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Embodied Arithmetic: The Grounding Metaphors

ARITHMETIC IS A LOT MORE THAN SUBITIZING and the elementary numerical capacities of monkeys and newborn babies. To understand what arithmetic is from a cognitive perspective, we need to know much more. Why does arithmetic have the properties it has? Where do the laws of arithmetic come from? What cognitive mechanisms are needed to go from what we are born with to full-blown arithmetic? Arithmetic may seem easy once you've learned it, but there is an awful lot to it from the perspective of the embodied mind.

What Is Special About Mathematics?

As subsystems of the human conceptual system, arithmetic in particular and mathematics in general are special in several ways. They are:

- precise,
- consistent,
- stable across time and communities,
- understandable across cultures,
- symbolizable,
- calculable,
- generalizable, and
- effective as general tools for description, explanation, and prediction in a vast number of everyday activities, from business to building to sports to science and technology.

Any cognitive theory of mathematics must take these special properties into account, showing how they are possible given ordinary human cognitive capacities. That is the goal of this chapter.

The Cognitive Capacities Needed for Arithmetic

We are born with a minimal innate arithmetic, part of which we share with other animals. It is not much, but we do come equipped with it. Innate arithmetic includes at least two capacities: (1) a capacity for subitizing—instantly recognizing small numbers of items—and (2) a capacity for the simplest forms of adding and subtracting small numbers. (By "number" here, we mean a *cardinal* number, a number that specifies how many objects there are in a collection.) When we subitize, we have already limited ourselves to a grouping of objects in our visual field and we are distinguishing how many objects there are in that grouping.

In addition, we and many animals (pigeons, parrots, raccoons, rats, chimpanzees) have an innate capacity for "numerosity"—the ability to make consistent rough estimates of the number of objects in a group.

But arithmetic involves more than a capacity to subitize and estimate. Subitizing is certain and precise within its range. But we have additional capacities that allow us to extend this certainty and precision. To do this, we must count. Here are the cognitive capacities needed in order to count, say, on our fingers:

- *Grouping capacity:* To distinguish what we are counting, we have to be able to group discrete elements visually, mentally, or by touch.
- *Ordering capacity:* Fingers come in a natural order on our hands. But the objects to be counted typically do not come in any natural order in the world. They have to be ordered—that is, placed in a sequence, as if they corresponded to our fingers or were spread out along a path.
- *Pairing capacity:* We need a cognitive mechanism that enables us to sequentially pair individual fingers with individual objects, following the sequence of objects in order.
- *Memory capacity:* We need to keep track of which fingers have been used in counting and which objects have been counted.
- *Exhaustion-detection capacity:* We need to be able to tell when there are "no more" objects left to be counted.
- *Cardinal-number assignment:* The last number in the count is an ordinal number, a number in a sequence. We need to be able to assign that ordinal number as the size—the cardinal number—of the group counted. That cardinal number, the size of the group, has no notion of sequence in it.

- *Independent-order capacity:* We need to realize that the cardinal number assigned to the counted group is independent of the order in which the elements have been counted. This capacity allows us to see that the result is always the same.

When these capacities are used within the subitizing range between 1 and 4, we get stable results because cardinal-number assignment is done by subitizing, say, subitizing the fingers used for counting.

To count beyond four—the range of the subitizing capacity—we need not only the cognitive mechanisms listed above but the following additional capacities:

- *Combinatorial-grouping capacity:* You need a cognitive mechanism that allows you to put together perceived or imagined groups to form larger groups.
- *Symbolizing capacity:* You need to be able to associate physical symbols (or words) with numbers (which are conceptual entities).

But subitizing and counting are the bare beginnings of arithmetic. To go beyond them, to characterize arithmetic operations and their properties, you need much richer cognitive capacities:

- *Metaphorizing capacity:* You need to be able to conceptualize cardinal numbers and arithmetic operations in terms of your experiences of various kinds—experiences with groups of objects, with the part-whole structure of objects, with distances, with movement and locations, and so on.
- *Conceptual-blending capacity*. You need to be able to form correspondences across conceptual domains (e.g., combining *subitizing* with *counting*) and put together different conceptual metaphors to form complex metaphors.

Conceptual metaphor and conceptual blending are among the most basic cognitive mechanisms that take us beyond minimal innate arithmetic and simple counting to the elementary arithmetic of natural numbers. What we have found is that there are two types of conceptual metaphor used in projecting from subitizing, counting, and the simplest arithmetic of newborns to an arithmetic of natural numbers.

The first are what we call *grounding metaphors*—metaphors that allow you to project from everyday experiences (like putting things into piles) onto abstract concepts (like addition). The second are what we call *linking metaphors,* which link arithmetic to other branches of mathematics—for example, metaphors that allow you to conceptualize arithmetic in spatial terms, linking, say, geometry to arithmetic, as when you conceive of numbers as points on a line.

Two Kinds of Metaphorical Mathematical Ideas

Since conceptual metaphors play a major role in characterizing mathematical ideas, grounding and linking metaphors provide for two types of metaphorical mathematical ideas:

1. Grounding metaphors yield *basic, directly grounded ideas*. Examples: addition as adding objects to a collection, subtraction as taking objects away from a collection, sets as containers, members of a set as objects in a container. These usually require little instruction.
2. Linking metaphors yield *sophisticated ideas*, sometimes called *abstract ideas*. Examples: numbers as points on a line, geometrical figures as algebraic equations, operations on classes as algebraic operations. These require a significant amount of explicit instruction.

