

The Cognitive Foundations of Mathematics:

The Role of Conceptual Metaphor

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In *The Tree of Knowledge*, biologists Humberto Maturana and Francisco Varela (1987) analyze the biological foundations of human cognition. A crucial component of their arguments is a simple but profound aphorism: *Everything said is said by someone*. It follows from this that any concept, idea, belief, definition, drawing, poem, or piece of music, has to be produced by a living human being, constrained by the peculiarities of his or her body and brain. The entailment is straightforward: without living human bodies with brains, there are no ideas — and that includes mathematical ideas. This chapter deals with the structure of mathematical ideas *themselves*, and with how their *inferential organization* is provided by everyday human cognitive mechanisms such as conceptual metaphor.

The Cognitive Study of Ideas and their Inferential Organization

The approach to Mathematical Cognition we take in this chapter is relatively new, and it differs in important ways from (but is complementary to) the ones taken by many of the authors in this Handbook. In order to avoid potential misunderstandings regarding the subject matter and goals of our piece, we believe that it is important to clarify these differences right upfront. The differences reside mainly on three fundamental aspects:

- a) The level at which the subject matter, namely, mathematical cognition, is defined and studied,
- b) the scope of what is considered to be “mathematics,” and
- c) the methods used to gather knowledge about the subject matter, mathematical cognition.

Most chapters in this volume focus on performance, abilities, stereotypes, learning, belief systems, neurological, or developmental disorders, and the effect of aging, involved in some aspect of mathematical *behavior* (usually basic arithmetic tasks). For instance, some of the authors analyze the nature and locus of the most basic of brain functions and locations that give rise to extremely basic number-related behaviors like subitizing, numeration, counting, estimating, and so on. Others study the developmental dimensions involved in the learning of the number concept, others the peculiarities of number processing, and others focus their efforts in studying mathematical abilities, problem-solving and performance investigating their psychological and biological underpinnings. What is common to these studies is the following:

- a) Subject matter: Their primary concern is some aspect of the psychological, neurological, or educational reality involved in some mathematical behavior, performance, or competence of a *person*. The subject matter is defined at the level of an *individual*, or at the level of an individual’s nervous system. Mathematics *per se* is untouched. It is not the primary concern.
- b) Scope: What it is usually meant by “mathematics,” is, in general, simple arithmetic, number processing, or numerical calculation. Occasionally, it also means basic geometry or basic algebraic thinking.

- c) Method: The methods of investigation are mainly standard empirical methods used in behavioral studies in psychology, studies with neuropsychological syndromes, and computational models of numerical processing.

We, of course, celebrate this work and, building partially on their findings, move on to a radically different set of questions *about* mathematics. And here we mean the *inferential organization* of mathematics *itself*, not just performances or behaviors of individuals in some numerical domain: If mathematics does build on human ideas, how can we give a cognitive account of what *is* mathematics, with all the precision and complexities of its theorems, axioms, formal definitions, and proofs? What is the nature of what is taken to be truth (i.e., a theorem)? And, how do we get from numbers and baby arithmetic (proto-addition and subtraction up to three items) to higher forms of mathematics: full-blown arithmetic with rational and real numbers, set theory, logic, analytic geometry, trigonometry, exponentials and logarithms, calculus, complex analysis, transfinite numbers, abstract algebra, and so on?

We believe that these are questions for cognitive science — the scientific study of the mind — not for mathematics per se. We are asking, which cognitive mechanisms are used in structuring mathematical ideas? And more specifically, what cognitive mechanisms can characterize the inferential organization observed in mathematical ideas themselves? At this point we need to clarify the notion of *inferential organization*.

Consider the following two linguistic expressions: “Christmas is still *ahead* of us” and “That cold winter took place way *back* in the 60’s.” Literally, these expressions don’t make any sense. “Christmas” is not something that can physically be in front of us in any measurable or observable way, and a “cold winter” is not something that can be

physically behind us. Hundreds of thousands of these expressions, whose meaning is not literal but metaphorical, can be observed in human everyday language. A branch of cognitive science, cognitive linguistics (and more specifically, cognitive semantics), has shown that these hundreds of thousands metaphorical expressions can be modeled by a relatively small number of *conceptual metaphors* (Lakoff, 1993). A crucial component of what is modeled is, precisely, their *inferential organization*. In the previous example, although the expressions use completely different words (i.e., the former refers to a location *ahead*, and the latter to a location *behind*), they are both linguistic manifestations of a single conceptual metaphor, namely, TIME EVENTS ARE THINGS IN UNIDIMENSIONAL SPACE¹, which maps (preserving transitivity) locations in front of ego with events in the future, co-locations with ego with events in the present, and locations behind ego with events in the past (Spatial construals of time are, of course, much more complex. For details see Lakoff, 1993; Lakoff & Johnson, 1999; Núñez, 1999; For experimental psychological studies based on priming paradigms see Gentner, 2001; Boroditski, 2000; Núñez, in preparation). For the purposes of this chapter, there are two very important aspects to keep in mind:

1. At this level of the cognitive analysis, what matters is not how *single individuals* learn how to use these metaphors, or how they use them under stressful situations, or how they may lose the ability to use them after a brain injury, and so on. What matters is to characterize (i.e., to model), across hundreds of linguistic expressions, the structure of the inferences that can be drawn from them. For example, if “Christmas is ahead of us,” we can infer

¹ Following a convention in cognitive linguistics, capitals here serve to denote the name of the conceptual mapping as such. Particular instances of these mappings, called metaphorical expressions (e.g., “she has a

that New Year's Eve (which takes place later in December) is not just ahead of us, but *further away in front of us*. Similarly, if “the cold winter took place way back in the 60's,” we can infer that last winter not only is behind us, but also *much closer to us*.

