# Introduction: Why Cognitive Science Matters to Mathematics

MATHEMATICS AS WE KNOW IT HAS BEEN CREATED and used by human beings: mathematicians, physicists, computer scientists, and economists—all members of the species *Homo sapiens*. This may be an obvious fact, but it has an important consequence. Mathematics as we know it is limited and structured by the human brain and human mental capacities. The only mathematics we know or can know is a brain-and-mind-based mathematics.

As cognitive science and neuroscience have learned more about the human brain and mind, it has become clear that the brain is not a general-purpose device. The brain and body co-evolved so that the brain could make the body function optimally. Most of the brain is devoted to vision, motion, spatial understanding, interpersonal interaction, coordination, emotions, language, and everyday reasoning. Human concepts and human language are not random or arbitrary; they are highly structured and limited, because of the limits and structure of the brain, the body, and the world.

This observation immediately raises two questions:

1. Exactly what mechanisms of the human brain and mind allow human beings to formulate mathematical ideas and reason mathematically?
2. Is brain-and-mind-based mathematics all that mathematics *is*? Or is there, as Platonists have suggested, a disembodied mathematics transcending all bodies and minds and structuring the universe—this universe and every possible universe?

Question 1 asks where mathematical ideas come from and how mathematical ideas are to be analyzed from a cognitive perspective. Question 1 is a scientific question, a question to be answered by cognitive science, the interdisciplinary science of the mind. As an empirical question about the human mind and brain, it cannot be studied purely within mathematics. And as a question for empirical science, it cannot be answered by an a priori philosophy or by mathematics itself. It requires an understanding of human cognitive processes and the human brain. Cognitive science matters to mathematics because only cognitive science can answer this question.

Question 1 is what this book is mostly about. We will be asking how normal human cognitive mechanisms are employed in the creation and understanding of mathematical ideas. Accordingly, we will be developing techniques of mathematical idea analysis.

But it is Question 2 that is at the heart of the philosophy of mathematics. It is the question that most people want answered. Our answer is straightforward:

- Theorems that human beings prove are within a human mathematical conceptual system.
- All the mathematical knowledge that we have or can have is knowledge within human mathematics.
- There is no way to know whether theorems proved by human mathematicians have any objective truth, external to human beings or any other beings.

The basic form of the argument is this:

1. The question of the existence of a Platonic mathematics cannot be addressed *scientifically*. At best, it can only be a matter of faith, much like faith in a God. That is, Platonic mathematics, like God, cannot in itself be perceived or comprehended via the human body, brain, and mind. Science alone can neither prove nor disprove the existence of a Platonic mathematics, just as it cannot prove or disprove the existence of a God.
2. As with the conceptualization of God, all that is possible for human beings is an understanding of mathematics in terms of what the human brain and mind afford. The only conceptualization that we can have of mathematics is a human conceptualization. Therefore, mathematics as we know it and teach it can only be humanly created and humanly conceptualized mathematics.

3. What human mathematics is, is an empirical scientific question, not a mathematical or a priori philosophical question.
4. Therefore, it is only through cognitive science—the interdisciplinary study of mind, brain, and their relation—that we can answer the question: What is the nature of the only mathematics that human beings know or can know?
5. Therefore, if you view the nature of mathematics as a scientific question, then mathematics *is* mathematics as conceptualized by human beings using the brain's cognitive mechanisms.
6. However, you may view the nature of mathematics itself not as a scientific question but as a philosophical or religious one. The burden of scientific proof is on those who claim that an external Platonic mathematics does exist, and that theorems proved in human mathematics are objectively true, external to the existence of any beings or any conceptual systems, human or otherwise. At present there is no known way to carry out such a scientific proof in principle.

This book aspires to tell you what human mathematics, conceptualized via human brains and minds, is like. Given the present and foreseeable state of our scientific knowledge, human mathematics *is* mathematics. What human mathematical concepts are is what mathematical concepts are.

We hope that this will be of interest to you whatever your philosophical or religious beliefs about the existence of a transcendent mathematics.

There is an important part of this argument that needs further elucidation. What accounts for what the physicist Eugene Wigner has referred to as "the unreasonable effectiveness of mathematics in the natural sciences" (Wigner, 1960)? How can we make sense of the fact that scientists have been able to find or fashion forms of mathematics that accurately characterize many aspects of the physical world and even make correct predictions? It is sometimes assumed that the effectiveness of mathematics as a scientific tool shows that mathematics itself exists *in the structure of the physical universe*. This, of course, is not a scientific argument with any empirical scientific basis.

We will take this issue up in detail in Part V of the book. Our argument, in brief, will be that whatever "fit" there is between mathematics and the world occurs in the minds of scientists who have observed the world closely, learned the appropriate mathematics well (or invented it), and fit them together (often effectively) using their all-too-human minds and brains.

Finally, there is the issue of whether human mathematics is an instance of, or an approximation to, a transcendent Platonic mathematics. This position presupposes a nonscientific faith in the existence of Platonic mathematics. We will argue that even this position cannot be true. The argument rests on analyses we will give throughout this book to the effect that human mathematics makes fundamental use of conceptual metaphor in characterizing mathematical concepts. Conceptual metaphor is limited to the minds of living beings. Therefore, human mathematics (which is constituted in significant part by conceptual metaphor) cannot be a part of Platonic mathematics, which—if it existed—would be purely literal.

Our conclusions will be:

1. Human beings can have no access to a transcendent Platonic mathematics, if it exists. A belief in Platonic mathematics is therefore a matter of faith, much like religious faith. There can be no scientific evidence for or against the existence of a Platonic mathematics.
2. The only mathematics that human beings know or can know is, therefore, a *mind-based mathematics,* limited and structured by human brains and minds. The only scientific account of the nature of mathematics is therefore an account, via cognitive science, of human mind-based mathematics. Mathematical idea analysis provides such an account.
3. Mathematical idea analysis shows that human mind-based mathematics uses conceptual metaphors as part of the mathematics itself.
4. Therefore human mathematics cannot be a part of a transcendent Platonic mathematics, if such exists.

These arguments will have more weight when we have discussed in detail what human mathematical concepts are. That, as we shall see, depends upon what the human body, brain, and mind are like. A crucial point is the argument in (3)—that conceptual metaphor structures mathematics as human beings conceptualize it. Bear that in mind as you read our discussions of conceptual metaphors in mathematics.

## Recent Discoveries about the Nature of Mind

In recent years, there have been revolutionary advances in cognitive science—advances that have an important bearing on our understanding of mathematics. Perhaps the most profound of these new insights are the following:

1. *The embodiment of mind*. The detailed nature of our bodies, our brains, and our everyday functioning in the world structures human concepts and human reason. This includes mathematical concepts and mathematical reason.
2. *The cognitive unconscious*. Most thought is unconscious—not repressed in the Freudian sense but simply inaccessible to direct conscious introspection. We cannot look directly at our conceptual systems and at our low-level thought processes. This includes most mathematical thought.
3. *Metaphorical thought*. For the most part, human beings conceptualize abstract concepts in concrete terms, using ideas and modes of reasoning grounded in the sensory-motor system. The mechanism by which the abstract is comprehended in terms of the concrete is called *conceptual metaphor*. Mathematical thought also makes use of conceptual metaphor, as when we conceptualize numbers as points on a line.

This book attempts to apply these insights to the realm of mathematical ideas. That is, we will be taking mathematics as a subject matter for cognitive science and asking how mathematics is created and conceptualized, especially how it is conceptualized metaphorically.

As will become clear, it is only with these recent advances in cognitive science that a deep and grounded mathematical idea analysis becomes possible. Insights of the sort we will be giving throughout this book were not even imaginable in the days of the old cognitive science of the disembodied mind, developed in the 1960s and early 1970s. In those days, thought was taken to be the manipulation of purely abstract symbols and all concepts were seen as literal—free of all biological constraints and of discoveries about the brain. Thought, then, was taken by many to be a form of symbolic logic. As we shall see in Chapter 6, symbolic logic is itself a mathematical enterprise that requires a cognitive analysis. For a discussion of the differences between the old cognitive science and the new, see *Philosophy in the Flesh* (Lakoff & Johnson, 1999) and *Reclaiming Cognition* (Núñez & Freeman, eds., 1999).

Mathematics is one of the most profound and beautiful endeavors of the imagination that human beings have ever engaged in. Yet many of its beauties and profundities have been inaccessible to nonmathematicians, because most of the cognitive structure of mathematics has gone undescribed. Up to now, even the basic ideas of college mathematics have appeared impenetrable, mysterious, and paradoxical to many well-educated people who have approached them. We

believe that cognitive science can, in many cases, dispel the paradoxes and clear away the shrouds of mystery to reveal in full clarity the magnificence of those ideas. To do so, it must reveal how mathematics is grounded in embodied experience and how conceptual metaphors structure mathematical ideas.

Many of the confusions, enigmas, and seeming paradoxes of mathematics arise because conceptual metaphors that are part of mathematics are not recognized as metaphors but are taken as literal. When the full metaphorical character of mathematical concepts is revealed, such confusions and apparent paradoxes disappear.

But the conceptual metaphors themselves do not disappear. They cannot be analyzed away. Metaphors are an essential part of mathematical thought, not just auxiliary mechanisms used for visualization or ease of understanding. Consider the metaphor that Numbers Are Points on a Line. Numbers don't have to be conceptualized as points on a line; there are conceptions of number that are not geometric. But the number line is one of the most central concepts in all of mathematics. Analytic geometry would not exist without it, nor would trigonometry.

Or take the metaphor that Numbers Are Sets, which was central to the Foundations movement of early-twentieth-century mathematics. We don't have to conceptualize numbers as sets. Arithmetic existed for over two millennia without this metaphor—that is, without zero conceptualized as being the empty set, 1 as the set containing the empty set, 2 as the set containing 0 and 1, and so on. But if we do use this metaphor, then forms of reasoning about sets can also apply to numbers. It is only by virtue of this metaphor that the classical Foundations of Mathematics program can exist.