This chapter is devoted to *grounding* metaphors. The rest of the book is devoted primarily to *linking* metaphors.

Incidentally, there is another type of metaphor that this book is not about at all: what we will call *extraneous* metaphors, or metaphors that have nothing whatever to do with either the grounding of mathematics or the structure of mathematics itself. Unfortunately, the term "metaphor," when applied to mathematics, has mostly referred to such extraneous metaphors. A good example of an extraneous metaphor is the idea of a "step function," which can be drawn to look like a staircase. The staircase image, though helpful for visualization, has nothing whatever to do with either the inherent content or the grounding of the mathematics. Extraneous metaphors can be eliminated without any substantive change in the conceptual structure of mathematics, whereas eliminating grounding or linking metaphors would make much of the conceptual content of mathematics disappear.

Preserving Inferences About Everyday Activities

Since conceptual metaphors preserve inference structure, such metaphors allow us to ground our understanding of arithmetic in our prior understanding of extremely commonplace physical activities. Our understanding of elementary arithmetic is based on a correlation between (1) the most basic literal aspects of arithmetic, such as subitizing and counting, and (2) everyday activities, such as collecting objects into groups or piles, taking objects apart and putting them together, taking steps, and so on. Such correlations allow us to form metaphors by which we greatly extend our subitizing and counting capacities.

One of the major ways in which metaphor preserves inference is via the preservation of image-schema structure. For example, the formation of a collection or pile of objects requires conceptualizing that collection as a container—that is, a bounded region of space with an interior, an exterior, and a boundary—either physical or imagined. When we conceptualize numbers as collections, we project the logic of collections onto numbers. In this way, experiences like grouping that correlate with simple numbers give further logical structure to an expanded notion of number.

The Metaphorizing Capacity

The metaphorizing capacity is central to the extension of arithmetic beyond mere subitizing, counting, and the simplest adding and subtracting. Because of its centrality, we will look at it in considerable detail, starting with the Arithmetic Is Object Collection metaphor. This is a grounding metaphor, in that it grounds our conception of arithmetic directly in an everyday activity.

No metaphor is more basic to the extension of our concept of number from the innate cardinal numbers to the natural numbers (the positive integers). The reason is that the correlation of grouping with subitizing and counting the elements in a group is pervasive in our experience from earliest childhood.

Let us now begin an extensive guided tour of everything involved in this apparently simple metaphor. As we shall see, even the simplest and most intuitive of mathematical metaphors is incredibly rich, and so the tour will be extensive.

Arithmetic As Object Collection

If a child is given a group of three blocks, she will naturally subitize them automatically and unconsciously as being three in number. If one is taken away, she will subitize the resulting group as two in number. Such everyday experiences of subitizing, addition, and subtraction with small collections of objects involve correlations between addition and adding objects to a collection and between subtraction and taking objects away from a collection. Such regular correlations, we hypothesize, result in neural connections between sensory-motor physical operations like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe, *constitute a conceptual metaphor* at the neural level—in this case, the metaphor that Arithmetic Is Object Collection. This metaphor, we hypothesize, is learned at an early age, prior to any formal arithmetic training. Indeed, arithmetic training assumes this unconscious conceptual (not linguistic!) metaphor: In teaching arithmetic, we all take it for granted that the adding and subtracting of numbers can be understood in terms of adding and taking away objects from collections. Of course, at this stage all of these are mental operations *with no symbols!* Calculating with symbols requires additional capacities.

The Arithmetic Is Object Collection metaphor is a precise mapping from the domain of physical objects to the domain of numbers. The metaphorical mapping consists of

1. the source domain of object collection (based on our commonest experiences with grouping objects);
2. the target domain of arithmetic (structured nonmetaphorically by subitizing and counting); and
3. a mapping across the domains (based on our experience subitizing and counting objects in groups). The metaphor can be stated as follows:

ARITHMETIC IS OBJECT COLLECTION

Source Domain OBJECT COLLECTION		*Target Domain* ARITHMETIC
Collections of objects of the same size	→	Numbers
The size of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The smallest collection	→	The unit (One)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

Linguistic Examples of the Metaphor

We can see evidence of this conceptual metaphor in our everyday language. The word *add* has the physical meaning of physically placing a substance or a number of objects into a container (or group of objects), as in "*Add* sugar to my coffee," "*Add* some logs to the fire," and "*Add* onions and carrots to the soup." Similarly, *take . . . from, take . . . out of,* and *take . . . away* have the physical meaning of removing a substance, an object, or a number of objects from some container or collection. Examples include "*Take* some books *out of* the box," "*Take* some water *from* this pot," "*Take away* some of these logs." By virtue of the Arithmetic Is Object Collection metaphor, these expressions are used for the corresponding arithmetic operations of addition and subtraction.

> If you *add* 4 apples *to* 5 apples, how many do you have? If you *take* 2 apples *from* 5 apples, how many apples are *left*? *Add* 2 *to* 3 and you have 5. *Take* 2 *from* 5 and you have 3 *left*.

It follows from the metaphor that adding yields something *bigger* (more) and subtracting yields something *smaller* (less). Accordingly, words like *big* and *small*, which indicate size for objects and collections of objects, are also used for numbers, as in "Which is *bigger*, 5 or 7?" and "Two is *smaller* than four." This metaphor is so deeply ingrained in our unconscious minds that we have to think twice to realize that numbers are not physical objects and so do not literally have a size.