2. Truth is always relative to the inferential organization of the mappings involved in the underlying conceptual metaphor. For instance, “last summer” can be conceptualized as being *behind us* as long as we operate with the conceptual metaphor TIME EVENTS ARE THINGS IN UNIDIMENSIONAL SPACE mentioned above, which determines a specific bodily orientation respect to metaphorically conceived events in time. Núñez and Sweetser (2001; in preparation) have shown, based on lexical, metaphorical, and gestural empirical evidence, that the details of that mapping are not universal. In the Aymara culture of the Andes, for instance, “last summer” is conceptualized as being in front of ego, not behind of ego, and “from now on” not as a frontwards motion but as backwards motion. As we will see, truth in mathematics also depends on the details of the underlying conceptual metaphors.

In sum, this chapter analyzes mathematical cognition from the perspective of the cognitive components of the inferential organization of mathematics itself (focusing mainly on conceptual metaphor), and not with the behavior or performance of individual subjects doing some form of mathematics. We believe that the approach we present here is not inconsistent with standard approaches in mathematical cognition. We think, however, that it is different in what concerns its subject matter, scope, and methodology:

great future in front of her”), are not written with capitals.

- a) Subject matter: The primary concern of this approach is mathematics *itself*.
The subject matter is defined at the level of the inferential organization of mathematical *ideas*. Behavior, performance, and competence of particular individuals are secondary.
- b) Scope: Arithmetic (or numerical calculations) is not given any privileged status. The goal is to study mathematics in all its manifestations, most of which are not numerical at all (e.g., topology, set theory, algebra, etc).
- c) Method: The methods of investigation used are mainly drawn from modeling in cognitive semantics. In particular we will be using a technique we called *Mathematical Idea Analysis* (Lakoff & Núñez, 2000).

As we said earlier, the approach we present here is still relatively new, and in many ways it is still going through exploratory phases. For those interested in studies involving behavior, performance, learning, brain injuries, and abilities in mathematical cognition, we believe that this approach should provide fruitful information for the elaboration of hypotheses that can be tested empirically.

The Cognitive Science *of* Mathematics

In the last fifteen years or so the field of cognitive semantics has produced many interesting findings regarding the basic mechanisms of human thought as they are manifested through language. Important discoveries such as force dynamics schemas (Talmy, 1988, 2003), frames Fillmore, 1982, 1985), prototypes of various kinds (Rosch, 1981, 1999), image schemas (Johnson, 1987; Lakoff, 1987), conceptual metaphor

(Reddy, 1979; Lakoff & Johnson, 1980/2003; Sweetser, 1990; Lakoff, 1993; Lakoff & Núñez, 1997; Núñez, 2000), conceptual metonymy (Lakoff & Johnson, 1980), and conceptual blends (Fauconnier & Turner, 1998, 2002) have provided new deep insights into the nature of human ideas. With this work in mind, the question we ask is: How are these basic mechanisms of thought (which manifest through language and gesture) used to characterize the inferential organization of mathematical ideas – ideas like exponentials, trigonometric functions, derivatives, and so on? We ask further how these mathematical ideas allow us to express in precise mathematical terms such ordinary ideas as proportion, difference, negation, change, reversal, recurrence, rotation, and even structure.

Moreover, we ask what the relation is between mathematical ideas and the symbolization of those ideas: Why do calculations mean what they do and why do they “work”? In our book, *Where Mathematics Comes From* (Lakoff & Núñez, 2000) we claim that the ensemble of those questions constitutes a new field of inquiry we called the *Cognitive Science of Mathematics* (see also Lakoff & Núñez, 1997). In the book we provide an in-depth analysis of such questions, and give preliminary answers to them. In addition, we outline the method of analysis we called *Mathematical Idea Analysis*. In short, *The Cognitive Science of Mathematics* asks foundational questions about the very nature of mathematics itself.

The present chapter can only give the barest suggestion of the answers to the questions we address in our book and a hint at how mathematical idea analysis works. Perhaps one of the most interesting findings in our research is that conceptual metaphors and conceptual blends are constitutive of the ideas of higher mathematics. In this essay

we will limit our discussion to conceptual metaphor, since this particular cognitive mechanism has been studied in depth for at least 25 years and has gathered evidence from a wide range of sources: psychological experiments (Gibbs, 1994); historical semantic change (Sweetser, 1990); spontaneous gesture (McNeill, 1992; Núñez & Sweetser, 2001); American Sign Language (Taub, 2001); child language development (C. Johnson, 1997); generalizations over polysemy (i.e., cases where the same word has multiple systematically related meanings ; Lakoff & Johnson, 1980/2003), generalizations over inference patterns (cases where source and target domains have corresponding inference patterns, Lakoff 1993); novel cases (new examples of conventional mappings, as in poetry song, advertisements, and so on; Lakoff & Turner, 1989); discourse coherence (Narayanan, 1997); cross-linguistic studies (Yu, 1998). For a thorough discussion of such evidence, see Lakoff and Johnson, 1999, Chapter 6).

In order to illustrate our arguments, we would like to consider a simple but deep example: *actual infinity*. As finite beings, we have no direct experience of infinity. Yet, via conceptual metaphor, we can extend our finite experiences metaphorically to create and conceptualize infinity as a *completed realized entity*, such as an infinite set, an infinite sequence, a point at infinity in projective geometry, an infinite sum, an infinite number, and even an infinite intersection of sets. Such cases of actual infinity are absolutely central to most of modern mathematics. It is important to bear in mind during the discussion that follows that conceptual metaphors are precisely stateable and that they preserve inferences, which is what allows them to play a central role in mathematics.