Conceptual metaphor is a cognitive mechanism for allowing us to reason about one kind of thing as if it were another. This means that metaphor is not simply a linguistic phenomenon, a mere figure of speech. Rather, it is a cognitive mechanism that belongs to the realm of thought. As we will see later in the book, "conceptual metaphor" has a technical meaning: It is a *grounded, inference-preserving cross-domain mapping*—a neural mechanism that allows us to use the inferential structure of one conceptual domain (say, geometry) to reason about another (say, arithmetic). Such conceptual metaphors allow us to apply what we know about one branch of mathematics in order to reason about another branch.

Conceptual metaphor makes mathematics enormously rich. But it also brings confusion and apparent paradox if the metaphors are not made clear or are taken to be literal truth. Is zero a point on a line? Or is it the empty set? Or both? Or is it just a number and neither a point nor a set? There is no one answer. Each

answer constitutes a choice of metaphor, and each choice of metaphor provides different inferences and determines a different subject matter.

Mathematics, as we shall see, layers metaphor upon metaphor. When a single mathematical idea incorporates a dozen or so metaphors, it is the job of the cognitive scientist to tease them apart so as to reveal their underlying cognitive structure.

This is a task of inherent scientific interest. But it also can have an important application in the teaching of mathematics. We believe that revealing the cognitive structure of mathematics makes mathematics much more accessible and comprehensible. Because the metaphors are based on common experiences, the mathematical ideas that use them can be understood for the most part in everyday terms.

The cognitive science of mathematics asks questions that mathematics does not, and cannot, ask about itself. How do we understand such basic concepts as infinity, zero, lines, points, and sets using our everyday conceptual apparatus? How are we to make sense of mathematical ideas that, to the novice, are paradoxical—ideas like space-filling curves, infinitesimal numbers, the point at infinity, and non-well-founded sets (i.e., sets that "contain themselves" as members)?

Consider, for example, one of the deepest equations in all of mathematics, the Euler equation, $e^{\pi i} + 1 = 0$, $e$ being the infinite decimal 2.718281828459045. . . , a far-from-obvious number that is the base for natural logarithms. This equation is regularly taught in elementary college courses. But what exactly does it mean? We are usually told that an exponential of the form $q^n$ is just the number $q$ multiplied by itself $n$ times; that is, $q \cdot q \cdot \ldots \cdot q$. This makes perfect sense for $2^5$, which would be $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which multiplies out to 32. But this definition of an exponential makes no sense for $e^{\pi i}$. There are at least three mysteries here.

1. What does it mean to multiply an infinite decimal like $e$ by itself? If you think of multiplication as an algorithmic operation, where do you start? Usually you start the process of multiplication with the last digit on the right, but there is no last digit in an infinite decimal.
2. What does it mean to multiply any number by itself $\pi$ times? $\pi$ is another infinite nonrepeating decimal. What could "$\pi$ times" for performing an operation mean?
3. And even worse, what does it mean to multiply a number by itself an imaginary ($\sqrt{-1}$) number of times?

And yet we are told that the answer is –1. The typical proof is of no help here. It proves that $e^{\pi i} + 1 = 0$ is true, but it does not tell you what $e^{\pi i}$ means! In the course of this book, we will.

In this book, unlike most other books about mathematics, we will be concerned not just with *what* is true but with what mathematical ideas *mean*, how they can be understood, and *why* they are true. We will also be concerned with the nature of mathematical truth from the perspective of a mind-based mathematics.

One of our main concerns will be the concept of infinity in its various manifestations: infinite sets, transfinite numbers, infinite series, the point at infinity, infinitesimals, and objects created by taking values of sequences "at infinity," such as space-filling curves. We will show that there is a single Basic Metaphor of Infinity that all of these are special cases of. This metaphor originates outside mathematics, but it appears to be the basis of our understanding of infinity in virtually all mathematical domains. When we understand the Basic Metaphor of Infinity, many classic mysteries disappear and the apparently incomprehensible becomes relatively easy to understand.

The results of our inquiry are, for the most part, not mathematical results but results in the cognitive science of mathematics. They are results about the human conceptual system that makes mathematical ideas possible and in which mathematics makes sense. But to a large extent they are not results reflecting the conscious thoughts of mathematicians; rather, they describe the *unconscious* conceptual system used by people who do mathematics. The results of our inquiry should not change mathematics in any way, but they may radically change the way mathematics is understood and what mathematical results are taken to mean.

Some of our findings may be startling to many readers. Here are some examples:

- Symbolic logic is not the basis of all rationality, and it is not absolutely true. It is a beautiful metaphorical system, which has some rather bizarre metaphors. It is useful for certain purposes but quite inadequate for characterizing anything like the full range of the mechanisms of human reason.
- The real numbers do not "fill" the number line. There is a mathematical subject matter, the hyperreal numbers, in which the real numbers are rather sparse on the line.
- The modern definition of *continuity* for functions, as well as the so-called *continuum*, do not use the idea of continuity as it is normally understood.

- So-called *space-filling curves* do not fill space.
- There is no absolute yes-or-no answer to whether 0.99999. . . . = 1. It will depend on the conceptual system one chooses. There is a mathematical subject matter in which 0.99999. . . . = 1, and another in which 0.99999. . . . ≠ 1.

These are not new mathematical findings but new ways of understanding well-known results. They are findings in the cognitive science of mathematics—results about the conceptual structure of mathematics and about the role of the mind in creating mathematical subject matters.

Though our research does not affect mathematical results in themselves, it does have a bearing on the understanding of mathematical results and on the claims made by many mathematicians. Our research also matters for the philosophy of mathematics. *Mind-based mathematics*, as we describe it in this book, is not consistent with any of the existing philosophies of mathematics: Platonism, intuitionism, and formalism. Nor is it consistent with recent postmodernist accounts of mathematics as a purely social construction. Based on our findings, we will be suggesting a very different approach to the philosophy of mathematics. We believe that the philosophy of mathematics should be consistent with scientific findings about the only mathematics that human beings know or can know. We will argue in Part V that the *theory of embodied mathematics*—the body of results we present in this book—determines an empirically based philosophy of mathematics, one that is coherent with the "embodied realism" discussed in Lakoff and Johnson (1999) and with "ecological naturalism" as a foundation for embodiment (Núñez, 1995, 1997).

Mathematics as we know it is human mathematics, a product of the human mind. Where does mathematics come from? It comes from us! We create it, but it is not arbitrary—not a mere historically contingent social construction. What makes mathematics nonarbitrary is that it uses the basic conceptual mechanisms of the embodied human mind as it has evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history.

By the time you finish this book, our reasons for saying this should be clear.

## The Structure of the Book

Part I is introductory. We begin in Chapter 1 with the brain's innate arithmetic—the ability to subitize (i.e., to instantly determine how many objects are in a very small collection) and do very basic addition and subtraction. We move

on in Chapter 2 to some of the basic results in cognitive science on which the remainder of the book rests. We then take up basic metaphors grounding our understanding of arithmetic (Chapter 3) and the question of where the laws of arithmetic come from (Chapter 4).

In Part II, we turn to the grounding and conceptualization of sets, logic, and forms of abstract algebra such as groups (Chapters 5, 6, and 7).

Part III deals with the concept of infinity—as fundamental a concept as there is in sophisticated mathematics. The question we ask is how finite human cognitive capacities and everyday conceptual mechanisms can give rise to the full range of mathematical notions of infinity: points at infinity, infinite sets, mathematical induction, infinite decimals, limits, transfinite numbers, infinitesimals, and so on. We argue that the concept of actual infinity is metaphorical in nature and that there is a single conceptual metaphor—the Basic Metaphor of Infinity (Chapter 8)—underlying most if not all infinite notions in mathematics (Chapters 8 through 11). We will then, in Part IV, point out the implications of this type of analysis for an understanding of the continuum (Chapter 12) and for continuity and the real numbers (Chapters 13 and 14).

At this point in the book, we take a break from our line of argumentation to address a commonly noticed apparent contradiction, which we name the Length Paradox. We call this interlude *le trou normand,* after the course in a rich French meal where a sorbet with calvados is served to refresh the palate.

We now have enough results for Part V, a discussion of an overall *theory of embodied mathematics* (Chapter 15) and a new philosophy of mathematics (Chapter 16).

To demonstrate the real power of the approach, we end the book with Part VI, a detailed case study of the equation that brings together the ideas at the heart of classical mathematics: $e^{\pi i} + 1 = 0$. To show exactly what this equation means, we have to look at the cognitive structure—especially the conceptual metaphors—underlying analytic geometry and trigonometry (Case Study 1), exponentials and logarithms (Case Study 2), imaginary numbers (Case Study 3), and the cognitive mechanisms combining them (Case Study 4).

We chose to place this case study at the end for three reasons. First, it is a detailed illustration of how the cognitive mechanisms described in the book can shed light on the structure of classical mathematics. We have placed it after our discussion of the philosophy of mathematics to provide an example to the reader of how a change in the nature of what mathematics is can lead to a new understanding of familiar mathematical results.

Second, it is in the case study that mathematical idea analysis comes to the fore. Though we will be analyzing mathematical ideas from a cognitive per-

spective throughout the book, the study of Euler's equation demonstrates the power of the analysis of ideas in mathematics, by showing how a single equation can bring an enormously rich range of ideas together—even though the equation itself contains nothing but numbers: $e$, $\pi$, $\sqrt{-1}$, 1, and 0. We will be asking throughout how mere *numbers* can express *ideas*. It is in the case study that the power of the answer to this question becomes clear.

Finally, there is an educational motive. We believe that classical mathematics can best be taught with a cognitive perspective. We believe that it is important to teach mathematical ideas and to explain why mathematical truths follow from those ideas. This case study is intended to illustrate to teachers of mathematics how this can be done.

We see our book as an early step in the development of a cognitive science of mathematics—a discipline that studies the cognitive mechanisms used in the human creation and conceptualization of mathematics. We hope you will find this discipline stimulating, challenging, and worthwhile.