Entailments of the Metaphor

The Arithmetic Is Object Collection metaphor has many entailments. Each arises in the following way: Take the basic truths about collections of physical objects. Map them onto statements about numbers, using the metaphorical mapping. The result is a set of "truths" about the natural numbers under the operations of addition and subtraction.

For example, suppose we have two collections, A and B, of physical objects, with A bigger than B. Now suppose we add the same collection C to each. Then A plus C will be a bigger collection of physical objects than B plus C. This is a fact about collections of physical objects of the same size. Using the mapping Numbers Are Collections of Objects, this physical truth that we experience in grouping objects becomes a mathematical truth about numbers: If A is greater than B, then $A + C$ is greater than $B + C$. All of the following truths about numbers arise in this way, via the metaphor Arithmetic Is Object Collection.

The Laws of Arithmetic Are Metaphorical Entailments

In each of the following cases, the metaphor Arithmetic Is Object Collection maps a property of the source domain of *object collections* (stated on the left) to a unique corresponding property of the target domain of *numbers* (stated on the right). This metaphor extends properties of the innate subitized numbers 1 through 4 to an indefinitely large collection of natural numbers. In the cases below, you can see clearly how properties of object collections are mapped by the metaphor onto properties of natural numbers in general.

MAGNITUDE		
Object collections have a magnitude	→	*Numbers* have a magnitude

STABILITY OF RESULTS FOR ADDITION		
Whenever you add a fixed *object collection* to a second fixed *object collection,* you get the same result.	→	Whenever you add a fixed *number* to another fixed *number*, you get the same result.

STABILITY OF RESULTS FOR SUBTRACTION		
Whenever you subtract a fixed *object collection* from a second fixed *object collection*, you get the same result.	→	Whenever you subtract a fixed *number* from another fixed *number*, you get the same result.

INVERSE OPERATIONS		
For *collections:* Whenever you subtract what you added, or add what you subtracted, you get the original *collection*.	→	For *numbers:* Whenever you subtract what you added, or add what you subtracted, you get the original *number*.

UNIFORM ONTOLOGY		
Object collections play three roles in addition. • what you add to something; • what you add something to; • the result of adding. Despite their differing roles, they all have the same nature with respect to the operation of the *addition* of *object collections*.	→	*Numbers* play three roles in addition. • what you add to something; • what you add something to; • the result of adding. Despite their differing roles, they all have the same nature with respect to the operation of the *addition* of *numbers.*

CLOSURE FOR ADDITION		
The process of *adding* an *object collection* to *another object collection* yields a *third object collection.*	→	The process of *adding* a *number* to a *number* yields a third *number.*

UNLIMITED ITERATION FOR ADDITION		
You can add *object collections* indefinitely.	→	You can add *numbers* indefinitely.

LIMITED ITERATION FOR SUBTRACTION		
You can subtract *object collections* from other *object collections* until nothing is left.	→	You can subtract *numbers* from other *numbers* until nothing is left.

SEQUENTIAL OPERATIONS		
You can do combinations of adding and subtracting *object collections*.	→	You can do combinations of adding and subtracting *numbers*.

Equational Properties

EQUALITY OF RESULT		
You can obtain the same resulting *object collection* via different operations.	→	You can obtain the same resulting *number* via different operations.

PRESERVATION OF EQUALITY		
For *object collections*, adding equals to equals yields equals.	→	For *numbers*, adding equals to equals yields equals.
For *object collections*, subtracting equals from equals yields equals.	→	For *numbers*, subtracting equals from equals yields equals.

COMMUTATIVITY		
For *object collections*, adding A to B gives the same result as adding B to A.	→	For *numbers*, adding A to B gives the same result as adding B to A.

ASSOCIATIVITY		
For *object collections*, adding B to C and then adding A to the result is equivalent to adding A to B and adding C to that result.	→	For *numbers*, adding B to C and then adding A to the result is equivalent to adding A to B and adding C to that result.

Relationship Properties

LINEAR CONSISTENCY		
For *object collections*, if A is bigger than B, then B is smaller than A.	→	For *numbers,* if A is greater than B, then B is less than A.

LINEARITY		
If A and B are two *object collections*, then either A is bigger than B, or B is bigger than A, or A and B are the same size.	→	If A and B are two *numbers*, then either A is greater than B, or B is greater than A, or A and B are the same magnitude.

SYMMETRY		
If *collection A* is the same size as *collection B*, then B is the same size as A.	→	If *number A* is the same magnitude as *number B*, then B is the same magnitude as A.

TRANSITIVITY		
For *object collections*, if A is bigger than B and B is bigger than C, then A is bigger than C.	→	For *numbers*, if A is greater than B and B is greater than C, then A is greater than C.

In order for there to be a metaphorical mapping from object collections to numbers, the entailments of such a mapping must be consistent with the properties of innate arithmetic. Innate arithmetic has some of these properties—for example, uniform ontology, linear consistency, linearity, symmetry, commutativity, and preservation of equality. The Arithmetic Is Object Collection metaphor will map the object-collection version of these properties onto the version of these properties in innate arithmetic (e.g., $2 + 1 = 1 + 2$).

