point. For all $x$, it will always be possible to find the specified $\epsilon$’s and $\delta$’s necessaries to satisfy the conditions for preservation of closeness. However, according to the everyday notion of continuity — *natural continuity* (Núñez & Lakoff, 1998) — as it was used by Kepler, Euler, and the inventors of infinitesimal calculus in the 17th Century, Newton and Leibniz, this function is *not* continuous. According to the inferential organization of natural continuity, certain conditions have to be met. For instance, in a naturally continuous line we are supposed to be able to tell how long the line is between to points. We are also supposed to be able to describe essential components of the motion of a point along that line. With the function $f(x) = x \sin \frac{1}{x}$ we can’t do that. Since the function “oscillates” infinitely many times as it “approaches” the point $(0, 0)$ we can’t really tell how long the line is between two points located on the left and right sides of the plane. Moreover, as the function “approaches” the origin $(0, 0)$ we can’t tell whether it will “cross” from the right plane to the left plane “going down” or “going up.” This function violates these properties of natural continuity and therefore it is not continuous. The point is that the formal $\epsilon$ - $\delta$ definition of continuity doesn’t capture the inferential organization of the human everyday notion of continuity, and it doesn’t generalize the notion of continuity either. The function $f(x) = x \sin \frac{1}{x}$ is $\epsilon$ - $\delta$ continuous but it is not naturally continuous.

The moral here is that what is characterized formally in mathematics leaves out a huge amount of inferential organization of the human ideas that constitute mathematics. As we will see, this is precisely what happens with the dynamic aspects of the
expressions we saw before, such as “approaching,” “tending to,” “going farther and farther,” “oscillating,” and so on. Motion, in those examples, is a genuine and constitutive manifestation of the nature of mathematical ideas. In pure mathematics, however, motion is not captured by formalisms and axiomatic systems.

4. Embodied Cognition

We can now look, from the perspective of embodied cognition, at the questions we asked earlier regarding the origin of motion in the above mathematical ideas. In the case of limits of infinite series, motion in “the sum $s_n$ approaches the limit 1 as $n$ tends to infinity” emerges metaphorically from the successive values taken by $n$ in the sequences as a whole. It is beyond the scope of this chapter to go into the details of the mappings involved in the various underlying conceptual metaphors that provide the required dynamic inferential organization (for details see Lakoff & Núñez, 2000). One can note, however, that there are many conceptual metaphors and metonymies involved.

Conceptual metaphor and conceptual metonymy

Conceputal metaphor and conceptual metonymy play a fundamental role in providing the inferential organization to the most basic pieces of our puzzle. For example, conceptual metonymies occur in cases where a partial sum is conceptualized as standing for the entire infinite sum; there are conceptual metaphors in cases where the sequence of these metonymical sums is conceptualized as a single trajector moving in space (the single case is indicated by the conjugation of the sum $s_n$ approaches in 3rd person singular); there are conceptual metaphors for conceiving infinity as a single location in space such that a metonymical $n$ (standing for the entire sequence of values) can “tend to;” there are conceptual metaphors for conceiving 1 as the result of the infinite sum (i.e., not as a mere natural number but as an infinitely precise real number); and so on. Notice that none of these ideas and expressions can be literal. The facts described in these sentences don’t exist in any real perceivable world. They are metaphorical in nature. It is important to understand that these conceptual metaphors and metonymies are not simply embellishments added on top of formalisms, or “aids” to understand these formalisms. They are in fact constitutive of the very bodily-grounded forms of sense-making that
make mathematical ideas possible. It is the inferential organization provided by our embodied understanding of “approaching” and “tending to” that is at the core of these mathematical ideas.

In what concerns our “oscillating” function example, the moving object is again one single holistic object (the trigonometric function in the Real plane) constructed metaphorically from infinitely many discrete real values for \( x \), which are progressively smaller in absolute terms. In this case motion takes place in a specific manner, towards the origin from two opposite sides (i.e., for negative and positive values of \( x \)) and always between the values \( y = 1 \) and \( y = -1 \). As we saw, a variation of this function, \( f(x) = x \sin(x) \), reveals deep cognitive incompatibilities between the dynamic notion of continuity implicit in the semantics of the notion of oscillation and the static \( \varepsilon-\delta \) definition of continuity coined in the second half of the 19th century (based on quantifiers and discrete Real numbers). These deep cognitive incompatibilities between dynamic-wholistic entities and static-discrete ones may explain important dimensions of the difficulties encountered by students all over the world when learning the modern technical version of the notions of limits and continuity (Núñez, Edwards, and Matos, 1999).

\textit{Fictive motion}

Now that we are aware of the metaphorical (and metonymical nature) of the mathematical ideas mentioned above, let us analyze more in detail the dynamic component of these ideas. From where do these ideas get motion? What cognitive mechanism is allowing us to conceive static entities in dynamic terms? The answer is \textit{fictive motion}.