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*Part I*

# The Embodiment of Basic Arithmetic

# 1

# The Brain's Innate Arithmetic

THIS BOOK ASKS A CENTRAL QUESTION: What is the cognitive structure of sophisticated mathematical ideas? What are the simplest mathematical ideas, and how do we build on them and extend them to develop complex mathematical ideas: the laws of arithmetic, set theory, logic, trigonometry, calculus, complex numbers, and various forms of infinity—transfinite numbers, infinitesimals, limits, and so on? Let us begin with the most fundamental aspects of number and arithmetic, the part we are all born with.

## Number Discrimination by Babies

The very idea that babies have mathematical capacities is startling. Mathematics is usually thought of as something inherently difficult that has to be taught with homework and exercises. Yet we come into life prepared to do at least some rudimentary form of arithmetic. Recent research has shown that babies have the following numerical abilities:

1. At three or four days, a baby can discriminate between collections of two and three items (Antell & Keating, 1983). Under certain conditions, infants can even distinguish three items from four (Strauss & Curtis, 1981; van Loosbroek & Smitsman, 1990).
2. By four and a half months, a baby "can tell" that one plus one is two and that two minus one is one (Wynn, 1992a).
3. A little later, infants "can tell" that two plus one is three and that three minus one is two (Wynn, 1995).

4. These abilities are not restricted to visual arrays. Babies can also discriminate numbers of sounds. At three or four days, a baby can discriminate between sounds of two or three syllables (Bijeljac-Babic, Bertoncini, & Mehler, 1991).
5. And at about seven months, babies can recognize the numerical equivalence between arrays of objects and drumbeats of the same number (Starkey, Spelke, & Gelman, 1990).

How do we know that babies can make these numerical distinctions? Here is one of the now-classic experimental procedures (Starkey & Cooper, 1980): Slides were projected on a screen in front of babies sitting on their mother's lap. The time a baby spent looking at each slide before turning away was carefully monitored. When the baby started looking elsewhere, a new slide appeared on the screen. At first, the slides contained two large black dots. During the trials, the baby was shown the same numbers of dots, though separated horizontally by different distances. After a while, the baby would start looking at the slides for shorter and shorter periods of time. This is technically called *habituation*; nontechnically, the baby got bored.

The slides were then changed without warning to three black dots. Immediately the baby started to stare longer, exhibiting what psychologists call a longer *fixation time*. The consistent difference of fixation times informs psychologists that the baby could tell the difference between two and three dots. The experiment was repeated with the three dots first, then the two dots. The results were the same. These experiments were first tried with babies between four and five months of age, but later it was shown that newborn babies at three or four days showed the same results (Antell & Keating, 1983). These findings have been replicated not just with dots but with slides showing objects of different shapes, sizes, and alignments (Strauss & Curtis, 1981). Such experiments suggest that the ability to distinguish small numbers is present in newborns, and thus that there is at least some innate numerical capacity.

The ability to do the simplest arithmetic was established using similar habituation techniques. Babies were tested using what, in the language of developmental psychology, is called the *violation-of-expectation paradigm*. The question asked was this: Would a baby at four and a half months expect, given the presence of one object, that the addition of one other object would result in the presence of two objects? In the experiment, one puppet is placed on a stage. The stage is then covered by a screen that pops up in front of it. Then the baby sees someone placing a second identical puppet behind the screen. Then the screen is lowered. If there are two puppets there, the baby shows no surprise;

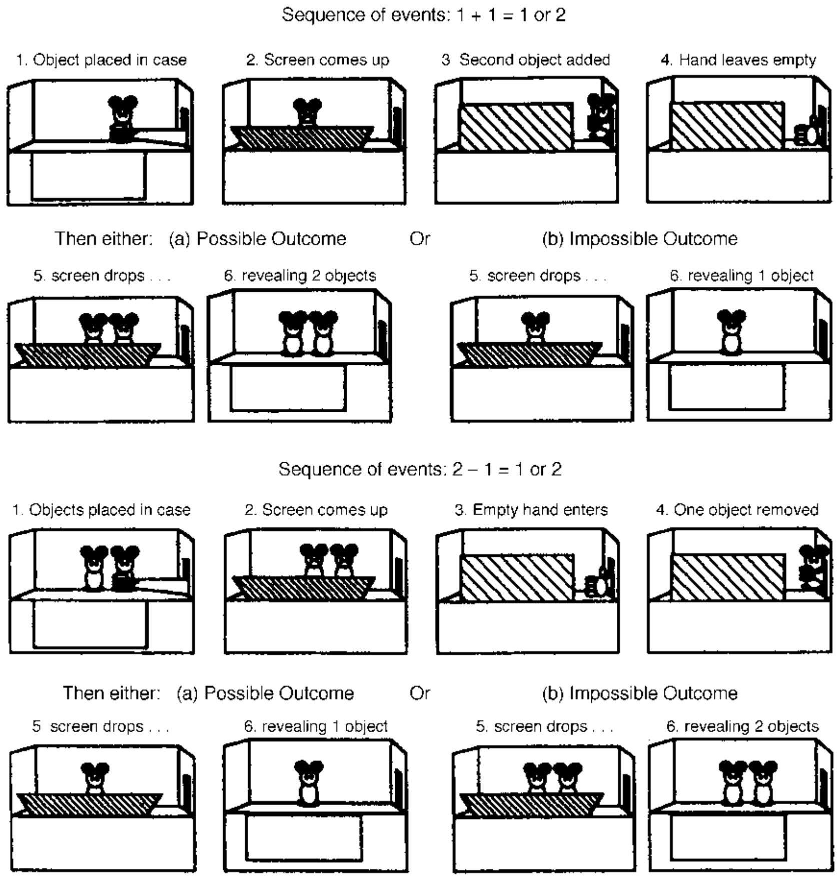


FIGURE 1.1 A usual design for studying the arithmetic capabilities of infants (Wynn, 1992a). These studies indicate that infants as young as four-and-a-half months old have a primitive form of arithmetic. They react normally to events in which 1 puppet + 1 puppet = 2 puppets or in which 2 puppets – 1 puppet = 1 puppet. But they exhibit startled reactions (e.g., systematically longer staring) for impossible outcomes in which 1 puppet + 1 puppet = 1 puppet or in which 2 puppets – 1 puppet = 2 puppets.

that is, it doesn't look at the stage any longer than otherwise. If there is only one puppet, the baby looks at the stage for a longer time. Presumably, the reason is that the baby expected two puppets, not one, to be there. Similarly, the baby stares longer at the stage if three puppets are there when the screen is lowered. The conclusion is that the baby can tell that one plus one is supposed to be two, not one or three (see Figure 1.1).

Similar experiments started with two puppets being placed on-stage, the screen popping up to cover them, and then one puppet being visibly removed

from behind the screen. The screen was then lowered. If there was only one puppet there, the babies showed no surprise; that is, they didn't look at the screen for any longer time. But if there were still two puppets on the stage after one had apparently been removed, the babies stared at the stage for a longer time. They presumably knew that two minus one is supposed to leave one, and they were surprised when it left two. Similarly, babies at six months expected that two plus one would be three and that three minus one would be two. In order to show that this was not an expectation based merely on the location of the puppets, the same experiment was replicated with puppets moving on turntables, with the same results (Koechlin, Dehaene, & Mehler, 1997). These findings suggest that babies use mechanisms more abstract than object location.

Finally, to show that this result had to do with abstract number and not particular objects, other experimenters had the puppets change to balls behind the screen. When two balls appeared instead of two puppets, four- and five-month-olds (unlike older infants) manifested no surprise, no additional staring at the stage. But when one ball or three balls appeared where two were expected, the babies did stare longer, indicating surprise (Simon, Hespos, & Rochat, 1995). The conclusion was that only number, not object identity, mattered.

In sum, newborn babies have the ability to discern the number of discrete, separate arrays of objects in space and the number of sounds produced sequentially (up to three or four). And at about five months they can distinguish correct from incorrect addition and subtraction of objects in space, for very small numbers.

The evidence that babies have these abilities is robust, but many questions remain open. What exactly are the mechanisms—neurophysiological, psychological, and others—underlying these abilities? What are the exact situational conditions under which these abilities can be confirmed experimentally? When an infant's expectations are violated in such experiments, exactly what expectation is being violated? How do these abilities relate to other developmental processes? And so on.

The experimental results to date do not give a complete picture. For example, there is no clear-cut evidence that infants have a notion of order before the age of fifteen months. If they indeed lack the concept of order before this age, this would suggest that infants can do what they do without realizing that, say, three is larger than two or that two is larger than one (Dehaene, 1997). In other words, it is conceivable that babies make the distinctions they make, but without a rudimentary concept of order. If so, when, exactly, does order emerge from the rudiments of baby arithmetic—and how?

Despite the evidence discussed, experimenters do not necessarily agree on how to answer these questions and how to interpret many of the findings. The

new field of baby arithmetic is going through the usual growing pains. (For a brief summary, see Bideaud, 1996.) But as far as the present book is concerned, what matters is that such abilities do exist at a very early age. We will refer to these abilities as *innate arithmetic*. (This term has also been used in Butterworth, 1999, p. 108.)

## Subitizing

All human beings, regardless of culture or education, can instantly tell at a glance whether there are one, two, or three objects before them. This ability is called *subitizing*, from the Latin word for "sudden." It is this ability that allows newborn babies to make the distinctions discussed above. We can subitize—that is, accurately and quickly discern the number of—up to about four objects. We cannot as quickly tell whether there are thirteen as opposed to fourteen objects, or even whether there are seven as opposed to eight. To do that takes extra time and extra cognitive operations—grouping the objects into smaller, subitizable groups and counting them. In addition to being able to subitize objects in arrays, we can subitize sequences. For example, given a sequence of knocks or beeps or flashes of light, we can accurately and quickly tell how many there are, up to five or six (Davis & Pérusse, 1988). These results are well established in experimental studies of human perception, and have been for half a century (Kaufmann, Lord, Reese, & Volkmann, 1949). Kaufmann et al. observed that subitizing was a different process from counting or estimating. Today there is a fair amount of robust evidence suggesting that the ability to subitize is inborn. A survey of the range of subitizing experiments can be found in Mandler and Shebo (1982).