However, this metaphor will also extend innate arithmetic, adding properties that the innate arithmetic of numbers 1 through 4 does not have, because of its limited range—namely, *closure* (e.g., under addition) and what follows from closure: unlimited iteration for addition, sequential operations, equality of result, and preservation of equality. The metaphor will map these properties from the domain of object collections to the expanded domain of number. The result is the elementary arithmetic of addition and subtraction for natural numbers, *which goes beyond innate arithmetic.*

Thus, the fact that there is an innate basis for arithmetic does not mean that all arithmetic is innate. Part of arithmetic arises from our experience in the world with object collections. The Arithmetic Is Object Collection metaphor arises naturally in our brains as a result of regularly using innate neural arithmetic while interacting with small collections of objects.

Extending Elementary Arithmetic

The version of the Arithmetic Is Object Collection metaphor just stated is limited to conceptualizing addition and subtraction of numbers in terms of addition and subtraction of collections. Operations in one domain (using only collections) are mapped onto operations in the other domain (using only numbers). There is no single operation characterized in terms of elements from both domains—that is, no single operation that uses both numbers *and* collections simultaneously.

But with multiplication, we *do* need to refer to numbers and collections simultaneously, since understanding multiplication in terms of collections requires performing operations on *collections* a certain *number* of times. This cannot be done in a domain with collections alone or numbers alone. In this respect, multiplication is cognitively more complex than addition or subtraction.

The cognitive mechanism that allows us to extend this metaphor from addition and subtraction to multiplication and division is *metaphoric blending*. This is not a new mechanism but simply a consequence of having metaphoric mappings.

Recall that each metaphoric mapping is characterized neurally by a fixed set of connections across conceptual domains. The results of inferences in the source domain are mapped to the target domain. If both domains, together with the mapping, are activated at once (as when one is doing arithmetic on object collections), the result is a metaphoric blend: the simultaneous activation of two domains with connections across the domains.

Two Versions of Multiplication and Division

Consider 3 times 5 in terms of collections of objects:

- Suppose we have 3 small collections of 5 objects each. Suppose we pool these collections. We get a single collection of 15 objects.
- Now suppose we have a big pile of objects. If we put 5 objects in a box 3 times, we get 15 objects in the box. This is repeated addition: We added 5 objects to the box repeatedly—3 times.

In the first case, we are doing multiplication by *pooling*, and in the second by *repeated addition*.

Division can also be characterized in two corresponding ways, *splitting up* and *repeated subtraction:*

- Suppose we have a single collection of 15 objects, then we can split it up into 3 collections of 5 objects each. That is, 15 divided by 3 is 5.
- Suppose again that we have a collection of 15 objects and that we repeatedly subtract 5 objects from it. Then, after 3 repeated subtractions, there will be no objects left. Again, 15 divided by 3 is 5.

In each of these cases we have used numbers with only addition and subtraction defined in order to characterize multiplication and division metaphorically in terms of object collection. From a cognitive perspective, we have used a metaphoric blend of object collections together with numbers to extend the Arithmetic Is Object Collection metaphor to multiplication and division.

We can state the pooling and iteration extensions of this metaphor precisely as follows:

THE POOLING/SPLITTING EXTENSION OF
THE ARITHMETIC IS OBJECT COLLECTION METAPHOR

Source Domain THE OBJECT-COLLECTION/ ARITHMETIC BLEND		*Target Domain* ARITHMETIC
The pooling of A subcollections of size B to form an overall collection of size C.	→	Multiplication ($A \cdot B = C$)
The splitting up of a collection of size C into A subcollections of size B.	→	Division ($C \div B = A$)

THE ITERATION EXTENSION OF THE ARITHMETIC IS OBJECT COLLECTION METAPHOR

Source Domain THE OBJECT-COLLECTION/ ARITHMETIC BLEND		*Target Domain* ARITHMETIC
The repeated addition (A times) of a collection of size B to yield a collection of size C.	→	Multiplication ($A \cdot B = C$)
The repeated subtraction of collections of size B from an initial collection of size C until the initial collection is exhausted. A is the number of times the subtraction occurs.	→	Division ($C \div B = A$)

Note that in each case, the result of the operation is given in terms of the size of the collection as it is understood in the source domain of collections. Since the result of a multiplication or division is always a *collection* of a given size, multiplication and division (in this metaphor) can be combined with the addition and subtraction of collections to give further results in terms of collections.

What is interesting about these two equivalent metaphorical conceptions of multiplication and division is that they are both defined relative to the number-collection blend, but *they involve different ways of thinking about* operating on collections.

These metaphors for multiplication and division map the properties of the source domain onto the target domain, giving rise to the most basic properties of multiplication and division. Let us consider the commutative, associative, and distributive properties.

COMMUTATIVITY FOR MULTIPLICATION

Pooling A collections of size B gives a collection of the same resulting size as pooling B collections of size A.	→	Multiplying A times B gives the same resulting number as multiplying B times A.

ASSOCIATIVITY FOR MULTIPLICATION		
Pooling *A* collections of size *B* and pooling that number of collections of size *C* gives a collection of the same resulting size as pooling the number of *A* collections of the size of the collection formed by pooling *B* collections of size *C*.	→	Multiplying *A* times the result of multiplying *B* times *C* gives the same number as multiplying *A* times *B* and multiplying the result times *C*.

DISTRIBUTIVITY OF MULTIPLICATION OVER ADDITION		
First, pool *A* collections of the size of the collection formed by adding a collection of size *B* to a collection of size *C*. This gives a collection of the same size as adding a collection formed by pooling *A* collections of size *B* to *A* collections of size *C*.	→	First, multiply *A* times the sum of *B* plus *C*. This gives the same number as adding the product of *A* times *B* to the product of *A* times *C*.

