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# Where Mathematics Comes From

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# Where Mathematics Comes From

## How the Embodied Mind Brings Mathematics into Being

George Lakoff
Rafael E. Núñez

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Designed by Rachel Hegarty

Library of Congress cataloging-in-publication data
Lakoff, George.
Where mathematics comes from : how the embodied mind brings mathematics into being / George Lakoff and Rafael E. Núñez.
p. cm.
Includes bibliographical references and index.
ISBN 0-465-03770-4
1. Number concept. 2. Mathematics—Psychological aspects.
3. Mathematics—Philosophy. I. Núñez, Rafael E., 1960– . II. Title.
QA141.15.L37 2000
510—dc21 00-034216
CIP

FIRST EDITION

00 01 02 03 / 10 9 8 7 6 5 4 3 2 1

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To Rafael's parents,
César Núñez and Eliana Errázuriz,

to George's wife,
Kathleen Frumkin,

and to the memory of
James D. McCawley

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# Acknowledgments

THIS BOOK WOULD NOT HAVE BEEN POSSIBLE without a lot of help, especially from mathematicians and mathematics students, mostly at the University of California at Berkeley.

Our most immediate debt is to Reuben Hersh, who has long been one of our heroes and who has supported this endeavor since its inception.

We have been helped enormously by conversations with mathematicians, especially Elwyn Berlekamp, George Bergman, Felix Browder, Martin Davis, Joseph Goguen, Herbert Jaeger, William Lawvere, Leslie Lamport, Lisa Lippincott, Giuseppe Longo, Robert Osserman, Jean Petitot, William Thurston, and Alan Weinstein. Other scholars concerned with mathematical cognition and education have made crucial contributions to our understanding, especially Paolo Boero, Stanislas Dehaene, Jean-Louis Dessalles, Laurie Edwards, Lyn English, Gilles Fauconnier, Jerry Feldman, Deborah Foster, Cathy Kessel, Jean Lassègue, Paolo Mancosu, João Filipe Matos, Srini Narayanan, Bernard Plancherel, Jean Retschitzki, Adrian Robert, and Mark Turner. In spring 1997 we gave a graduate seminar at Berkeley, called "The Metaphorical Structure of Mathematics: Toward a Cognitive Science of Mathematical Ideas," as a way of approaching the task of writing this book. We are grateful to the students and faculty in that seminar, especially Mariya Brodsky, Mireille Broucke, Karen Edwards, Ben Hansen, Ilana Horn, Michael Kleber, Manya Raman, and Lisa Webber. We would also like to thank the students in George Lakoff's course on The Mind and Mathematics in spring 2000, expecially Lydia Chen, Ralph Crowder, Anton Dochterman, Markku Hannula, Jeffrey Heer, Colin Holbrook, Amit Khetan, Richard Mapplebeck-Palmer, Matthew Rodriguez, Eric Scheel, Aaron Siegel, Rune Skoe, Chris Wilson, Grace Wang, and Jonathan Wong. Brian Denny, Scott Nailor, Shweta Narayan, and Katherine Talbert helped greatly with their detailed comments on a preliminary version of the manuscript.

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We are not under the illusion that we can thank our partners, Kathleen Frumkin and Elizabeth Beringer, sufficiently for their immense support in this enterprise, but we intend to work at it.

The great linguist and lover of mathematics James D. McCawley, of the University of Chicago, died unexpectedly before we could get him the promised draft of this book. As someone who gloried in seeing dogma overturned and who came to intellectual maturity having been taught that mathematics provided foundations for linguistics, he would have delighted in the irony of seeing arguments for the reverse.

It has been our great good fortune to be able to do this research in the magnificent, open intellectual community of the University of California at Berkeley. We would especially like to thank the Institute of Cognitive Studies and the International Computer Science Institute for their support and hospitality over many years.

Rafael Núñez would like to thank the Swiss National Science Foundation for fellowship support that made the initial years of this research possible. George Lakoff offers thanks to the university's Committee on Research for grants that helped in the preparation and editing of the manuscript.