The classic subitizing experiment involves reaction time and accuracy. A number of items are flashed before subjects for a fraction of a second and they have to report as fast as they can how many there are. As you vary the number of items presented (the independent variable), the reaction time (the dependent variable) is roughly about half a second (actually about 600 milliseconds) for arrays of three items. After that, with arrays of four or five items, the reaction time begins increasing linearly with the number of items presented (see Figure 1.2). Accuracy varies according to the same pattern: For arrays of three or four items, there are virtually no errors. Starting with four items, the error rate rises linearly with the number of items presented. These results hold when the objects presented are in different spatial locations. When they overlap spatially, as with concentric circles, the results no longer hold (Trick & Pylyshyn, 1993, 1994).

There is now a clear consensus that subitizing is not merely a pattern-recognition process. However, the neural mechanism by which subitizing works is

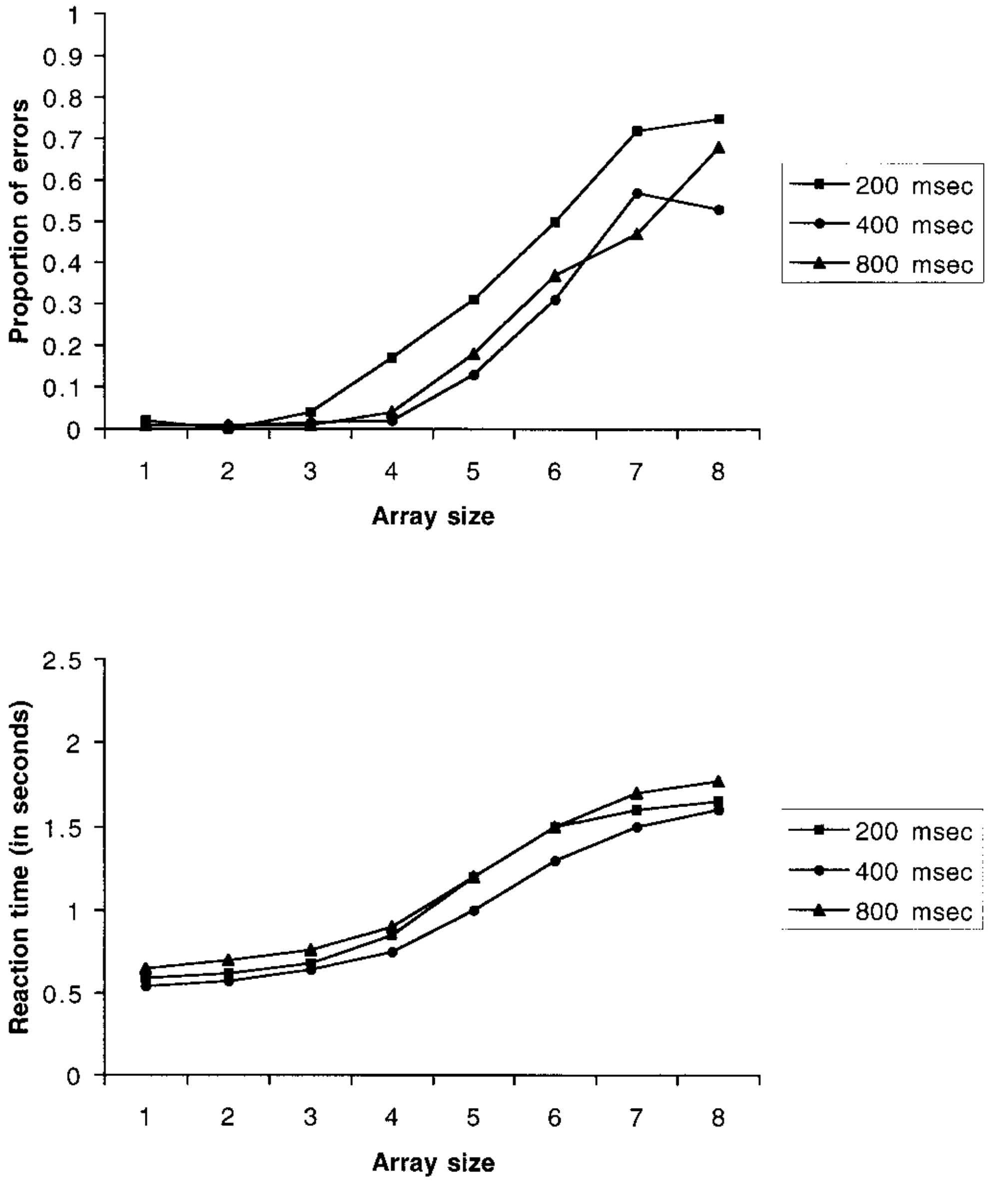


FIGURE 1.2 Fifty years ago, experimental studies established the human capacity for making quick, error-free, and precise judgments of the numerosity of small collections of items. The capacity was called *subitizing*. The figure shows results for these judgments under three experimental conditions. The levels of accuracy (top graphic) and reaction time (bottom graphic) stay stable and low for arrays of sizes of up to four items. The numbers increase dramatically for larger arrays (adapted from Mandler & Shebo, 1982).

still in dispute. Randy Gallistel and Rochel Gelman have claimed that subitizing is just very fast counting—serial processing, with visual attention placed on each item (Gelman & Gallistel, 1978). Stanislas Dehaene has hypothesized instead that subitizing is all-at-once; that is, it is accomplished via "parallel preattentive processing," which does not involve attending to each item one at a time (Dehaene, 1997). Dehaene's evidence for his position comes from patients with brain damage that prevents them from attending to things in their environment serially and therefore from counting them. They can nonetheless subitize accurately and quickly up to three items (Dehaene & Cohen, 1994).

## The Numerical Abilities of Animals

Animals have numerical abilities—not just primates but also raccoons, rats, and even parrots and pigeons. They can subitize, estimate numbers, and do the simplest addition and subtraction, just as four-and-a-half-month-old babies can. How do we know? Since we can't ask animals directly, indirect evidence must be gathered experimentally.

Experimental methods designed to explore these questions have been conceived for more than four decades (Mechner, 1958). Here is a demonstration for showing that rats can learn to perform an activity a given number of times. The task involves learning to estimate the number of times required. The rats are first deprived of food for a while. Then they are placed in a cage with two levers, which we will call *A* and *B*. Lever *B* will deliver food, but only after lever *A* has been pressed a certain fixed number of times—say, four. If the rat presses *A* the wrong number of times or not at all and then presses *B*, it is punished. The results show that rats learn to press *A* about the right number of times. If the number of times is eight, the rats learn to press a number close to that—say, seven to nine times.

To show that the relevant parameter is number and not just duration of time, experimenters conceived further manipulations: They varied the degree of food deprivation. As a result, some of the rats were very hungry and pressed the lever much faster, in order to get food quickly. But despite this, they still learned to press the lever close to the right number of times (Mechner & Guevrekian, 1962). In other series of experiments, scientists showed that rats have an ability to learn and generalize when dealing with numbers or with duration of time (Church & Meck, 1984).

Rats show even more sophisticated abilities, extending across different action and sensory modalities. They can learn to estimate numbers in association not just with motor actions, like pressing a bar, but also with the perception of tones

or light flashes. This shows that the numerical estimation capacity of rats is not limited to a specific sensory modality: It applies to number independent of modality. Indeed, modalities can be combined: Following the presentation of a sequence of, say, two tones synchronized with two light flashes (for a total of four events), the rats will systematically respond to four (Church & Meck, 1984).

Nonhuman primates display abilities that are even more sophisticated. Rhesus monkeys in the wild have arithmetic abilities similar to those of infants, as revealed by studies with the violation-of-expectation paradigm. For example, a monkey was first presented with one eggplant placed in an open box. Then a partition was placed in front of the eggplant, blocking the monkey's view. Then a second eggplant was placed in the box, in such a way that the monkey could see it being put there. The partition was then removed to reveal either one or two eggplants in the box. The monkey looked significantly longer at the "impossible" one-eggplant case, reacting even more strongly than the babies. According to the primatologists' interpretation, this was an indication that the monkey expected to see two eggplants in the box (Hauser, MacNeilage, & Ware, 1996).

In another line of research, primatologists have found that a chimpanzee can do arithmetic, combining simple physical fractions: one-quarter, one-half, and three-quarters. When one-quarter of an apple and one-half glass of a colored liquid were presented as a stimulus, the chimpanzee would choose as a response a three-quarter disc over a full disc (Woodruff & Premack, 1981).

Chimpanzees have even been taught to use numerical symbols, although the training required years of painstaking work. About twenty years ago at Kyoto University, Japanese primatologists trained chimpanzees to use arbitrary visual signs to characterize collections of objects (and digits to characterize numbers). One of their best "students," a chimpanzee named Ai, learned to report the kind, color, and numerosity of collections of objects. For instance, she would appropriately select sequences of signs, like "pencil-red-three" for a group of three red pencils and "toothbrush-blue-five" for a collection of five blue toothbrushes (Matsuzawa, 1985). Reaction-time data indicate that beyond the numbers three or four, Ai used a very humanlike serial form of counting. Recently Ai has made improvements, mastering the labeling of collections of objects up to nine, as well as ordering digits according to their numerical size (Matsuzawa, 1997).

Other researchers have shown that chimpanzees are also able to calculate using numerical symbols. For example, Sarah Boysen succeeded, through years of training, in teaching her chimpanzee, Sheba, to perform simple comparisons and additions. Sheba was progressively better able to match a collection of objects with the corresponding Arabic numeral, from 0 to 9. Sheba also learned to choose, from among several collections of objects, the one correctly matching a

given numeral. Later, using an ingenious experimental design, Boysen demonstrated that Sheba was able to mentally perform additions using symbols alone. For example, given symbols "2" and "4," Sheba would pick out the symbol "6" as a result (Boysen & Capaldi, 1993; Boysen & Berntson, 1996). Although these impressive capacities require years of training, they show that our closest relative, the chimpanzee, shares with us a nontrivial capacity for at least some innate arithmetic along with abilities that can be learned through long-term, explicit, guided training.