Fictive motion is a fundamental embodied cognitive mechanism through which we unconsciously (and effortlessly) conceptualize static entities in dynamic terms, as when we say \textit{the road goes along the coast}. The road itself doesn’t actually move anywhere. It is simply standing still. But we may conceive it as moving “along the coast.” Fictive motion was first studied by Len Talmy (1996), via the analysis of linguistic expressions taken from everyday language in which static scenes are described in dynamic terms. The following are linguistic examples of fictive motion:
• The Equator passes through many countries

• The boarder between Switzerland and Germany runs along the Rhine.

• The California coast goes all the way down to San Diego

• After Corvisart, line 6 reaches Place d’Italie.

• The fence stops right after the tree.

• Unlike Tokyo, in Paris there is no metro line that goes around the city.

Motion, in all these cases, is fictive, imaginary, not real in any literal sense. Not only these expressions use verbs of action, but they provide precise descriptions of the quality, manner, and form of motion. In all cases of fictive motion there is a trajector (the moving agent) and a landscape (the space in which the trajector moves). Sometimes the trajector may be a real object (e.g., the road goes; the fence stops), and sometimes it is metaphorical (e.g., the Equator passes through; the boarder runs). In fictive motion, real world trajectors don’t move but they have the potential to move or the potential to enact movement (e.g., a car moving along that road). In mathematics proper, however, the trajector has always a metaphorical component. That is, the trajector as such can’t be literally capable or incapable of enacting movement, because the very nature of the trajectory is imagined via metaphor (Núñez, 2003). For example, a point in the Cartesian Plane is an entity that has location (determined by its coordinates) but has no extension. So when we say “point P moves from A to B” we are ascribing motion to a metaphorically entity that only has location. First, as we saw earlier, entities which have only location (i.e., points) don’t exist in the real world, so as such, they don’t have the potential to move or not to move in any literal sense. They simply don’t exist in the real world. They are metaphorical entities. Second, literally speaking, in the Cartesian Plane point A and point B are distinct locations, and no point can change location while preserving its identity. That is, the trajector (point P, uniquely determined by its coordinates) can’t preserve its identity throughout the process of motion from A to B, since that would mean that it is changing the very properties that are defining it, namely, its coordinates.

With this basic understanding of how conceptual metaphor and fictive motion work together, we are in a position to see the embodied cognitive mechanisms underlying the mathematical expressions we saw earlier. Here we have similar expressions:
• \( \sin \frac{1}{x} \) oscillates more and more as \( x \) approaches zero
• \( g(x) \) never goes beyond 1
• If there exists a number \( L \) with the property that \( f(x) \) gets closer and closer to \( L \) as \( x \) gets larger and larger; \( \lim_{x \to \infty} f(x) = L \).

In these examples fictive motion operates on a network of precise conceptual metaphors, such as numbers are locations in space (which allows us to conceive numbers in terms of spatial positions), to provide the inferential structure required to conceive mathematical functions as having motion and directionality. Conceptual metaphor generates a purely imaginary entity in a metaphorical space, and fictive motion makes it a moving trajector in this metaphorical space. Thus, the progressively smaller numerical values taken by \( x \) which determine numerical values of \( \sin \frac{1}{x} \), are via the conceptual metaphor Numbers Are Locations in Space conceptualized as spatial locations. The now metaphorical spatial locus of the function (i.e., the “line” drawn in the plane) now becomes available for fictive motion to act upon. The progressively smaller numerical values taken by \( x \) (now metaphorically conceptualized as locations progressively closer to the origin) determine corresponding metaphorical locations in space for \( \sin \frac{1}{x} \). In this imaginary space, via conceptual metaphor and fictive motion now \( \sin \frac{1}{x} \) can “oscillate” more and more as \( x \) “approaches” zero.

**Dead metaphors?**

Up to now, we have analyzed some mathematical ideas through classic methods in cognitive linguistics, such as conceptual metaphor and fictive motion. We have studied the inferential organization modeling linguistic expressions. But so far no much have been said of actual people speaking, writing, explaining, learning, or gesturing when doing mathematics. The analysis has been almost exclusively at the level of written and oral linguistic expressions. The remaining task now is to show that these linguistic expressions are not, as some scholars have suggested, mere instances of so-called dead metaphors, that is, expressions that once in the past had a metaphorical dimension but that now, after centuries of usage, have lost their metaphorical component becoming “dead.” The question that remains open is, could it be the case that the mathematical
expressions we have seen, once were metaphorical expressions but now have become literal expressions whose meaningful origins speakers (mathematicians) don’t know anymore? If that is indeed the case, these expressions would be very much like those Latin or Greek etymologies underlying many English words which may have been known by speakers at a certain point in history, but whose original meaning is now lost. Is this what is happening to cases such as “approaching limits” and “oscillating functions”? Maybe all what we have in the mathematical expressions we have examined, is simply a story of dead metaphors, with no psychological reality whatsoever. As we will see, the study of human gesture provides embodied convergent evidence showing that this is not the case at all. Via a detailed investigation of bodily motion (mainly hands and arms) and speech co-production, gesture analysis show that the conceptual metaphors and fictive motion involved in the mathematical ideas analyzed above, far from being dead, do have an embodied psychological reality that unfolds in real-time.