The Basic Metaphor of Infinity

At least since the time of Aristotle, there have been two concepts of infinity, *potential* infinity and *actual* infinity. Suppose you start to count: 1, 2, 3, ... and you imagine you go on indefinitely without stopping. That is an instance of potential infinity, infinity without an end. On the other hand consider the set of *all* natural numbers. No one could ever enumerate all of them; the enumeration would go on without end. Yet we conceptualize a set containing *all of them*, even though the enumeration has never and could never produce them all. That is an instance of *actual infinity* — an infinite completed thing!

In *Where Mathematics Comes From* we hypothesize that the idea of “actual infinity” in mathematics is metaphorical, that all the diverse ideas using actual infinity make use of the ultimate metaphorical *result* of a process without end. Literally, there is no such thing; if the process does not end, there can be no such “ultimate result.” But the very human mechanism of metaphor allows us to conceptualize and construct the “result” of an infinite process — in terms of the only way we have for conceptualizing the result of a process—in terms of a process that *does* have an end.

We hypothesize that all cases of actual infinity —from infinite sets to points at infinity to limits of infinite series to infinite intersections to least upper bounds—are special cases of a single general conceptual metaphor in which processes that go on indefinitely (that is, without end) are conceptualized as having an end and an ultimate result. We call this metaphor the *BASIC METAPHOR OF INFINITY* — or the BMI for short (Lakoff & Núñez, 2000. For details regarding how the BMI applies to Georg Cantor’s transfinite cardinal numbers, see Núñez, in press).

A conceptual metaphor is a cross-domain mapping (in the cognitive sense of “mapping”) from one conceptual domain to another, where inferences from the source domain are mapped to the target. The source domain of the BMI is the domain of iterative processes with end, that is, what linguists call *perfective aspect* (Comrie, 1976). That is, the source domain consists of an ordinary iterative process with an indefinite (though finite) number of iterations with a completion and resultant state. The target domain of the BMI is the domain of processes without end, that is, processes having *imperfective aspect*. In itself, without the metaphorical mapping, the target domain characterizes *potential infinity*. The effect of the BMI is to add a metaphorical completion to the process that goes on and on indefinitely, so that it is seen as *having a final result* — an infinite *thing*. This metaphorical addition is indicated in boldface in the statement of the conceptual mapping given below.

The source and target domains are alike in certain essential ways:

- Both have an initial state.
- Both have an iterative process with an unspecified number of iterations.
- Both have a resultant state after each iteration.

In the metaphor, the initial state, the iterative process, and the result after each iteration are mapped onto the corresponding elements of the target domain. But the crucial effect of the metaphor is *to add to the target domain the completion of the process and its resulting state*. It is this last part of the metaphor that allows us to conceptualize the ongoing process in terms of a completed process, and so to produce the concept of actual infinity. The following is the mapping of the BASIC METAPHOR OF INFINITY.

The Basic Metaphor of Infinity

| Source Domain Completed Iterative Processes | | Target Domain Iterative Processes That Go On and On |
|---|---|--|
| The Beginning State | → | The Beginning State |
| State resulting from the initial stage of the process. | → | State resulting from the initial stage of the process. |
| The process: From a given intermediate state, produce the next state. | → | The process: From a given intermediate state, produce the next state. |
| The intermediate result after that iteration of the process. | → | The intermediate result after that iteration of the process. |
| The Final Resultant State | → | “The Final Resultant State” (actual infinity) |
| Entailment E: The final resultant state is unique and follows every nonfinal state. | → | Entailment: The final resultant state is unique and follows every nonfinal state. |

Notice that the source domain of the metaphor has something that does not correspond to anything in the *literal* target domain, namely, a final resultant state. The inferential organization of this conceptual mapping functions so as to impose a “final resultant state” on an unending process. The literal unending process is given on the right-hand side of the top three arrows. The metaphorically imposed final resultant state (which characterizes what is unique about actual infinity) is indicated in boldface on the right side of the fifth line of the mapping.

In addition there is a crucial entailment that arises in the source domain and that is imposed by the metaphor on the target domain. In any completed process the final resultant state is *unique*. The fact that it is the *final* state of the process means that:

- There is no earlier final state; that is, there is no distinct previous state within the process that both follows the completion stage of the process yet precedes the final state of the process.
- Similarly, there is no later final state of the process; that is, there is no other state of the process that both results from the completion of the process and follows the final state of the process. Any such putative state would have to be “outside the process” as we conceptualize it.

Thus, the uniqueness of the final state of a complete process is a fact of human cognition, not a fact about some transcendental truth. That is, it follows from the way our brains and bodies allow us to conceptualize completed processes.

THE BASIC METAPHOR OF INFINITY maps this uniqueness property for final resultant states of processes onto actual infinity. Actual infinity, as characterized by *any given application of the BMI* is unique.

What results from the BMI is a metaphorical creation that does not occur literally: A process that goes on and on indefinitely and yet has a unique final resultant state, a state “at infinity.” This metaphor allows us to conceptualize “potential” unending infinity, which has neither end nor result, in terms of a familiar kind of process that has a unique result. Via the BMI, infinity is converted from an open-ended process to a specific, unique entity with a precise inferential organization. (for details see Lakoff & Núñez, 2000, Chapter 8).