## The Inferior Parietal Cortex

Up to now, we have mainly discussed human numerical capacities that are nonsymbolic, where numbers of objects and events were involved, but not symbols for those numbers. Numbers are, of course, distinct from numerals, the symbols for numbers. The capacity for using numerals is more complex than that for number alone, as we shall discuss below in detail. Moreover, there are two aspects of the symbolization of number: written symbols (say, Arabic numerals like "6") and words, both spoken and written, for those symbols (say, "six"). The words and the numerals have different grammatical structure. The grammatical structure of the words is highly language-dependent, as can be seen from the English "eighty-one" versus the French "quatre-vingt-un" ("four-twenty-one"). Thus, the capacity for naming numbers involves a capacity for number plus two symbolic capacities—one for written numerals and one for characterizing the structure of the (typically complex) words for those numerals.

There is a small amount of evidence suggesting that the inferior parietal cortex is involved in symbolic numerical abilities. One bit of evidence comes from patients with *Epilepsia arithmetices,* a rare form of epileptic seizure that occurs when doing arithmetic calculations. About ten cases in the world have been studied. In each case, the electroencephalogram (EEG) showed abnormalities in the inferior parietal cortex. From the moment the patients started doing even very simple arithmetic calculations, their brain waves showed abnormal rhythmic discharges and triggered epileptic fits. Other intellectual activities, such as reading, had no ill effects. (For discussion, see Dehaene, 1997, p. 191.)

A second piece of evidence comes from Mr. M, a patient of Laurent Cohen and Stanislas Dehaene, who has a lesion in the inferior parietal cortex. Mr. M cannot tell what number comes between 3 and 5, but he can tell perfectly well what letter comes between A and C and what day comes between Tuesday and Thursday. Knowledge of number sequence has been lost, but other sequential information is unaffected.

Mr. M can correctly give names to numerals. Shown the symbol 5, he can respond "five." But he cannot do simple arithmetic. He says that two minus one makes two, that nine minus eight is seven, that three minus one makes four. He has lost the sense of the structure of integers. He also fails "bisection" tasks—deciding which number falls in a given interval. Between three and five, he places three. Between ten and twenty, he places thirty, then corrects his answer to twenty-five. Yet his rote arithmetic memory is intact. He knows the multiplication table by heart; he knows that three times nine is twenty-seven, but fails when the result of addition goes beyond ten. Asked to add 8 + 5, he cannot break 5 down into 2 + 3, to give (8 + 2) + 3, or 10 + 3. He has lost every intuition about arithmetic, but he preserves rote memory. He can perform simple rote calculations, but he does not understand them.

The inferior parietal cortex is a highly associative area, located anatomically where neural connections from vision, audition, and touch come together—a location appropriate for numerical abilities, since they are common to all sensory modalities. Lesions in this area have been shown to affect not only arithmetic but also writing, representing the fingers of the hand, and distinguishing right from left. Mr. M has all these disabilities. However, some patients have only one or another, which suggests that the inferior parietal cortex is divided into microregions associated with each.

Dehaene (1997) asks why these capacities come together in a single region. "What," he asks, "is the relationship between numbers, writing, fingers, and space?" He speculates as follows: Numbers are connected to fingers because children learn to count on their fingers. Numbers are related to writing because they are symbolized by written numerals. Numbers are related to space in various ways; subitizing, for example, requires objects to be distributed over space, and integers are conceptualized as being spread in space over a number line. And mathematical talent often correlates with spatial abilities. Thus, Dehaene reasons that, despite limited evidence at present, it makes sense to conclude that basic arithmetic abilities make major use of the inferior parietal cortex.

Other mathematical abilities appear to involve other areas of the brain. For example, the prefrontal cortex, which is involved in complex structuring—complex motor routines, plans, and so on—seems to be used in complex arithmetic calculation, though not in rote memory (say, for multiplication tables). Patients with frontal lesions have difficulty using the multiplication algorithm, adding when they should multiply, forgetting to carry over, not processing digits in the right order; in short, they are unable to carry out complex sequential operations correctly.

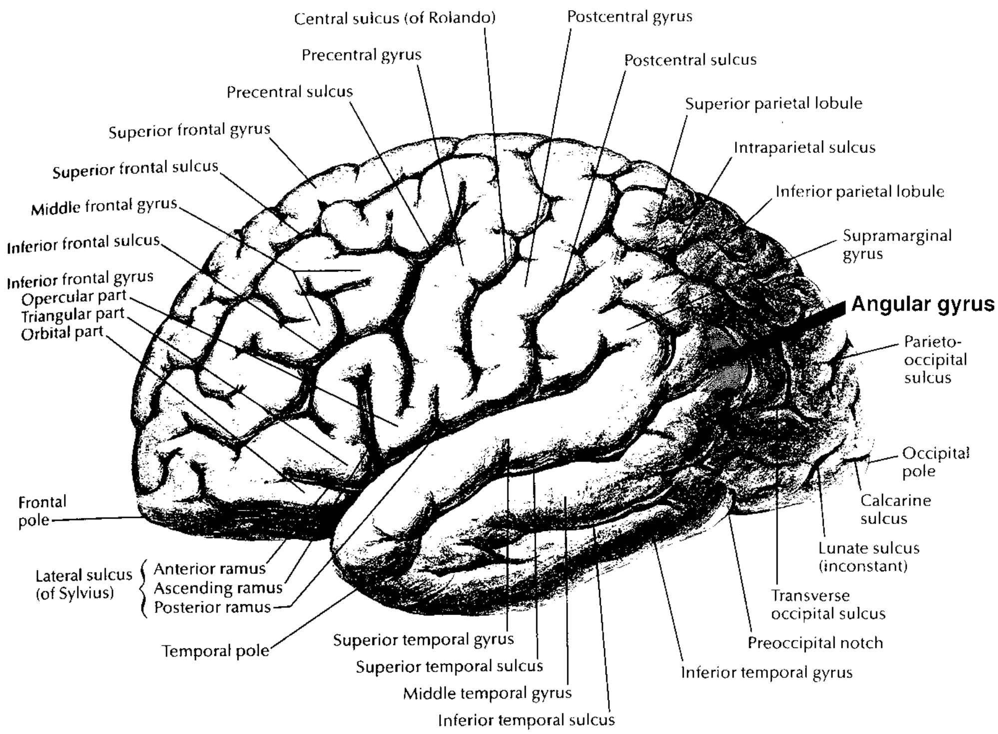


FIGURE 1.3 The left hemisphere of the human brain, showing where the angular gyrus is in the inferior parietal cortex. This is an area in which brain lesions appear to severely affect arithmetic capacities.

The capacity for basic arithmetic is separate from the capacity for rote memorization of addition and multiplication tables. Such rote abilities seem to be subcortical, associated with the basal ganglia. One patient of Cohen and Dehaene has a lesion of the basal ganglia and has lost many rote abilities: A former teacher and devout Christian, she can no longer recite the alphabet, familiar nursery rhymes, or the most common prayers. She has also lost the use of memorized addition and multiplication tables. Yet, with her inferior parietal cortex intact, she retains other nonrote arithmetic abilities. She can compare two numbers and find which number falls in between them. And although she does not remember what two times three is, she can calculate it by mentally counting three groups of two objects. She has no trouble with subtraction. This suggests that rote mathematical abilities involve the subcortical basal ganglia and cortico-subcortical loops.

Not only is rote calculation localized separately from basic arithmetic abilities but algebraic abilities are localized separately from the capacity for basic arith-

metic. Dehaene (1997) cites a patient with a Ph.D. in chemistry who has *acalculia*, the inability to do basic arithmetic. For example, he cannot solve $2 \cdot 3$, $7 - 3$, $9 \div 3$, or $5 \cdot 4$. Yet he can do abstract algebraic calculations. He can simplify $(a \cdot b) / (b \cdot a)$ into 1 and $a \cdot a \cdot a$ into $a^3$, and could recognize that $(d/c) + a$ is not generally equal to $(d + a) / (c + a)$. Dehaene concludes that algebraic calculation and arithmetic calculation are processed in different brain regions.

## From Brains to Minds and from Basic Arithmetic to Mathematics

Very basic arithmetic uses at least the following capacities: subitizing, perception of simple arithmetic relationships, the ability to estimate numerosity with close approximation (for bigger arrays), and the ability to use symbols, calculate, and memorize short tables. At present, we have some idea of what areas of the brain are active when we use such capacities, and we have some idea of which of these capacities are innate (for a general discussion, see Butterworth, 1999).

But that is not very much. First, it is not much of mathematics. In fact, when compared to the huge edifice of mathematics it is almost nothing. Second, knowing *where* is far from knowing *how*. To know what parts of the brain "light up" when certain tasks are performed is far from knowing the neural mechanism by which those tasks are performed. Identifying the parts of the brain involved is only a small, albeit crucial, part of the story.

For us, the hard question is how we go from such simple capacities to sophisticated forms of mathematics and how we employ ordinary cognitive mechanisms to do so. In the next chapter, we will introduce the reader to the basic cognitive mechanisms that are needed to begin answering these questions and that we will refer to throughout the book.

# 2

# A Brief Introduction to the Cognitive Science of the Embodied Mind

## The Cognitive Unconscious

Perhaps the most fundamental, and initially the most startling, result in cognitive science is that most of our thought is unconscious—that is, fundamentally inaccessible to our direct, conscious introspection. Most everyday thinking occurs too fast and at too low a level in the mind to be thus accessible. Most cognition happens backstage. That includes mathematical cognition.

We all have systems of concepts that we use in thinking, but we cannot consciously inspect our conceptual inventory. We all draw conclusions instantly in conversation, but we cannot consciously look at each inference and our own inference-drawing mechanisms while we are in the act of inferring on a massive scale second by second. We all speak in a language that has a grammar, but we do not consciously put sentences together word by word, checking consciously that we are following the grammatical rules of our language. To us, it seems easy: We just talk, and listen, and draw inferences without effort. But what goes on in our minds behind the scenes is enormously complex and largely unavailable to us.