Finally, we can think of few better places in the world to brainstorm and find sustenance than O Chame, Café Fanny, Bistro Odyssia, Café Nefeli, The Musical Offering, Café Strada, and Le Bateau Ivre.

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# Preface

We are cognitive scientists—a linguist and a psychologist—each with a long-standing passion for the beautiful ideas of mathematics. As specialists within a field that studies the nature and structure of ideas, we realized that despite the remarkable advances in cognitive science and a long tradition in philosophy and history, there was still no discipline of *mathematical idea analysis* from a cognitive perspective—no cognitive science of mathematics.

With this book, we hope to launch such a discipline.

A discipline of this sort is needed for a simple reason: Mathematics is deep, fundamental, and essential to the human experience. As such, it is crying out to be understood.

It has not been.

Mathematics is seen as the epitome of precision, manifested in the use of symbols in calculation and in formal proofs. Symbols are, of course, just symbols, not ideas. The intellectual content of mathematics lies in its ideas, not in the symbols themselves. In short, the intellectual content of mathematics does not lie where the mathematical rigor can be most easily seen—namely, in the symbols. Rather, it lies in human ideas.

But mathematics by itself does not and cannot *empirically* study human ideas; human cognition is simply not its subject matter. It is up to cognitive science and the neurosciences to do what mathematics itself cannot do—namely, apply the science of mind to human mathematical ideas. That is the purpose of this book.

One might think that the nature of mathematical ideas is a simple and obvious matter, that such ideas are just what mathematicians have consciously taken them to be. From that perspective, the commonplace formal symbols do as good a job as any at characterizing the nature and structure of those ideas. If that were true, nothing more would need to be said.

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But those of us who study the nature of concepts within cognitive science know, from research in that field, that the study of human ideas is not so simple. Human ideas are, to a large extent, grounded in sensory-motor experience. Abstract human ideas make use of precisely formulatable cognitive mechanisms such as conceptual metaphors that import modes of reasoning from sensory-motor experience. It is *always* an empirical question just what human ideas are like, mathematical or not.

The central question we ask is this: How can cognitive science bring systematic *scientific rigor* to the realm of human mathematical ideas, which lies outside the rigor of mathematics itself? Our job is to help make precise what mathematics itself cannot—the nature of mathematical ideas.

Rafael Núñez brings to this effort a background in mathematics education, the development of mathematical ideas in children, the study of mathematics in indigenous cultures around the world, and the investigation of the foundations of embodied cognition. George Lakoff is a major researcher in human conceptual systems, known for his research in natural-language semantics, his work on the embodiment of mind, and his discovery of the basic mechanisms of everyday metaphorical thought.

The general enterprise began in the early 1990s with the detailed analysis by one of Lakoff's students, Ming Ming Chiu (now a professor at the Chinese University in Hong Kong), of the basic system of metaphors used by children to comprehend and reason about arithmetic. In Switzerland, at about the same time, Núñez had begun an intellectual quest to answer these questions: How can human beings understand the idea of actual infinity—infinity conceptualized as a thing, not merely as an unending process? What is the concept of actual infinity in its mathematical manifestations—points at infinity, infinite sets, infinite decimals, infinite intersections, transfinite numbers, infinitesimals? He reasoned that since we do not encounter actual infinity directly in the world, since our conceptual systems are finite, and since we have no cognitive mechanisms to perceive infinity, there is a good possibility that metaphorical thought may be necessary for human beings to conceptualize infinity. If so, new results about the structure of metaphorical concepts might make it possible to precisely characterize the metaphors used in mathematical concepts of infinity. With a grant from the Swiss NSF, he came to Berkeley in 1993 to take up this idea with Lakoff.

We soon realized that such a question could not be answered in isolation. We would need to develop enough of the foundations of mathematical idea analysis so that the question could be asked and answered in a precise way. We would need to understand the cognitive structure not only of basic arithmetic but also of sym-

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bolic logic, the Boolean logic of classes, set theory, parts of algebra, and a fair amount of classical mathematics: analytic geometry, trigonometry, calculus, and complex numbers. That would be a task of many lifetimes. Because of other commitments, we had only a few years to work on the project—and only part-time.