In *Where Mathematics Comes From*, we dedicate several chapters to showing that a wide range of mathematical concepts use actual infinity, and that they can be precisely formulated using the BMI. The cases covered include infinite sets, points at infinity (in projective and inversive geometries), mathematical induction, infinite decimals, infinite sums, transfinite numbers, infinitesimal numbers, infinite intersections, least upper bounds, and limits of sequences of numbers. The general technique is to specify precisely what the parameters of the iterative process are. For example, in the case of the infinite set of natural numbers, we let the iterative process be: produce the next integer from a prior set of integers, and form a new set containing the new integer and the integers in the prior set. At the metaphorical final resultant state, we conceptualize the set of *all* the natural numbers. Depending on the nature of the process involved (e.g., iterative sums, iterative nesting of sets, etc.), and on the way in which the elements of the processes are parameterized, different instantiations of the BMI occur. It is important to mention that in many ways, the BMI plays a similar role to the various axioms of infinity in mathematics. The BMI and the axioms of infinity serve to “create” mathematical infinities. The main fundamental difference, however, is that the BMI is heavily constrained by findings in cognitive science (i.e., it has to be consistent with state of the art knowledge about the peculiarities of the human brain, the properties of human language, and so on), whereas the axioms in mathematics don’t have to meet the requirements of any of these empirical constraints. Axioms of infinity in mathematics are simply “made up” to assure the logical existence of mathematical infinities.

To get a sense of how conceptual metaphors work in mathematics, we will consider a relatively simple and well-know *apparent* paradox that allow us to see (1) the

metaphorical structures that constitute the most fundamental of mathematical ideas, and (2) the underlying inferential organization that make it appear as a paradox². The example, which is taken from the domain of curves and functions in the Cartesian Plane, has served as experimental material in research in mathematics education (Fischbein, Tirosh, and Hess, 1979) as well as in cognitive development (Núñez, 1993).

Curves, Functions, and Limits: The problem

There is a classical problem that involves the following mathematical construction, as given in figure 1. Start at stage 1 with a semicircle of diameter 1, extending from 0 to 1 on the X -axis of the Cartesian Plane. The perimeter of the semicircle is of length $\pi/2$. The center will be at $x = 1/2$ and the semicircle is above the X -axis.

[Insert figure 1 about here]

At stage 2, divide the diameter in half and form two semicircles extending from 0 to $1/2$ and $1/2$ to 1. The two centers will be at $x = 1/4$ and $x = 3/4$. See figure 2. The perimeter of each semicircle is $\pi/4$. The total length of both perimeters is $\pi/2$. The length of each diameter is $1/2$. The total length of both diameters is 1.

[Insert figure 2 about here]

At stage 3, divide the diameters in half again to form two more semicircles. There will now be four semicircles (see Figure 3). The centers will be at $x = 1/8$, $x = 3/8$, $x =$

² Due to constraints of space, in what follows we will only give a general characterization of the underlying conceptual metaphors without the details of the mappings involved.

$5/8$, and $x = 7/8$. The perimeter of each semicircle will be $\pi/8$. The total length of all four perimeters is $\pi/2$. The length of each diameter is $1/4$. The total length of all four diameters is 1.

Continue this process without stopping. This is an infinite process without an end. At every stage n , there will be a bumpy curve made up of 2^{n-1} semicircles, whose total length is $\pi/2$, and where all the diameters taken together constitute a segment of length 1. As n gets larger, the bumpy curve gets closer and closer to the diameter line, with the area between the bumpy curve and the diameter line getting smaller and smaller. But the length of the bumpy curve remains $\pi/2$ at all stages, while the length of the diameter line remains 1 at all stages. As n approaches infinity, the area between the bumpy curve and the diameter line approaches zero, while the lengths of the curve and the line remain constant at $\pi/2$ and 1 respectively.

What happens at $n = \infty$?

Lengths, Functions, and Sets of Points

At $n = \infty$, there is no area between the bumpy curve and the diameter line. They occupy the same place in space. Yet the bumpy curve is still of length $\pi/2$ and the diameter line is still of length 1. How is this possible? The bumpy curve and the diameter line appear to have become the same line, but with two different lengths! And as we know, a single line should have only a single length. Situations like this one provide perfect cases for the cognitive study of the inferential organization involved in mathematical conceptual systems.

A clearer statement of the problem will reveal why the apparent paradox arises. In the construction, there is an infinite sequence of curves approaching a limit. But sequences that have limits are sequences of *numbers*, as characterized by the BMI given above. How can one get from limits of sequences of *numbers* to limits of sequences of *curves*?

To do so, we will have to operate with a central metaphor developed in the late 19th Century, in which *naturally continuous* curves and lines were reconceptualized in a fundamentally different way. Up to the work of Cauchy, Weierstrass, Dedekind and others in the 19th Century, continuity predicated on holistic and dynamic entities such as “lines” or “planes” moving or extending over a background space. The work by Kepler and Euler, as well as the one by Newton and Leibniz, the inventors of Calculus in the 17th Century, built upon this notion of continuity. Euler, for instance, described (natural) continuity as “freely leading the hand.” This conception of continuity, which is the same students bring to math classes before they are exposed to calculus, changed dramatically via the introduction of new conceptual metaphors in which a space was conceived as constituted by sets of discrete elements called “points”: SPACE IS A SET OF POINTS. Incidentally, motion in this metaphorical space completely disappeared and it was replaced by strict statements involving static existential and universal quantifiers operating on discrete sets of points (for details see Núñez & Lakoff, 1998; Núñez, Edwards, & Matos, 1999; Lakoff & Núñez, 2000). Given the inferential organization of this new metaphor SPACE IS A SET OF POINTS, we can pick an appropriate sequence of functions $f_1(x), f_2(x), \dots$, and conceptualize the i^{th} bumpy curve as a set of ordered pairs of real numbers $(x, f_i(x))$ in the unit square in the Cartesian Plane. The first semi-circle

will be represented by the set of ordered pairs of real numbers $\{(x, f_1(x))\}$. Via this metaphor we are able to replace the sequence of geometric curves by a sequence of sets of ordered pairs of real numbers. In short, we have gone, via metaphor, from the geometry of natural space to a different mathematical domain consisting of sets and numbers.