Perhaps the most startling realization of all is that we have unconscious memory. The very idea of an unconscious memory seems like a contradiction in terms, since we usually think of remembering as a conscious process. Yet

hundreds of experimental studies have confirmed that we remember without being aware that we are remembering—that experiences we don't recall do in fact have a detectable and sometimes measurable effect on our behavior. (For an excellent overview, see Schacter, 1996.)

What has not been done so far is to extend the study of the cognitive unconscious to mathematical cognition—that is, the way we implicitly understand mathematics as we do it or talk about it. A large part of unconscious thought involves automatic, immediate, implicit rather than explicit understanding—making sense of things without having conscious access to the cognitive mechanisms by which you make sense of things. Ordinary everyday mathematical sense-making is not in the form of conscious proofs from axioms, nor is it always the result of explicit, conscious, goal-oriented instruction. Most of our everyday mathematical understanding takes place without our being able to explain exactly what we understood and how we understood it. Indeed, when we use the term "understanding" throughout this book, this automatic unconscious understanding is the kind of understanding we will be referring to, unless we say otherwise.

Therefore, this book is not about those areas of cognitive science concerned with conscious, goal-oriented mathematical cognition, like conscious approaches to problem solving or to constructing proofs. Though this book may have implications for those important fields, we will not discuss them here.

Our enterprise here is to study everyday mathematical understanding of this automatic unconscious sort and to ask a crucial question: How much of mathematical understanding makes use of the same kinds of conceptual mechanisms that are used in the understanding of ordinary, nonmathematical domains? Are the same cognitive mechanisms that we use to characterize ordinary ideas also used to characterize mathematical ideas?

We will argue that a great many cognitive mechanisms that are not specifically mathematical are used to characterize mathematical ideas. These include such ordinary cognitive mechanisms as those used for the following ordinary ideas: basic spatial relations, groupings, small quantities, motion, distributions of things in space, changes, bodily orientations, basic manipulations of objects (e.g., rotating and stretching), iterated actions, and so on.

To be more specific, we will suggest that:

- Conceptualizing the technical mathematical concept of a class makes use of the everyday concept of a collection of objects in a bounded region of space.
- Conceptualizing the technical mathematical concept of recursion makes use of the everyday concept of a repeated action.

- Conceptualizing the technical mathematical concept of complex arithmetic makes use of the everyday concept of rotation.
- Conceptualizing derivatives in calculus requires making use of such everyday concepts as motion, approaching a boundary, and so on.

From a nontechnical perspective, this should be obvious. But from the technical perspective of cognitive science, one must ask:

*Exactly what everyday concepts and cognitive mechanisms are used in exactly what ways in the unconscious conceptualization of technical ideas in mathematics?*

*Mathematical idea analysis*, as we will be developing it, depends crucially on the answers to this question. Mathematical ideas, as we shall see, are often grounded in everyday experience. Many mathematical ideas are ways of mathematicizing ordinary ideas, as when the idea of a derivative mathematicizes the ordinary idea of instantaneous change.

Since the cognitive science of mathematics is a new discipline, not much is known for sure right now about just how mathematical cognition works. Our job in this book is to explore how the general cognitive mechanisms used in everyday nonmathematical thought can create mathematical understanding and structure mathematical ideas.

## Ordinary Cognition and Mathematical Cognition

As we saw in the previous chapter, it appears that all human beings are born with a capacity for subitizing very small numbers of objects and events and doing the simplest arithmetic—the arithmetic of very small numbers. Moreover, if Dehaene (1997) is right, the inferior parietal cortex, especially the angular gyrus, "plays a crucial role in the mental representation of numbers as quantities" (p. 189). In other words, there appears to be a part of the brain innately specialized for a sense of quantity—what Dehaene, following Tobias Dantzig, refers to as "the number sense."

But there is a lot more to mathematics than the arithmetic of very small numbers. Trigonometry and calculus are very far from "three minus one equals two." Even realizing that zero is a number and that negative numbers are numbers took centuries of sophisticated development. Extending numbers to the rationals, the reals, the imaginaries, and the hyperreals requires an enormous cognitive apparatus and goes well beyond what babies and animals, and even a normal adult without instruction, can do. The remainder of this book will be concerned with the embodied cognitive capacities that allow one to go from in-

nate basic numerical abilities to a deep and rich understanding of, say, college-level mathematics.

From the work we have done to date, it appears that such advanced mathematical abilities are not independent of the cognitive apparatus used outside mathematics. Rather, it appears that the cognitive structure of advanced mathematics makes use of the kind of conceptual apparatus that is the stuff of ordinary everyday thought. This chapter presents prominent examples of the kinds of everyday conceptual mechanisms that are central to mathematics—especially advanced mathematics—as it is embodied in human beings. The mechanisms we will be discussing are (a) image schemas, (b) aspectual schemas, (c) conceptual metaphor, and (d) conceptual blends.

## Spatial Relations Concepts and Image Schemas

Every language has a system of spatial relations, though they differ radically from language to language. In English we have prepositions like *in, on, through, above,* and so on. Other languages have substantially different systems. However, research in cognitive linguistics has shown that spatial relations in a given language decompose into conceptual primitives called *image schemas*, and these conceptual primitives appear to be universal.

For example, the English word *on*, in the sense used in "The book is *on* the desk," is a composite of three primitive image schemas:

- The *Above schema* (the book is *above* the desk)
- The *Contact schema* (the book is *in contact with* the desk)
- The *Support schema* (the book is *supported by* the desk)

The Above schema is orientational; it specifies an orientation in space relative to the gravitational pull one feels on one's body. The Contact schema is one of a number of topological schemas; it indicates the absence of a gap. The Support schema is force-dynamic in nature; it indicates the direction and nature of a force. In general, static image schemas fall into one of these categories: orientational, topological, and force-dynamic. In other languages, the primitives combine in different ways. Not all languages have a single concept like the English *on*. Even in a language as close as German, the *on* in *on the table* is rendered as *auf*, while the *on* in *on the wall* (which does not contain the Above schema) is translated as *an*.

A common image schema of great importance in mathematics is the *Container schema*, which occurs as the central part of the meaning of words like *in* and *out*. The Container schema has three parts: an Interior, a Boundary, and an

Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides, in this case an Inside and an Outside. This structure is topological in the sense that the boundary can be made larger, smaller, or distorted and still remain the boundary of a Container schema.

To get schemas for the concepts In and Out, more must be added to the Container schema. The concept In requires that the Interior of the Container schema be "profiled"—that is, highlighted or activated in some way over the Exterior and Boundary. In addition, a figure/ground distinction must be added. For example, in a sentence like "The car is in the garage," the garage is the ground; that is, it is the landmark relative to which the car (the figure) is located. In cognitive linguistics, the ground in an image schema is called the *Landmark*, and the figure is called the *Trajector*. Thus, the In schema has the structure:

- Container schema, with Interior, Boundary, and Exterior
- Profiled: the Interior
- Landmark: the Interior

Image schemas have a special cognitive function: They are both *per*ceptual and *con*ceptual in nature. As such, they provide a bridge between language and reasoning on the one hand and vision on the other. Image schemas can fit visual perception, as when we see the milk as being *in* the glass. They can also be imposed on visual scenes, as when we see the bees swarming *in* the garden, where there is no physical container that the bees are in. Because spatial-relations terms in a given language name complex image schemas, image schemas are the link between language and spatial perception.

In addition, complex image schemas like *In* have built-in spatial "logics" by virtue of their image-schematic structures. Figure 2.1 illustrates the spatial logic built into the Container schema. In connection with this figure, consider the following two statements:

1. Given two Container schemas *A* and *B* and an object *X*, if *A* is *in B* and *X* is *in A*, then *X* is *in B*.
2. Given two Container schemas *A* and *B* and an object *Y*, if *A* is *in B* and *Y* is *outside of B*, then *Y* is *outside of A*.

We don't have to perform deductive operations to draw these conclusions. They are self-evident simply from the images in Figure 2.1. Because image