So we adopted an alternative strategy. We asked, What would be the minimum background needed

- to answer Núñez's questions about infinity,
- to provide a serious beginning for a discipline of mathematical idea analysis, and
- to write a book that would engage the imaginations of the large number of people who share our passion for mathematics and want to understand what mathematical ideas are?

As a consequence, our discussion of arithmetic, set theory, logic, and algebra are just enough to set the stage for our subsequent discussions of infinity and classical mathematics. Just enough for that job, but not trivial. We seek, *from a cognitive perspective,* to provide answers to such questions as, Where do the laws of arithmetic come from? Why is there a unique empty class and why is it a subclass of all classes? Indeed, why is the empty class a class at all, if it cannot be a class *of* anything? And why, in formal logic, does every proposition follow from a contradiction? Why should anything at all follow from a contradiction?

From a cognitive perspective, these questions cannot be answered merely by giving definitions, axioms, and formal proofs. That just pushes the question one step further back: How are those definitions and axioms understood? To answer questions at this level requires an account of ideas and cognitive mechanisms. Formal definitions and axioms are *not* basic cognitive mechanisms; indeed, they themselves require an account in cognitive terms.

One might think that the best way to understand mathematical ideas would be simply to ask mathematicians what they are thinking. Indeed, many famous mathematicians, such as Descartes, Boole, Dedekind, Poincaré, Cantor, and Weyl, applied this method to themselves, introspecting about their own thoughts. Contemporary research on the mind shows that as valuable a method as this can be, it can at best tell a partial and not fully accurate story. Most of our thought and our systems of concepts are part of the cognitive unconscious (see Chapter 2). We human beings have no direct access to our deepest forms of understanding. The analytic techniques of cognitive science are necessary if we are to understand how we understand.

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One of the great findings of cognitive science is that our ideas are shaped by our bodily experiences—not in any simpleminded one-to-one way but indirectly, through the grounding of our entire conceptual system in everyday life. The cognitive perspective forces us to ask, Is the system of mathematical ideas also grounded indirectly in bodily experiences? And if so, exactly how?

The answer to questions as deep as these requires an understanding of the cognitive superstructure of a whole nexus of mathematical ideas. This book is concerned with how such cognitive superstructures are built up, starting for the most part with the commonest of physical experiences.

To make our discussion of classical mathematics tractable while still showing its depth and richness, we have limited ourselves to one profound and central question: What does Euler's classic equation, $e^{\pi i} + 1 = 0$, mean? This equation links all the major branches of classical mathematics. It is proved in introductory calculus courses. The equation itself mentions only numbers and mathematical operations on them. What is lacking, from a cognitive perspective, is an analysis of the *ideas* implicit in the equation, the *ideas* that characterize those branches of classical mathematics, the way those *ideas* are linked in the equation, and why the truth of the equation follows from those *ideas*. To demonstrate the utility of mathematical idea analysis for classical mathematics, we set out to provide an initial idea analysis for that equation that would answer all these questions. This is done in the case-study chapters at the end of the book.

To show that mathematical idea analysis has some importance for the philosophy of mathematics, we decided to apply our techniques of analysis to a pivotal moment in the history of mathematics—the arithmetization of real numbers and calculus by Dedekind and Weierstrass in 1872. These dramatic developments set the stage for the age of mathematical rigor and the Foundations of Mathematics movement. We wanted to understand exactly what ideas were involved in those developments. We found the answer to be far from obvious: The modern notion of mathematical rigor and the Foundations of Mathematics movement both rest on a sizable collection of crucial conceptual metaphors.

In addition, we wanted to see if mathematical idea analysis made any difference at all in how mathematics is understood. We discovered that it did: What is called the *real-number line* is not a line as most people understand it. What is called the *continuum* is not continuous in the ordinary sense of the term. And what are called *space-filling curves* do not fill space as we normally conceive of it. These are not mathematical discoveries but discoveries about how mathematics is conceptualized—that is, discoveries in the cognitive science of mathematics.

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Though we are not primarily concerned here with mathematics education, it *is* a secondary concern. Mathematical idea analysis, as we seek to develop it, asks what theorems *mean* and *why* they are true *on the basis of what they mean*. We believe it is important to reorient mathematics teaching more toward understanding mathematical ideas and understanding *why* theorems are true.