MULTIPLICATIVE IDENTITY		
Pooling one collection of size *A* results in a collection of size *A*.	→	Multiplying one times *A* yields *A*
Pooling *A* collections of size one yields a collection of size *A*.	→	Multiplying *A* times one yields *A*.

INVERSE OF MULTIPLICATION		
Splitting a collection of size *A* into *A* subcollections yields subcollections of size one.	→	Dividing *A* by *A* yields one.

In each case, a true statement about collections is projected by the metaphor in the pooling/splitting extension onto the domain of numbers, yielding a true statement about arithmetic. The same will work for iterative extension (i.e., repeated addition and repeated subtraction).

Thus, the Arithmetic Is Object Collection metaphor extends our understanding of number from the subitized numbers of innate arithmetic and from simple counting to the arithmetic of the natural numbers, grounding the extension of arithmetic in our everyday experience with groups of physical objects.

Zero

The Arithmetic Is Object Collection metaphor does, however, leave a problem. What happens when we subtract, say, seven from seven? The result cannot be understood in terms of a collection. In our everyday experience, the result of taking a collection of seven objects from a collection of seven objects is an absence of any objects at all—*not a collection of objects*. If we want the result to be a number, then in order to accommodate the Arithmetic Is Object Collection metaphor we must conceptualize the absence of a collection *as a collection*. A new conceptual metaphor is necessary. What is needed is a metaphor that creates something out of nothing: From the absence of a collection, the metaphorical mapping creates a unique collection of a particular kind—a collection with no objects in it.

THE ZERO COLLECTION METAPHOR

The lack of objects to form a collection	→	The empty collection

Given this additional metaphor as input, the Arithmetic Is Object Collection metaphor will then map the empty collection onto a number—which we call "zero."

This new metaphor is of a type common in mathematics, which we will call an *entity-creating metaphor*. In the previous case, the conceptual metaphor *creates* zero as an actual number. Although zero is an extension of the object-collection metaphor, it is not a natural extension. It does not arise from a correlation between the experience of collecting and the experience of subitizing and doing innate arithmetic. It is therefore an artificial metaphor, concocted ad hoc for the purpose of extension.

Once the metaphor Arithmetic Is Object Collection is extended in this way, more properties of numbers follow as entailments of the metaphor.

ADDITIVE IDENTITY

Adding the empty collection to a collection of size A yields a collection of size A.	→	Adding zero to A yields A.
Adding a collection of size A to the empty collection yields a collection of size A.	→	Adding A to zero yields A.

INVERSE OF ADDITION

Taking a collection of size *A* away from a collection of size *A* yields the empty collection.	→	Subtracting *A* from *A* yields zero.

These metaphors ground our most basic extension of arithmetic—from the innate cardinal numbers to the natural numbers plus zero. As is well known, this understanding of number still leaves gaps: It does not give a meaningful characterization of 2 minus 5 or 2 divided by 3. To fill those gaps we need further entity-creating metaphors, e.g., metaphors for the negative numbers. We will discuss such metaphors shortly.

At this point, we have explored only one of the basic grounding metaphors for arithmetic. There are three more to go. It would be unnecessarily repetitive to go into each in the full detail given above. Instead, we will sketch only the essential features of these metaphors.

Arithmetic As Object Construction

Consider such commonplaces of arithmetic as these: "Five is *made up of* two plus three." "You can *factor* 28 *into* 7 times 4." "If you *put* 2 and 2 *together*, you get 4."

How is it possible to understand a number, which is an abstraction, as being "made up," or "composed of," other numbers, which are "put together" using arithmetic operations? What we are doing here is conceptualizing numbers as wholes made up of parts. The parts are other numbers. And the operations of arithmetic provide the patterns by which the parts fit together to form wholes. Here is the metaphorical mapping used to conceptualize numbers in this way.

ARITHMETIC IS OBJECT CONSTRUCTION

Source Domain OBJECT CONSTRUCTION		*Target Domain* ARITHMETIC
Objects (consisting of ultimate parts of unit size)	→	Numbers
The smallest whole object	→	The unit (one)
The size of the object	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
Acts of object construction	→	Arithmetic operations
A constructed object	→	The result of an arithmetic operation

A whole object	→	A whole number
Putting objects together with other objects to form larger objects	→	Addition
Taking smaller objects from larger objects to form other objects	→	Subtraction

As in the case of Arithmetic Is Object Collection, this metaphor can be extended in two ways via metaphorical blending: fitting together/splitting up and iterated addition and subtraction.

THE FITTING TOGETHER/SPLITTING UP EXTENSION

The fitting together of A parts of size B to form a whole object of size C	→	Multiplication ($A \cdot B = C$)
The splitting up of a whole object of size C into A parts of size B, a number that corresponds in the blend to an object of size A, which is the result	→	Division ($C \div B = A$)

THE ITERATION EXTENSION

The repeated addition (A times) of A parts of size B to yield a whole object of size C	→	Multiplication ($A \cdot B = C$)
The repeated subtraction of parts of size B from an initial object of size C until the initial object is exhausted. The result, A, is the number of times the subtraction occurs.	→	Division ($C \div B = A$)

Fractions are understood metaphorically in terms of the characterizations of division (as splitting) and multiplication (as fitting together).