Now that spaces, curves, and points have been replaced metaphorically by sets, ordered pairs, and numbers, can we use the characterization of limits of sequences of numbers, as given by the BMI. For each number x between 0 and 1, there will be a sequence of numbers y — y_1, y_2, y_3, \dots — given by the values of y in the functions $f_1(x) = y_1, f_2(x) = y_2, f_3(x) = y_3, \dots$. Each of these sequences of y -values defined for the number x will have a limit as the y 's get smaller and smaller — namely, zero. Thus, for each real number x between 0 and 1, there will be a sequence of ordered pairs $(x, y_1), (x, y_2), (x, y_3), \dots$ that converges to $(x, 0)$ (see Figure 4).

[Insert figure 4 about here]

But a subtle shift has occurred. We have replaced each bumpy curve by a bumpy-curve-set consisting of ordered pairs of numbers (x, y) , with $y = f(x)$, where x ranges over all the real numbers between 0 and 1. But what converges to a limit is not this sequence of bumpy-curve-sets. Instead, we have an infinity of convergent sequences of y -values — one from each member of the sequence of bumpy-curve-sets — for each number x between 0 and 1. The limit of each such sequence is the pair $(x, 0)$. The set of all such limits is the set of ordered pairs of numbers $\{(x, 0)\}$ where x is a real number between 0

and 1. This set of ordered pairs of numbers corresponds, via the metaphors used, to the diameter line.

But this set is a *set of limits of sequences of ordered pairs* of numbers. What we wanted was the limit of a sequence of curves, that is, the *limit of a sequence of sets of ordered pairs* of numbers. Those are very different things conceptually.

To get what we want from what we have, we must operate with a new metaphor, which we will call the LIMIT-SET METAPHOR: THE LIMIT OF A SEQUENCE OF SETS IS THE SET OF THE LIMITS OF THE SEQUENCES. Only via such a metaphor can we get the diameter line *to be* the limit of the sequence of bumpy-curve-sets.

Two conceptual metaphors have provided the necessary inferential organization:

- CURVES (AND LINES) ARE SETS OF ORDERED PAIRS OF NUMBERS, and
- THE LIMIT-SET METAPHOR.

If we operate with these two conceptual metaphors, then the sequence of bumpy curves can be reconceptualized as a sequence of bumpy-curve-sets consisting of ordered pairs of numbers. That sequence will have as its limit the set of ordered pairs $\{(x, 0)\}$, where x is between 0 and 1. This set has the *same* elements as the set of ordered pairs of numbers representing the diameter line under the metaphor CURVES (AND LINES) ARE SETS OF ORDERED PAIRS OF NUMBERS. Mathematically speaking, there is an axiom (i.e., the axiom of extensionality) that imposes the “truth” that a set is uniquely determined by its members. Via this artificially concocted axiom, the two sets become “identical.”

Cognitively speaking, however, the two sets are radically different. Here we can see that the LIMIT-SET METAPHOR is one of the sources of the apparent paradox.

What Is the Length of a Set?

In order to characterize the limit of a sequence of curves, we have had to metaphorically reconceptualize each curve as a set — a set of ordered pairs of numbers. The reason is that limits of sequences are technically defined only for numbers, not for geometric curves. But now a problem arises. What is “length” for such a set?

In physical space as we experience it every day, there are natural lengths, like the length of your arm or your foot. Hence, we have units of measurements like “one foot.” But when curves are replaced by sets, we no longer have natural lengths. Sets, literally, have no lengths. To characterize the “length” of such a set, we will need a relation between the set and a number called its “length.” In general, curves in the Cartesian Plane have all sorts of numerical properties — the area under the curve, the curvature at each point, the tangent at each point, and so on. Once geometric curves are replaced by sets, then all those properties of the curves will have to be replaced by relations between the sets and numbers.

The Length Function

The inferential organization of the length of a line segment $[a, b]$ along a number line is metaphorically provided by the absolute value of the difference between the numbers, namely, $|b - a|$. This is extended via the Pythagorean Theorem to any line segment oriented at any angle in the Cartesian Plane. Suppose its endpoints are (a_1, b_1) , and (a_2, b_2) . Its length is $\sqrt{|a_2 - a_1|^2 + |b_2 - b_1|^2}$.

What about the *length of a curve*? Choose a finite number of points along the curve (including the end points). Draw the sequence of straight lines connecting those

points. Call it a *partition* of the curve. The length of the partition is the sum of the lengths of the straight lines in the partition (see Figure 5). Think of the length of the curve via the BMI appropriately parameterized as the *least upper bound of the set of the lengths of all partitions of the curve*. This gives us a *length function* for every curve.

[Insert Figure 5 about here]

Now think of the line segments as measuring sticks. As the measuring sticks get shorter and shorter, they measure the length of the curve more and more accurately. The length of the curve is *the limit of measurements as the length of the measuring sticks approaches zero*. Let us call this the CURVE LENGTH METAPHOR.

The Sources of the Apparent Paradox

The appearance of the paradox comes from two sources with mutually inconsistent inferential organization:

1. A set of expectations about naturally continuous curves, and
2. The metaphors used to characterize curves in formal mathematics.

It should come as no surprise that our normal expectations are violated by the metaphors of formal mathematics.

Let us start with our normal expectations, that is, with the conceptual apparatus structured by the inferential organization of natural continuity.

- Length, curvature at each point, and the tangent at each point are inherent properties of a naturally continuous curve.
- Identical curves should have identical properties.

- Nearly identical curves should have nearly identical properties.
- If a sequence of curves converges to a limit curve, the sequence of properties of those curves should converge to the properties of the limit curve.