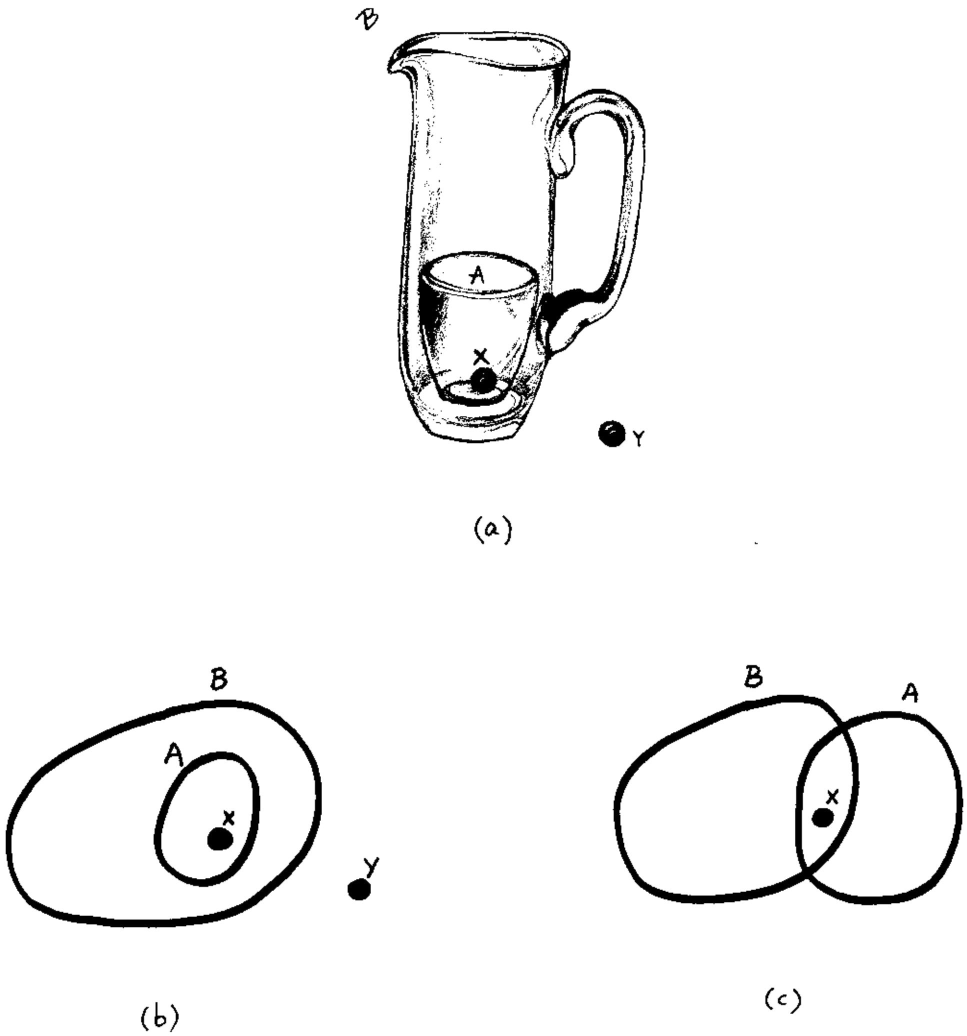


FIGURE 2.1 The logic of cognitive Container schemas. In (a), one cognitive Container schema, *A*, occurs inside another, *B*. By inspection, one can see that if *X* is in *A*, then *X* is in *B*. Similarly, if *Y* is outside *B*, then *Y* is outside *A*. We conceptualize physical containers in terms of cognitive containers, as shown in (b), which has the same logic as (a). However, conceptual containers, being part of the mind, can do what physical containers usually cannot—namely, form intersections, as in (c). In that case, an imagined entity *X* can be in two Container schemas *A* and *B* at once. Cognitive Container schemas are used not only in perception and imagination but also in conceptualization, as when we conceptualize bees as swarming *in* the garden. Container schemas are the cognitive structures that allow us to make sense of familiar Venn diagrams (see Figure 2.4).

schemas have spatial logics built into their imagistic structure, they can function as spatial concepts and be used directly in spatial reasoning. Reasoning about space seems to be done directly in spatial terms, using image schemas rather than symbols, as in mathematical proofs and deductions in symbolic logic.

Ideas do not float abstractly in the world. Ideas can be created only by, and instantiated only in, brains. Particular ideas have to be generated by neural structures in brains, and in order for that to happen, exactly the right kind of neural processes must take place in the brain's neural circuitry. Given that image schemas are conceptual in nature—that is, they constitute ideas with a structure of a very special kind—they must arise through neural circuitry of a very special kind.

Terry Regier (1996) has used the techniques of structured connectionism to build a computational neural model of a number of image schemas, as part of a neural simulation of the learning of spatial-relations terms in various languages. The research involved in Regier's simulation makes certain things clear. First, topographic maps of the visual field are needed in order to link cognition to vision. Second, a visual "filling-in" mechanism (Ramachandran & Gregory, 1991), in which activation spreads from outside to inside in a map of the visual field, will, in combination with other neural structures required, yield the topological properties of the Container schema. Third, orientation-sensitive cell assemblies found in the visual cortex are employed by orientational schemas. Fourth, map comparisons, requiring neural connections across maps, are needed. Such map-comparison structures are the locus of the relationship between the Trajector and the Landmark. Whatever changes are made in future models of spatial-relations concepts, it appears that at least these features will be needed.

Here is the importance of this for embodied mathematics: The concept of containment is central to much of mathematics. Closed sets of points are conceptualized as containers, as are bounded intervals, geometric figures, and so on. The concept of orientation is equally central. It is used in notions like angles, direction of change (tangents to a curve), rotations, and so on. The concepts of containment and orientation are not special to mathematics but are used in thought and language generally. Like any other concepts, these arise only via neural mechanisms in the right kind of neural circuitry. It is of special interest that the neural circuitry we have evolved for other purposes is an inherent part of mathematics, which suggests that embodied mathematics does not exist independently of other embodied concepts used in everyday life. Instead, mathematics makes use of our adaptive capacities—our ability to adapt other cognitive mechanisms for mathematical purposes.

Incidentally, the visual system of the brain, where such neural mechanisms as orientational cell assemblies reside, is not restricted to vision. It is also the

locus of mental imagery. Mental imagery experiments, using fMRI techniques, have shown that much of the visual system, down to the primary visual cortex, is active when we create mental imagery without visual input. The brain's visual system is also active when we dream (Hobson, 1988, 1994). Moreover, congenitally blind people, most of whom have the visual system of the brain intact, can perform visual imagery experiments perfectly well, with basically the same results as sighted subjects, though a bit slower (Marmor & Zaback, 1976; Carpenter & Eisenberg, 1978; Zimler & Keenan, 1983; Kerr, 1983). In short, one should not think of the visual system as operating purely on visual input. Thus, it makes neurological sense that structures in the visual system can be used for conceptual purposes, even by the congenitally blind.

Moreover, the visual system is linked to the motor system, via the prefrontal cortex (Rizzolatti, Fadiga, Gallese, & Fogassi, 1996; Gallese, Fadiga, Fogassi, & Rizzolatti, 1996). Via this connection, motor schemas can be used to trace out image schemas with the hands and other parts of the body. For example, you can use your hands to trace out a seen or imagined container, and correspondingly you can visualize the structure of something whose shape you trace out with your hands in the dark. Thus, congenitally blind people can get "visual" image-schematic information from touch. Image schemas are kinesthetic, going beyond mere seeing alone, even though they use neural structures in the visual system. They can serve general conceptual purposes and are especially well suited for a role in mathematical thought.

There are many image schemas that characterize concepts important for mathematics: centrality, contact, closeness, balance, straightness, and many, many more. Image schemas and their logics are essential to mathematical reasoning.

## Motor Control and Mathematical Ideas

One might think that nothing could be further from mathematical ideas than motor control, the neural system that governs how we move our bodies. But certain recent discoveries about the relation between motor control and the human conceptual system suggest that our neural motor-control systems may be centrally involved in mathematical thought. Those discoveries have been made in the field of structured connectionist neural modeling.

Building on work by David Bailey (1997), Srini Narayanan (1997) has observed that neural motor-control programs all have the same superstructure:

- *Readiness:* Before you can perform a bodily action, certain conditions of readiness have to be met (e.g., you may have to reorient your body, stop doing something else, rest for a moment, and so on).

- *Starting up:* You have to do whatever is involved in beginning the process (e.g., to lift a cup, you first have to reach for it and grasp it).
- *The main process:* Then you begin the main process.
- *Possible interruption and resumption:* While you engage in the main process, you have an option to stop, and if you do stop, you may or may not resume.
- *Iteration or continuing:* When you have done the main process, you can repeat or continue it.
- *Purpose:* If the action was done to achieve some purpose, you check to see if you have succeeded.
- *Completion:* You then do what is needed to complete the action.
- *Final state:* At this point, you are in the final state, where there are results and consequences of the action.

This might look superficially like a flow diagram used in classical computer science. But Narayanan's model of motor-control systems differs in many significant respects: It operates in real time, is highly resource- and context-dependent, has no central controller or clock, and can operate concurrently with other processes, accepting information from them and providing information to them. According to the model, these are all necessary properties for the smooth function of a neural motor-control system.

One might think the motor-control system would have nothing whatever to do with concepts, especially abstract concepts of the sort expressed in the grammars of languages around the world. But Narayanan has observed that this general motor-control schema has the same structure as what linguists have called *aspect*—the general structuring of events. Everything that we perceive or think of as an action or event is conceptualized as having that structure. We reason about events and actions in general using such a structure. And languages throughout the world all have means of encoding such a structure in their grammars. What Narayanan's work tells us is that *the same neural structure used in the control of complex motor schemas can also be used to reason about events and actions* (Narayanan, 1997).

We will call such a structure an *Aspect schema*.

One of the most remarkable of Narayanan's results is that exactly the same general neural control system modeled in his work can carry out a complex bodily movement when providing input to muscles, or carry out a rational inference when input to the muscles is inhibited. What this means is that neural control systems for bodily motions have the same characteristics needed for rational inference in the domain of aspect—that is, the structure of events.

Among the logical entailments of the aspectual system are two inferential patterns important for mathematics:

- The stage characterizing the completion of a process is further along relative to the process than any stage within the process itself.
- There is no point in a process further along than the completion stage of that process.

These fairly obvious inferences, as we shall see in Chapter 8 on infinity, take on considerable importance for mathematics.

Verbs in the languages of the world have inherent aspectual structure, which can be modified by various syntactic and morphological means. What is called *imperfective aspect* focuses on the internal structure of the main process. *Perfective aspect* conceptualizes the event as a whole, not looking at the internal structure of the process, and typically focusing on the completion of the action. Some verbs are inherently imperfective, like *breathe* or *live*. The iterative activity of breathing and the continuous activity of living—as we conceptualize them—do not have completions that are part of the concept. Just as the neural motor-control mechanism governing breathing does not have a completion (stopping, as in holding one's breath, is quite different from completion), so the concept is without a notion of completion. Death follows living but is *not* the *completion* of living, at least in our culture. Death is conceptualized, rather, as the cutting-off of life, as when a child is killed in an auto accident: Death follows life, but life is not completed. And you can say, "I have lived" without meaning that your life has been completed. Thus, an inherently imperfective concept is one that is conceptualized as being open-ended—as not having a completion.

There are two ways in which processes that have completions can be conceptualized: The completion may be either (1) internal to the process or (2) external to the process. This is not a matter of how the natural world really works but of how we conceptualize it and structure it through language. Take an example of case 1: If you jump, there are stages of jumping—namely, taking off, moving through the air, and landing. Landing completes the process of jumping. The completion, landing, is conceptualized as part of the jumping, as *internal* to what "jump" means. There is a minimally contrasting case that exemplifies case 2: flying. In the everyday concept of flying, as with birds and planes, landing is part of the conceptual frame. Landing follows flying and is a completion of flying. But *landing* is not conceptualized as part of *flying*. *Landing* is a completion of *flying* but it is *external* to *flying*. The distinction between an internal completion, as in *jump,* and an external completion, as in *fly*, is crucial in aspect.