FRACTIONS

A part of a unit object (made by splitting a unit object into n parts)	→	A simple fraction ($1/n$)
An object made by fitting together m parts of size $1/n$	→	A complex fraction (m/n)

These additional metaphorical mappings yield an important entailment about number based on a truth about objects.

If you split a unit object into n parts and then you fit the n parts together again, you get the unit object back.	→	If you divide 1 by n and multiply the result by n, you get 1. That is, $1/n \cdot n = 1$.

In other words, $1/n$ is the multiplicative inverse of n.

As in the case of the object-collection metaphor, a special additional metaphor is needed to conceptualize zero. Since the lack of an object is not an object, it should not, strictly speaking, correspond to a number. The zero object metaphor is thus an artificial metaphor.

THE ZERO OBJECT METAPHOR

The Lack of a Whole Object	→	Zero

The object-construction metaphor is intimately related to the object-collection metaphor. The reason is that constructing an object necessarily requires collecting the parts of the object together. Every whole made up of parts is a collection of the parts, with the added condition that the parts are assembled according to a certain pattern. Since object construction is a more specific version of object collection, the metaphor of Arithmetic As Object Construction is a more specific version of the metaphor of Arithmetic As Object Collection. Accordingly, the object-construction metaphor has all the inferences of the object-collection metaphor—the inferences we stated in the previous section. It differs in that it is extended to characterize fractions and so has additional inferences—for example, $(1/n) \cdot n = 1$.

It also has metaphorical entailments that characterize the decomposition of numbers into parts.

Whole objects are composites of their parts, put together by certain operations.	→	Whole numbers are composites of their parts, put together by certain operations.

It is this metaphorical entailment that gives rise to the field of number theory, the study of which numbers can be decomposed into other numbers and operations on them.

The Measuring Stick Metaphor

The oldest (and still often used) method for designing buildings or physically laying out dimensions on the ground is to use a measuring stick or string—a stick or string taken as a unit. These are physical versions of what in geometry are called *line segments*. We will refer to them as "physical segments." A distance can be measured by placing physical segments of unit length end-to-end and counting them. In the simplest case, the physical segments are body parts: fingers, hands, forearms, arms, feet, and so on. When we put physical segments end-to-end, the result is another physical segment, which may be a real or envisioned tracing of a line in space.

In a wide range of languages throughout the world, this concept is represented by a classifier morpheme. In Japanese, for example, the word *hon* (literally, "a long, thin thing") is used for counting such long, thin objects as sticks, canes, pencils, candles, trees, ropes, baseball bats, and so on—including, of course, rulers and measuring tapes. Even though English does not have a single word for the idea, it is a natural human concept.

THE MEASURING STICK METAPHOR

Source Domain THE USE OF A MEASURING STICK		*Target Domain* ARITHMETIC
Physical segments (consisting of ultimate parts of unit length)	→	Numbers
The basic physical segment	→	One
The length of the physical segment	→	The size of the number
Longer	→	Greater
Shorter	→	Less
Acts of physical segment placement	→	Arithmetic operations
A physical segment	→	The result of an arithmetic operation

Putting physical segments together end-to-end with other physical segments to form longer physical segments	→	Addition
Taking shorter physical segments from larger physical segments to form other physical segments	→	Subtraction

As in the previous two metaphors, there are two ways of characterizing multiplication and division: fitting together/dividing up and iterated addition and subtraction.

THE FITTING TOGETHER/DIVIDING UP EXTENSION		
The fitting together of *A* physical segments of length *B* to form a line segment of length *C*	→	Multiplication ($A \cdot B = C$)
The splitting up of a physical segment *C* into *A* parts of length *B*. *A* is a number that corresponds in the blend to a physical segment of length *A*, which is the result.	→	Division ($C \div B = A$)

THE ITERATION EXTENSION		
The repeated addition (*A* times) of *A* physical segments of length *B* to form a physical segment of length *C*.	→	Multiplication ($A \cdot B = C$)
The repeated subtraction of physical segments of length *B* from an initial physical segment of length *C* until nothing is left of the initial physical segment. The result, *A*, is the number of times the subtraction occurs.	→	Division ($C \div B = A$)

As in the case of the object-construction metaphor, the physical segment metaphor can be extended to define fractions.

FRACTIONS		
A part of a physical segment (made by splitting a single physical segment into n equal parts)	→	A simple fraction ($1/n$)
A physical segment made by fitting together (end-to-end) m parts of size $1/n$	→	A complex fraction (m/n)

Just as in the object-construction metaphor, this metaphor needs to be extended in order to get a conceptualization of zero.

The lack of any physical segment	→	Zero

Up to this point, the measuring stick metaphor looks very much like the object-construction metaphor: A physical segment can be seen as a physical object, even if it is an imagined line in space. But physical segments are very special "constructed objects." They are unidimensional and they are continuous. In their abstract version they correspond to the line segments of Euclidean geometry. As a result, the blend of the source and target domains of this metaphor has a very special status. It is a blend of line (physical) segments with numbers specifying their length, which we will call the Number/Physical Segment blend.

Moreover, once you form the blend, a fateful entailment arises. Recall that the metaphor states that Numbers Are Physical Segments, and that given this metaphor you can characterize natural numbers, zero, and positive complex fractions (the rational numbers) in terms of physical segments. That is, for every positive rational number, this metaphor (given a unit length) provides a unique physical segment. The metaphorical mapping is unidirectional. It does not say that for any line segment at all, there is a corresponding number.