The reason we have these expectations is that we metaphorically conceptualize curves as *objects in space* and properties that are inherent to a curve as *parts of the curve*. For example, we naturally understand the curvature at a point in a curve as being *inherent to the curve*. If we think of a curve as being traced out by a point in motion (as did the brilliant mathematicians Kepler and Euler), we think of the direction of motion at each point (mathematicized as the tangent to the curve) as *inherent to the curve*. If we think of curves as objects and their inherent properties as parts of those objects, then as the curves get very close to each other, their properties should get correspondingly close. When the curves are so close that they cannot be distinguished, their properties should also be indistinguishably close.

The conceptual metaphors that characterize post 19th Century formal mathematics, when taken together, violate these expectations. Here are the relevant metaphors.

- FUNCTIONS ARE SETS OF ORDERED PAIRS OF REAL NUMBERS.
- REAL NUMBERS ARE LIMITS OF SEQUENCES OF RATIONAL NUMBERS (uses the BMI)
- CURVES (AND LINES) ARE SETS OF POINTS.
- POINTS IN THE (CARTESIAN) PLANE ARE ORDERED PAIRS OF NUMBERS
- THE LIMIT METAPHOR (uses the BMI for limits of sequences of numbers)

- THE LIMIT-SET METAPHOR (defines the limit of a sequence of curves as the set of point-by-point limits, as in Figure 4)
- PROPERTIES OF CURVES ARE FUNCTIONS
- SPATIAL DISTANCE (between points a and b on a line) IS NUMERICAL DIFFERENCE ($|b - a|$)
- THE CURVE LENGTH METAPHOR (uses the BMI)
- CLOSENESS (BETWEEN TWO CURVES) IS A NUMBER (defined by a metric, which assigns numbers to pairs of functions)

It should be clear why such metaphors violate the expectations discussed above. Curves are not physical objects, they are sets. Inherent properties are not parts, they are functions from one entity to a distinct entity. When two “curves” (sets) are “close” (have a small number assigned by a metric), there is no reason to think that their “properties” (numbers assigned to them by functions) should also be “close.”

Moreover, the LIMIT-SET METAPHOR that defines limits for curves says nothing about properties (like curvature, tangent, and length). From the perspective of the metaphors inherent in the formal mathematics, there is no reason to think that properties like curvature, tangent, and length *should* necessarily converge when the curves converge point by point.

Tangents and Length

Imagine measuring the length of a semi-circle on one of the bumpy curves, using measuring sticks that get shorter and shorter. If there were n semi-circles on that bumpy

curve, the measurements of each semi-circle would approach $\pi/2n$ as a limit. The total length, n times $\pi/2n$, is always $\pi/2$.

As the measuring sticks get shorter, they change direction and eventually approach the orientations of tangents to the curve. The CURVE LENGTH METAPHOR thus provides a link between lengths of curves and orientations of tangents, which in turn are characterized by the first derivative of the function defining the curve.

Compare the semi-circles with the diameter line. There the measuring sticks are always flat, with tangents at zero degrees. Correspondingly, the first derivative is zero at each point.

The Bumpy Curves in Function Space

A *function space* is defined by the metaphor that A FUNCTION IS A POINT IN A SPACE. The metaphor entails that there is a “distance” between the “points,” that is, the functions. By itself, that metaphor does not tell us how “close” the “points” are to one another. For this, one needs a metric, a function from pairs of functions to numbers. The numbers are understood as metaphorically measuring the “distance” between the functions.

All sorts of metrics are possible, providing that they meet three conditions on distance d : $d(a, a) = 0$, $d(a, b) = d(b, a)$, and $d(a, b) + d(b, c) \geq d(a, c)$. In the field of *functional analysis*, metrics are defined so as to reflect properties of functions. To get an idea of how this works, imagine the bumpy curves and the diameter line as being points in a space. Imagine the metric over that space as being defined in the following way.

- (1) The distance between any two functions $f(x)$ and $g(x)$ is defined as the sum of

- (i) the maximum difference in the values of the functions, plus
- (ii) the average difference in the values of the derivatives of the functions.

Formally, this is written:

$$d(f, g) = \sup_x |f(x) - g(x)| + \int_0^1 (|f'(x) - g'(x)|) dx$$

Via the metaphors CURVES ARE SETS OF POINTS and FUNCTIONS ARE ORDERED PAIRS OF REAL NUMBERS, let $g(x)$ be the diameter line and let $f(x)$ vary over the bumpy curves. As the bumpy curves get closer to the diameter line, the maximum distance (the first term of the sum) between each bumpy curve $f_i(x)$ and the diameter line $g(x)$ approaches zero. The second term of the sum does not, however, approach zero. It represents the average difference between the values of the tangents at each value (x). In the diameter line $g(x)$ the tangents are always zero, so $g'(x) = 0$ for all x . Since the tangents on each bumpy curve go through the same range of values, the average of the absolute values of the tangents will be the same for each bumpy curve. Thus term (ii) will be a non-zero constant when $g(x)$ is the diameter line and $f(x)$ is any bumpy curve.

According to this new inferential organization, we obtain the following meaningful entailment:

- Curves that are close in the Cartesian plane point-by-point, but *not* in their tangents, are not “close” in the function space defined by this metric.

In this function space, the metric given in (1) takes into account more than the difference between the values of the functions. It also considers the crucial factor that keeps the

length of the bumpy curves from converging to the length of diameter line, namely, the difference in the behavior of the tangents. In this metaphorical function space, the sequence of “bumpy curve” points do *not* get close to the diameter-line points as n approaches infinity.