5. Enter Gesture
We are now in a position to analyze mathematical expressions like the ones we saw before, but this time focusing on the gesture production of the speaker. For the purposes of this chapter, an important distinction we need to make concerns the gestures that refer to real objects in the real world, and gestures that refer to some abstract idea that in itself doesn’t exist in the real world. An example of the first group is shown in Figure 3, which shows renown physicist Professor Richard Feynman giving a lecture on physics of particles at Cornell University many years ago⁵. In this sequence he is talking about particles moving in all directions at very high speeds (Figure 3, a through e), and a few milliseconds later he completes his utterance by saying once in a while hit (Figure 3f).

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⁵ The time code shown at the bottom right of each freeze is expressed in minutes:seconds:frames. Each frame unit corresponds to approximately 33 milliseconds (there are 29.97 frames per second).
The series of gestures shown in the five first pictures correspond to the characterization of random movements of particles at high speeds. The precise finger pointing shown in figure 4f occurs when he says *once in a while hit* (the stroke of the gesture). The point here is that although Prof. Feynman’s talk was about a very abstract domain (i.e., particle physics), it is still the case that with his finger he is *indexing* a “particle,” an object with location, extension, and mass, which does exist in the real world. The trajector in this dynamic scene is, an extremely small and fast object, but nonetheless a real entity in the real world.

The following gestures—which involve pure mathematical notions—are similar in many respects, but they are even more abstract, in the sense that the entities that are indexed with the various handshapes are purely imaginary entities, like points and numbers in mathematics. Unlike particles in physics, these entities *do not exist* in the real world. For instance, Figure 4 shows a professor of mathematics teaching how to solve equations by “approximating” a root (a solution to quadratic equations).
While explaining this idea he says *you want to find sequences which get closer and closer*. As he says *sequences* (Figure 4a through 4e, which takes about 1 second) he iteratively moves (as if he is depicting steps) his right hand frontward, while pointing his palm towards his body and forming a modified version (to accommodate the thick pen he is using) of the handshape called *baby O* in American Sign Language and in gesture studies. The frontward hand iteration is co-produced with the utterance of the word *sequence*, which refers to the entity getting “closer and closer.” Notice that what “gets closer” is not an entity relative to him, but an entity “getting closer” to a specific “location,” which is, metaphorically, a numerical value in front of him. Grammatically, this is done via the iterative use of the word “closer” together with the conjunction “and.”

Figure 5 shows a professor of mathematics teaching about convergence of sequences of real numbers in a university level class. In this particular situation, he is talking about a case in which the values of an infinite sequence do not get closer and closer to a single value as \( n \) increases, but they “oscillate” between two fixed values. His right hand, with the palm towards his left has a *baby O* handshape. In this gesture the touching tip of the index and the thumb are metaphorically *indexing* a metonymical value standing for the values in the sequence as \( n \) increases (it is almost as if the subject is carefully holding a very tiny object with those two fingers). Holding that fixed

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6 In its prototypical form the index finger and the thumb are touching and are slightly bent while the other three fingers are bent
handshape, he moves his right arm horizontally back and forth while he says *still oscillates*.

Figure 5. A professor of mathematics teaching a university level class on convergence of sequences of real numbers. Here he is referring to a case in which the numbers of a sequence “oscillates.”