But the blend of source and target domains goes beyond the metaphor itself and has new entailments. When you form a blend of physical segments and numbers, constrained by the measuring stick metaphor, then within the blend there is a one-to-one correspondence between physical segments and numbers. The fateful entailment is this: Given a fixed unit length, it follows that for every physical segment there is a number.

Now consider the Pythagorean theorem: In $A^2 + B^2 = C^2$, let C be the hypotenuse of a right triangle and A and B be the lengths of the other sides. Let $A = 1$ and $B = 1$. Then $C^2 = 2$. The Pythagoreans had already proved that C could not be expressed as a fraction—that is, that it could not be a rational number—

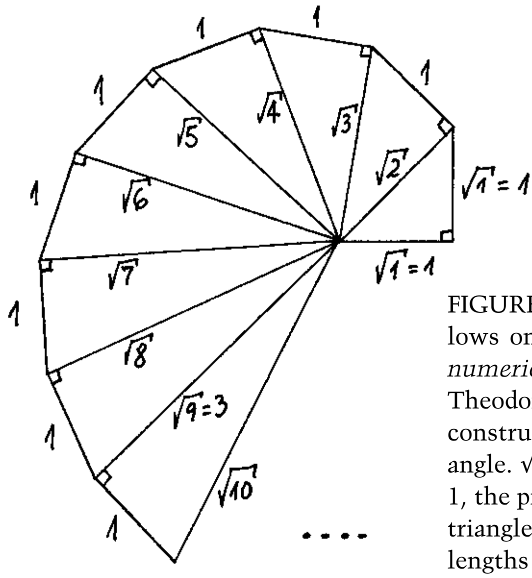

FIGURE 3.1 The measuring stick metaphor allows one to form physical segments of particular *numerical* lengths. In the diagram, taken from Theodorus of Cyrene (fourth century B.C.), $\sqrt{2}$ is constructed from the unit length 1 and a right triangle. $\sqrt{3}$ is then constructed from the unit length 1, the previously constructed length $\sqrt{2}$, and a right triangle. And so on. Without the metaphor, the lengths are just lengths, not numbers.

a ratio of physical lengths corresponding to integers. They assumed that only natural numbers and their ratios (the rational numbers) existed and that the length C was not a number at all; they called it an *incommensurable*—without ratio, that is, without a common measure.

But Eudoxus (c. 370 B.C.) observed, implicitly using the Number/Physical Segment blend, that corresponding to the hypotenuse in this triangle there must be a number: $C = \sqrt{2}$! This conclusion could not have been reached using numbers by themselves, taken literally. If you assume that only rational numbers exist and you prove that $\sqrt{2}$ cannot be a rational number, then it could just as well follow (as it did initially for the Pythagoreans) that 2 does not exist—that is, that 2 does not have any square root. But if, according to the Number/Physical Segment blend, there must exist a number corresponding to the length of every physical segment, then and only then must $\sqrt{2}$ exist as a number!

It was the measuring stick metaphor and the Number/Physical Segment blend that gave birth to the irrational numbers.

Arithmetic As Motion Along a Path

When we move in a straight line from one place to another, the path of our motion forms a physical segment—an imagined line tracing our trajectory. There is a simple relationship between a path of motion and a physical segment. The origin of the motion corresponds to one end of a physical segment; the endpoint of the motion corresponds to the other end of the physical segment; and the path of motion corresponds to the rest of the physical segment.

Given this correspondence between motions and physical segments, there is a natural metaphorical correlate to the measuring stick metaphor for arithmetic, namely, the metaphor that Arithmetic Is Motion Along a Path. Here is how that metaphor is stated.

ARITHMETIC IS MOTION ALONG A PATH

Source Domain MOTION ALONG A PATH		*Target Domain* ARITHMETIC
Acts of moving along the path	→	Arithmetic operations
A point-location on the path	→	The result of an arithmetic operation
The origin, the beginning of the path	→	Zero
Point-locations on a path	→	Numbers
A point-location	→	One
Further from the origin than	→	Greater than
Closer to the origin than	→	Less than
Moving from a point-location *A* away from the origin, a distance that is the same as the distance from the origin to a point-location *B*	→	Addition of *B* to *A*
Moving toward the origin from *A*, a distance that is the same as the distance from the origin to *B*	→	Subtraction of *B* from *A*

This metaphor can be extended to multiplication and division by means of iteration over addition and subtraction.

THE ITERATION EXTENSION

Starting at the origin, move *A* times in the direction away from the origin a distance that is the same as the distance from the origin to *B*.	→	Multiplication ($A \cdot B = C$)

Starting at C, move toward the origin distances of length B repeatedly A times.	→	Division ($C \div B = A$)

FRACTIONS

Starting at 1, find a distance d such that by moving distance d toward the origin repeatedly n times, you will reach the origin. $1/n$ is the point-location at distance d from the origin.	→	A simple fraction ($1/n$)
Point-location reached moving from the origin a distance $1/n$ repeatedly m times.	→	A complex fraction (m/n)

As we mentioned, the Arithmetic Is Motion metaphor corresponds in many ways to the measuring stick metaphor. But there is one major difference. In all the other metaphors that we have looked at so far, including the measuring stick metaphor, there had to be some entity-creating metaphor added to get zero. However, when numbers are point-locations on a line, the origin is by its very nature a point-location. When we designate zero as the origin, it is already a point-location.