In Figures 1-4, we represented the sequence of functions as curves — bumpy curves. This was a *metaphorical* representation of the functions, using the metaphors POINTS IN THE PLANE ARE ORDERED PAIRS OF NUMBERS and FUNCTIONS ARE SETS OF ORDERED PAIRS OF NUMBERS, which are part of the inferential organization of the Cartesian Plane. This spatial representation of the function gave the illusion that, as n approached infinity, the bumpy curves “approached” (came indefinitely close to) the diameter line. But this metaphorical image leads one to ignore the derivatives (the tangents) of the functions, which are crucial to the question of length. In this sense, this particular metaphorical representation of these functions in the Cartesian Plane are degenerate: they leave out crucial information. But in the function space defined by the metaphor Functions Are Points in Space and metric (1), this crucial information is included and it becomes clear that the bumpy curve functions do *not* come close to the diameter line function. There is not even the appearance of paradox here. Under this metric, curves that are close *both* point-by-point *and* in their tangents will be represented by points that are close in this metaphorical function space.

Conceptual Metaphor and Paradox

In the above discussion, we described the inferential organization involved in the conceptualization of the bumpy curves and the diameter lines as functions so that we

could use the theory of function spaces to show that the bumpy curves do not really converge to the diameter lines. However, we do not have to bring functions into the discussion at all. Suppose we just look at the curves in geometric terms as curves. Then the appearance of paradox remains, since the area under the bumpy curves does converge to zero and since the radii defining the heights of the bumpy curves also converge to zero. However, the length of the bumpy curves remains constant at $\pi/2$. The reason is that the curvatures of the semi-circles, far from converging to zero, increase without bound. Curvature is a property of the curve. A sequence of curves can converge to another curve only if all its properties also converge. The appearance of paradox arises because we are not paying attention to the nonconvergent properties.

Most people tend not pay attention to curvature and tangents in this case (Fischbein, Tirosh, Hess, 1979; Núñez, 1993). Moreover, most educated adults tend not to stop with the finite cases, but to move to the infinite case (via an inappropriate parameterization of the BMI, we hypothesize), which is where the “paradox” appears. But, as we said earlier, the BMI is a general conceptual metaphor, with an unlimited range of possible special cases. Which version you get, depends on how you characterize and parameterize the special case. If you were to try to plug curvatures or tangents into the BMI for the bumpy curves, it wouldn’t give you paradoxical inferences, because the entailment of the BMI in such a case would not give you convergence. But what is salient for most people in this example is not curvature or tangents, but rather the constant lengths on the one hand and on the other, the distance between the curves (characterized by the radii) and the area under the bumpy curve — both of which converge to zero.

To get the appearance of paradox, you have to operate with a version of the BMI highlighting the decreasing distance between the curves, while ignoring curvatures and tangents. The point here is that the general version of the BMI is cognitively real and can be applied in a way that is at odds with conventional mathematics. But there are other special cases of the BMI that are constitutive of conventional mathematics itself. The history of mathematics shows us that often these are precisely the cases developed in the field to deal with problems, paradoxes, and inconsistency. Depending on how the BMI is parameterized, one gets different results. From a cognitive perspective there is nothing strange about this. The same general metaphor may be fleshed out in different ways for different purposes — in some cases defining an aspect of mathematics, in other cases contradicting conventional mathematics.

Mathematical Idea Analysis

It should now have become clear why conceptual metaphor is central to the analysis of the inferential organization of mathematical ideas. In modern geometry, for example, space is not a medium or a background in which one locates things. A space is a set and points are not *locations* but entities that are *members* of that set and therefore *constitute* that very space. A geometric figure, like a circle or a sphere, is not an entity located *in* space, but rather a set of the points that make up the space itself. Thus, for example, consider two spheres that touch at a point. According to the inferential organization of our ordinary conceptual system, the spheres are distinct entities, and touch at a point-

location. But, in this metaphor, the spheres are two sets of points, *sharing* a point *in common*. A point that is constitutive of both spheres!

This is a simple example of how different metaphorical mathematical ideas can be from our ordinary conceptual system. This difference is often the cognitive reason of why some mathematical entities and facts are so counter-intuitive and difficult to learn. In the examples of the bumpy curve analyzed above, however, the mathematical concepts are metaphorically complex and the analysis is anything but obvious. A serious cognitive analysis of the inferential organization of metaphorical ideas is simply necessary if one is to understand the conceptual structure of mathematics itself.

In *Where Mathematics Comes From*, we take up even deeper cases of mathematical idea analysis, cases where certain aspects (but not all) of the inferential organization of everyday ideas are reconceptualized in terms of mathematical ideas, which allows for a mathematicization of everyday concepts. A simple case is the concept of *difference*, which is metaphorically conceptualized in terms of *distance* between points in a space and mathematicized in terms of the arithmetical operation of subtraction — the subtraction of one number from another, where the numbers metaphorically represent lengths of lines in space. This way of mathematicizing difference is ubiquitous not only in mathematics, but also in hundreds of disciplines applying mathematics to their subject matters, from descriptive and inferential statistics, to economics, biology, physics, psychology, and political science.

Another easy example is the concept of *change*, mathematicized in terms of derivatives. There is a general metaphor outside of mathematics that change is motion in space from one location to another. Qualities are represented conceptually as dimensions

in space, degrees of qualities as distances along these dimensions, time as a spatial dimension, and change of a quality as movement from one point in that dimension to another. Instantaneous change is then conceived of as average change of location over an infinitesimally small interval.

In our book, we give much more complex examples. Exponentiation is shown to express the inferential organization of the concept of change in proportion to size. Trigonometric functions are shown to characterize recurrence. And so on. It is via this means that we explain why mathematics can work in the sciences. Scientists ultimately categorize the phenomena they experience operating with the inferential organization of ordinary everyday concepts, like size, proportion, change, and recurrence. The inferential organization provided by the metaphor system of mathematics, as precisely formulated, allows us to mathematicize these concepts and perform calculations. Conceptual metaphors preserve inferences, and algorithmic calculations encode those inferences. Conceptual metaphors thus play a central role in permitting the calculation of predictions based on conceptual inferences.