In addition, we see our job as helping to make mathematical ideas precise in an area that has previously been left to "intuition." Intuitions are not necessarily vague. A cognitive science of mathematics should study the precise nature of clear mathematical intuitions.

## The Romance of Mathematics

In the course of our research, we ran up against a mythology that stood in the way of developing an adequate cognitive science of mathematics. It is a kind of "romance" of mathematics, a mythology that goes something like this.

- Mathematics is abstract and disembodied—yet it is real.
- Mathematics has an objective existence, providing structure to this universe and any possible universe, independent of and transcending the existence of human beings or any beings at all.
- Human mathematics is just a part of abstract, transcendent mathematics.
- Hence, mathematical proof allows us to discover transcendent truths of the universe.
- Mathematics is part of the physical universe and provides rational structure to it. There are Fibonacci series in flowers, logarithmic spirals in snails, fractals in mountain ranges, parabolas in home runs, and $\pi$ in the spherical shape of stars and planets and bubbles.
- Mathematics even characterizes logic, and hence structures reason itself—any form of reason by any possible being.
- To learn mathematics is therefore to learn the language of nature, a mode of thought that would have to be shared by any highly intelligent beings anywhere in the universe.
- Because mathematics is disembodied and reason is a form of mathematical logic, reason itself is disembodied. Hence, machines can, in principle, think.

It is a beautiful romance—the stuff of movies like *2001*, *Contact*, and *Sphere*. It initially attracted us to mathematics.

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But the more we have applied what we know about cognitive science to understand the cognitive structure of mathematics, the more it has become clear that this romance cannot be true. Human mathematics, the only kind of mathematics that human beings know, cannot be a subspecies of an abstract, transcendent mathematics. Instead, it appears that mathematics as we know it arises from the nature of our brains and our embodied experience. As a consequence, *every* part of the romance appears to be false, for reasons that we will be discussing.

Perhaps most surprising of all, we have discovered that a great many of the most fundamental mathematical ideas are inherently metaphorical in nature:

- The *number line*, where numbers are conceptualized metaphorically as points on a line.
- Boole's *algebra of classes*, where the formation of classes of objects is conceptualized metaphorically in terms of algebraic operations and elements: plus, times, zero, one, and so on.
- *Symbolic logic*, where reasoning is conceptualized metaphorically as mathematical calculation using symbols.
- *Trigonometric functions*, where angles are conceptualized metaphorically as numbers.
- The *complex plane*, where multiplication is conceptualized metaphorically in terms of rotation.

And as we shall see, Núñez was right about the centrality of conceptual metaphor to a full understanding of infinity in mathematics. There are two infinity concepts in mathematics—one literal and one metaphorical. The literal concept ("in-finity"—lack of an end) is called "potential infinity." It is simply a process that goes on without end, like counting without stopping, extending a line segment indefinitely, or creating polygons with more and more sides. No metaphorical ideas are needed in this case. Potential infinity is a useful notion in mathematics, but the main event is elsewhere. The idea of "actual infinity," where infinity becomes a *thing*—an infinite set, a point at infinity, a transfinite number, the sum of an infinite series—is what is really important. Actual infinity is fundamentally a metaphorical idea, just as Núñez had suspected. The surprise for us was that *all* forms of actual infinity—points at infinity, infinite intersections, transfinite numbers, and so on—appear to be special cases of just one Basic Metaphor of Infinity. This is anything but obvious and will be discussed at length in the course of the book.

As we have learned more and more about the nature of human mathematical cognition, the Romance of Mathematics has dissolved before our eyes. What has

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emerged in its place is an even more beautiful picture—a picture of what mathematics really is. One of our main tasks in this book is to sketch that picture for you.

None of what we have discovered is obvious. Moreover, it requires a prior understanding of a fair amount of basic cognitive semantics and of the overall cognitive structure of mathematics. That is why we have taken the trouble to write a book of this breadth and depth. We hope you enjoy reading it as much as we have enjoyed writing it.

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