Moreover, this metaphor provides a natural extension to negative numbers—let the origin be somewhere on a pathway extending indefinitely in both directions. The negative numbers will be the point-locations on the other side of zero from the positive numbers along the same path. This extension was explicitly made by Rafael Bombelli in the second half of the sixteenth century. In Bombelli's extension of the point-location metaphor for numbers, positive numbers, zero, and negative numbers are all point-locations on a line. This made it commonplace for European mathematicians to think and speak of the concept of a number *lying between* two other numbers—as in *zero lies between minus one and one*. Conceptualizing all (real) numbers metaphorically as point-locations on the same line was crucial to providing a uniform understanding of number. These days, it is hard to imagine that there was ever a time when such a metaphor was not commonly accepted by mathematicians!

The understanding of numbers as point-locations has come into our language in the following expressions:

How *close* are these two numbers?
37 is *far away from* 189,712.

4.9 is *near* 5.
The result is *around* 40.
Count up *to* 20, without *skipping* any numbers.
Count *backward* from 20.
Count *to* 100, *starting at* 20.
Name all the numbers *from* 2 *to* 10.

The linguistic examples are important here in a number of respects. First, they illustrate how the language of motion can be recruited in a systematic way to talk about arithmetic. The conceptual mappings characterize what is systematic about this use of language. Second, these usages of language provide evidence for the existence of the conceptual mapping—evidence that comes not only from the words but also from what the words mean. The metaphors can be seen as stating generalizations not only over the use of the words but also over the inference patterns that these words supply from the source domain of motion, which are then used in reasoning about arithmetic.

We have now completed our initial description of the four basic grounding metaphors for arithmetic. Let us now turn to the relation between arithmetic and elementary algebra.

The Fundamental Metonymy of Algebra

Consider how we understand the sentence "When the pizza delivery boy comes, give him a good tip." The conceptual frame is Ordering a Pizza for Delivery. Within this frame, there is a role for the Pizza Delivery Boy, who delivers the pizza to the customer. In the situation, we do not know which *individual* will be delivering the pizza. But we need to conceptualize, make inferences about, and talk about that individual, whoever he is. Via the Role-for-Individual metonymy, the role "pizza delivery boy" comes to stand metonymically for the particular individual who fills the role—that is, who happens to deliver the pizza today. "Give him a good tip" is an instruction that applies to the individual, whoever he is.

This everyday conceptual metonymy, which exists outside mathematics, plays a major role in mathematical thinking: It allows us to go from concrete (case by case) arithmetic to general algebraic thinking. When we write "$x + 2 = 7$," x is our notation for a role, Number, standing for an individual number. "$x + 2 = 7$" says that whatever number x happens to be, adding 2 to it will yield 7.

This everyday cognitive mechanism allows us to state general laws like "$x + y = y + x$," which says that adding a number y to another number x yields the same result as adding *x* to *y*. It is this metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about.

Clear examples of how we unconsciously use and master the Fundamental Metonymy of Algebra are provided by many passages in this very chapter. In fact, every time we have written (and every time you have read and understood) an expression such as "If collection *A* is the same size as collection *B*," or "adding zero to *A* yields *A*," we have been implicitly making use of the Fundamental Metonymy of Algebra. It is this cognitive mechanism that permits general proofs in mathematics—for example, proofs about any number, whatever it is.

The Metaphorical Meanings of One and Zero

The four grounding metaphors mentioned so far—Object Collection, Object Construction, the Measuring Stick, and Motion Along a Line—contain metaphorical characterizations of zero and one. Jointly, these metaphors characterize the symbolic meanings of zero and one. In the collection metaphor, zero is the empty collection. Thus, zero can connote *emptiness*. In the object-construction metaphor, zero is either the lack of an object, the absence of an object or, as a result of an operation, the destruction of an object. Thus, zero can mean *lack*, *absence*, or *destruction*. In the measuring stick metaphor, zero stands for the *ultimate in smallness*, the lack of any physical segment at all. In the motion metaphor, zero is the origin of motion; hence, zero can designate an *origin*. Hence, zero, in everyday language, can symbolically denote emptiness, nothingness, lack, absence, destruction, ultimate smallness, and origin.

In the collection metaphor, one is the collection with a lone member and, hence, symbolizes *individuality* and *separateness* from others. In the object-construction metaphor, one is a whole number and, by virtue of this, signifies *wholeness*, *unity*, and *integrity*. In the measuring stick metaphor, one is the length specifying the unit of measure. In this case, one signifies a *standard*. And in the motion metaphor, one indicates the first step in a movement. Hence, it symbolizes a *beginning*. Taken together, these metaphors give one the symbolic values of individuality, separateness, wholeness, unity, integrity, a standard, and a beginning. Here are some examples.

- *Beginning:* One small step for a man; one great step for mankind.
- *Unity:* E pluribus unum ("From many, one").
- *Integrity:* Fred and Ginger danced as one.
- *Origin:* Let's start again from zero.

- *Emptiness:* There's zero in the refrigerator.
- *Nothingness:* I started with zero and made it to the top.
- *Destruction:* This nullifies all that we have done.
- *Lack (of ability):* That new quarterback is a big zero.

These grounding metaphors therefore explain why zero and one, which are literally numbers, have the symbolic values that they have. But that is not the real importance of these metaphors for mathematics. The real importance is that they explain how innate arithmetic gets extended systematically to give arithmetic distinctive properties that innate arithmetic does not have. Because of their importance we will give the four grounding metaphors for arithmetic a name: *the 4Gs*. We now turn to their implications.