Conclusion

Mathematics is a human enterprise. It uses the same conceptual mechanisms of thought as in other intellectual domains, which shows a remarkable optimal use of a human's limited and highly constrained biological resources. To understand the inferential organization that makes mathematics what it is, is to understand how the human mind uses everyday cognitive mechanisms in very special and sophisticated ways.

Mathematics has very unique features. It is abstract (i.e., not directly perceivable through the senses), precise, consistent, stable, calculable, generalizable, and effective as general tools for description, explanation and prediction in a vast number of everyday activities. It is the inferential organization provided by conceptual metaphor, as used to constitute mathematics, that plays a fundamental role in making all this possible.

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Figure 1

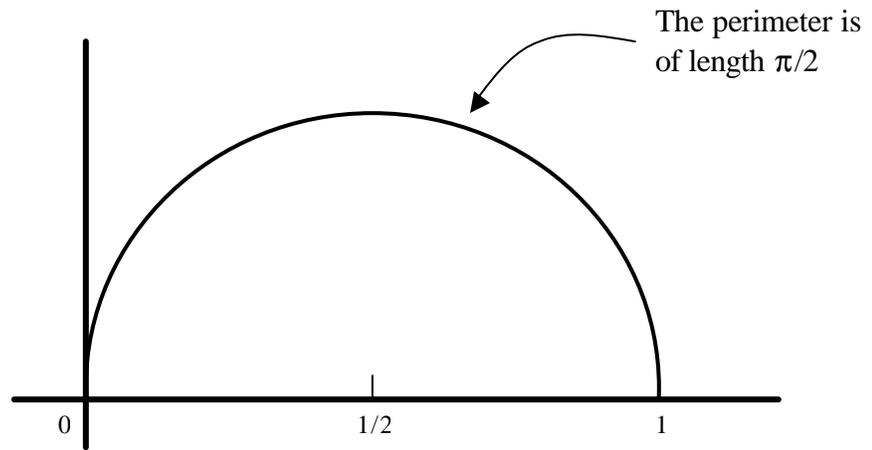


Figure 2

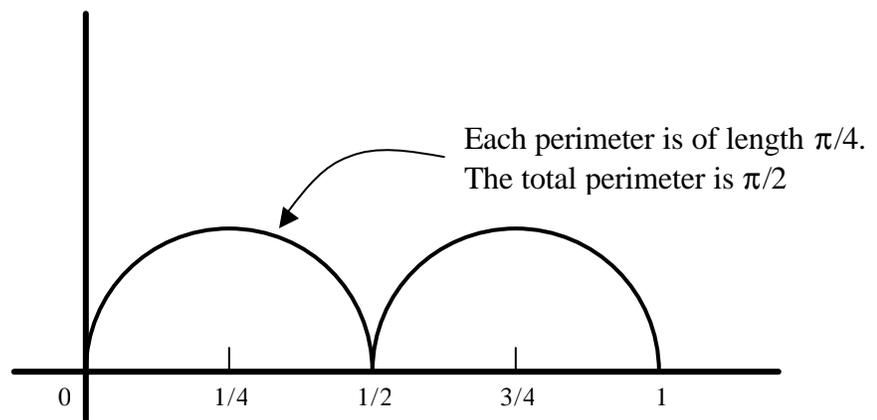


Figure 3

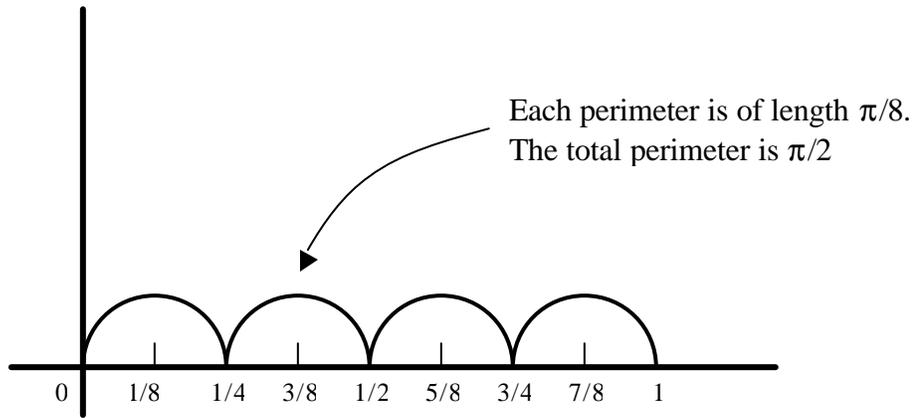


Figure 4

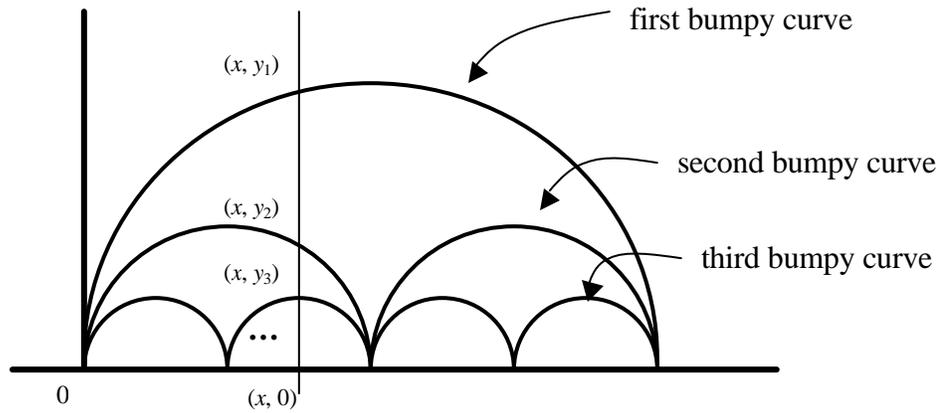


Figure 5