Hands and arms are essential body parts involved in gesturing. But often it is also the entire body that participates in enacting the inferential structure of an idea. In the following example (Figure 6) a professor of mathematics is teaching a university level course involving fundamental notions in calculus. In this scene he is talking about some particular theorems regarding monotone sequences. As he is talking about an unbounded monotone sequence, he is referring to the important property of being “going in one direction.” As he says this he is producing iterative unfolding circles with his right hand, and at the same time he is walking forward, accelerating at each step (Figure 6a through 6e). His right hand, with the palm toward his chest, displays a shape called *tapered O* (Thumb relatively extended and touching the upper part of an extended index bent in right angle, like the other fingers), which he keeps in a relatively fixed position while doing the iteration circular movement. Two seconds later he completes the sentence by saying *it takes off to infinity* at the very moment when his right arm is fully extended and his hand shape has shifted to an extended shape called *B spread* with a fully (almost over) extension, and the tips of the fingers pointing forward at eye-level.
Sometimes, when the sequence exhibits a peculiar property, handshapes adopt specific forms that match the meaning of those properties. In Figure 7 we see the same professor talking this time about a situation where the sequence is constant. As he begins to say *so that would be just like a constant sequence*, his dominant hand (the right one, with which he has been writing) raises and curls back and forms a slightly modified *tapered O* handshape (with the fingers a little bit bent pointing downwards), his elbow is bent in 90 degrees and his wrist is maximally bent with the palm oriented towards down (Fig. 7a). Then while keeping that handshape he extends his elbow (and wrist) producing a small and fast frontward (and slightly downward) motion with his right hand. In the meantime his left hand, with palm towards right, rises slowly, forming a *five* handshape with fingers extended flat (Fig. 7b). As he says the word *constant*, he abruptly stops the forward motion with his right hand marking a location situated a couple of inches in front of his open five left hand (Fig. 7c). While keeping his left hand totally fixed holding the same five handshape, he iterates a couple of times the same frontward movement with his right hand always stopping sharply at the same location, just a few inches from the open
palm of his left hand. These abruptly stopped movements performed with the curled handshape while referring to a constant sequence sharply contrast with the smooth openended fully extended arm, hand and fingers, of the previous example produced when referring to an unbounded monotone sequence (Fig. 6).

Figure 7. A professor of mathematics in a university level class talking about a constant sequence.

It is important to notice that in the last three cases the blackboard is full of mathematical expressions containing formalisms like the ones we saw earlier (e.g., static existential and universal quantifiers): formalisms, which have no indication or reference to motion. The gestures (and the linguistic expressions used), however, tell us a very different conceptual story. In these cases, the mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about. In the example shown in Figure 5, the oscillating gesture matches, and it is produced synchronically with, the linguistic expressions used. In the following one (Figure 6), the iterative unfolding circular gesture matches the inferential structure of the description of the iteration involved in the monotone sequence, and the entire body moves forwards as the sequence unfolds. Since the sequence is unbounded, it “takes off to infinity,” idea which is precisely characterized in a synchronous way with the full frontal extension of the arm and the hand. In the last example (Figure 7) the curled shaped hand moves slightly forward but “hits” repeatedly the same location never being able to go further, thus embodying the meaning of a constant sequence always unfolding the same number as outcome.
In sum, what we have learned from these gesture examples is the following:

- First, gestures provide converging evidence for the psychological and embodied reality of the linguistic expressions analyzed with classic techniques in cognitive linguistics (e.g., metaphor and fictive motion). In these cases gesture analysis show that the mathematical metaphorical expressions we saw earlier are not cases of dead metaphors. The above gestures show that the dynamism involved in these ideas have full psychological and cognitive reality.

- Second, these gestures show that the fundamental dynamic contents involving infinite sequences, limits, continuity, and so on, are in fact constitutive of the inferential structure of these ideas. Formal language in mathematics, however, is not as rich as everyday language and cannot capture the full complexity of the inferential structure of those mathematical ideas. It is cognitive science’s job (via, in this case, a detailed study of gesture production) to characterize the full richness of mathematical ideas.

6. Conclusion

One of the main points of this chapter was to show that the study of gestures can be an extremely powerful tool for complementing the investigation of the (cognitive, metaphorical) foundations of mathematics. Mathematics, which is perhaps the most abstract conceptual system we can think of, is ultimately grounded in the nature of our bodies, language, and cognition. Conceptual metaphor and fictive motion, being a manifestation of extremely fast, highly efficient, and effortless cognitive mechanisms that preserve inferences, play a fundamental role in bringing many mathematical concepts into being. We analyzed several cases involving dynamic language in limits and continuity in calculus, domains in which, according to the corresponding formal definitions and axioms, no motion should exist at all. Via the study of gestures, we were able to see that the conceptual metaphors underlying the linguistic metaphorical expressions in calculus were not simply cases of “dead” expressions. Thanks to the analysis of mathematicians’ gesture-speech co-production we were able to provide
convergent evidence supporting the psychological and cognitive reality of the conceptual metaphors underlying the notions of limits and continuity of functions (and their inferential organization). Building on gestures studies we were able to tell that these mathematics professors, not only were using metaphorical linguistic expressions, but that they were in fact—in real time—thinking dynamically! It is important to understand that these gestures are not mere teaching aids or explicit attempts to make abstract things more concrete (in fact, as we said, those dynamic gestures embody an inferential organization which is different from the ones expressed by the formalisms). Those gestures are in fact vivid manifestations of the fundamental inferential organization that constitute fundamental ideas in calculus such as limits and continuity.

References