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Aspectual ideas occur throughout mathematics. A rotation through a certain number of degrees, for example, is conceptualized as a process with a starting point and an ending point. The original notion of continuity for a function was conceptualized in terms of a continuous process of motion—one without intermediate ending points. The very idea of an algorithmic process of calculation involves a starting point, a process that may or may not be iterative, and a well-defined completion. As we shall see in Chapters 8 through 11, all notions of infinity and infinitesimals use aspectual concepts.

## The Source-Path-Goal Schema

Every language includes ways of expressing spatial sources (e.g., "from") and goals (e.g., "to," "toward") and paths intermediate between them (e.g., "along," "through," "across"). These notions do not occur isolated from one another but, rather, are part of a larger whole, the *Source-Path-Goal schema*. It is the principal image schema concerned with motion, and it has the following elements (or *roles*):

- A trajector that moves
- A source location (the starting point)
- A goal—that is, an intended destination of the trajector
- A route from the source to the goal
- The actual trajectory of motion
- The position of the trajector at a given time
- The direction of the trajector at that time
- The actual final location of the trajector, which may or may not be the intended destination.

Extensions of this schema are possible: the speed of motion, the trail left by the thing moving, obstacles to motion, forces that move one along a trajectory, additional trajectors, and so on.

This schema is topological in the sense that a path can be expanded or shrunk or deformed and still remain a path. As in the case of the Container schema, we can form spatial relations from this schema by the addition of profiling and a Trajector-Landmark relation. The concept expressed by *to* profiles the goal and identifies it as the landmark relative to which the motion takes place. The concept expressed by *from* profiles the source, taking the source as the landmark relative to which the motion takes place.

The Source-Path-Goal schema also has an internal spatial logic and built-in inferences (see Figure 2.2):

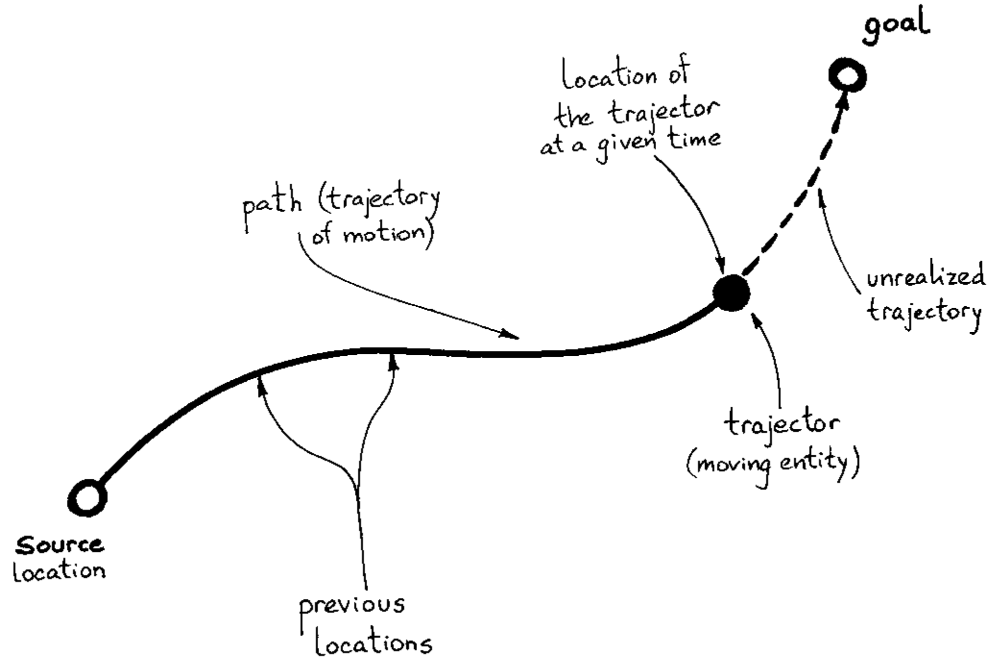


FIGURE 2.2 The Source-Path-Goal schema. We conceptualize linear motion using a conceptual schema in which there is a moving entity (called a trajector), a source of motion, a trajectory of motion (called a path), and a goal with an unrealized trajectory approaching that goal. There is a logic inherent in the structure of the schema. For example, if you are at a given location on a path, you have been at all previous locations on that path.

- If you have traversed a route to a current location, you have been at all previous locations on that route.
- If you travel from $A$ to $B$ and from $B$ to $C$, then you have traveled from $A$ to $C$.
- If there is a direct route from $A$ to $B$ and you are moving along that route toward $B$, then you will keep getting closer to $B$.
- If $X$ and $Y$ are traveling along a direct route from $A$ to $B$ and $X$ passes $Y$, then $X$ is further from $A$ and closer to $B$ than $Y$ is.

The Source-Path-Goal schema is ubiquitous in mathematical thought. The very notion of a directed graph (see Chapter 7), for example, is an instance of the Source-Path-Goal schema. Functions in the Cartesian plane are often conceptualized in terms of motion along a path—as when a function is described as "going up," "reaching" a maximum, and "going down" again.

One of the most important manifestations of the Source-Path-Goal schema in natural language is what Len Talmy (1996, 2000) has called *fictive motion*. In

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one form of fictive motion, a line is thought of in terms of motion tracing that line, as in sentences like "The road *runs* through the woods" or "The fence *goes* up the hill." In mathematics, this occurs when we think of two lines "*meeting* at a point" or the graph of a function as "*reaching* a minimum at zero."

## Conceptual Composition

Since image schemas are conceptual in nature, they can form complex composites. For example, the word "into" has a meaning—the *Into schema*—that is the composite of an *In schema* and a *To schema.* The meaning of "out of" is the composite of an *Out schema* and a *From schema.* These are illustrated in Figure 2.3. Formally, they can be represented in terms of correspondences between elements of the schemas that are part of the composite.

The following notations indicate composite structures.

The Into schema

- The In schema: A Container schema, with the Interior profiled and taken as Landmark
- The To schema: A Source-Path-Goal schema, with the Goal profiled and taken as Landmark
- Correspondences: (Interior; Goal) and (Exterior; Source)

The Out-of schema

- The Out schema: A Container schema, with the Exterior profiled and taken as Landmark
- The From schema: A Source-Path-Goal schema, with the Source profiled and taken as Landmark
- Correspondences: (Interior; Source) and (Exterior; Goal)

## Conceptual Metaphor

Metaphor, long thought to be just a figure of speech, has recently been shown to be a central process in everyday thought. Metaphor is not a mere embellishment; it is the basic means by which abstract thought is made possible. One of the principal results in cognitive science is that abstract concepts are typically understood, via metaphor, in terms of more concrete concepts. This phenome-

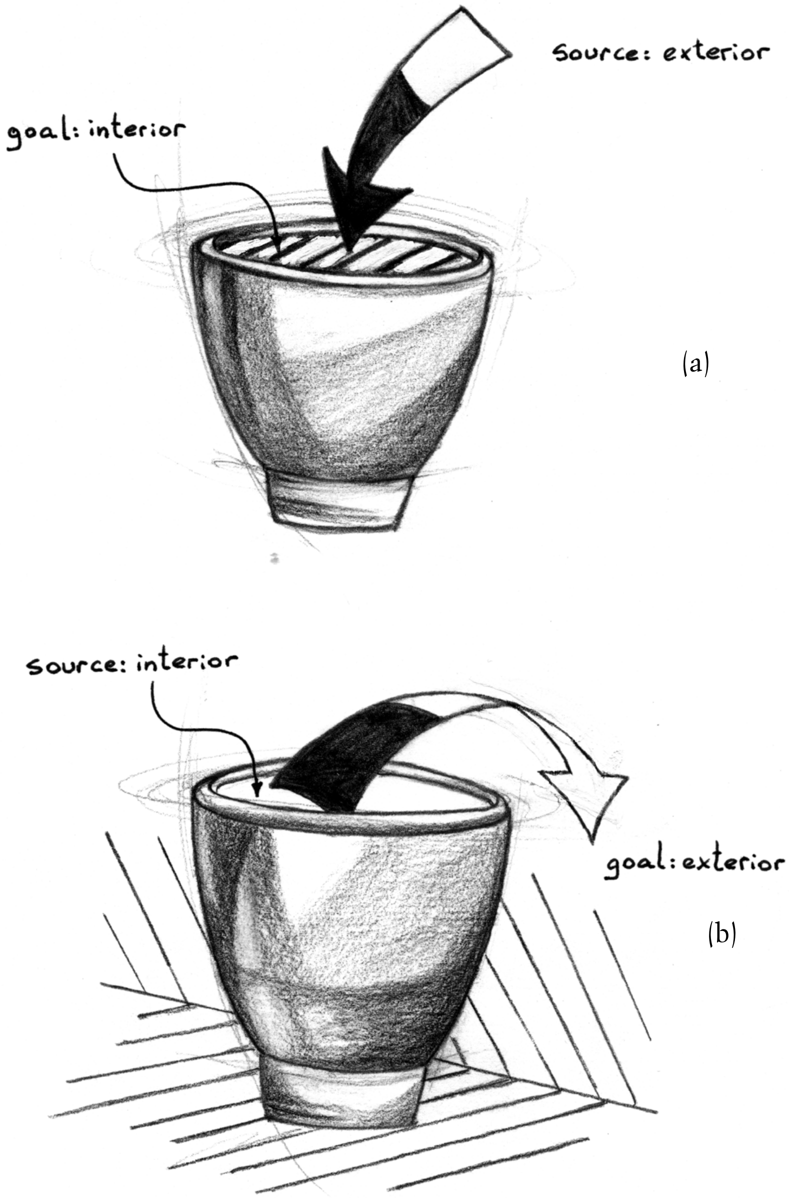


FIGURE 2.3 Conceptual composition of schemas. The English expressions "into" and "out of" have composite meanings. "In" profiles the interior of a Container schema, while "out" profiles the exterior. "To" profiles the goal of the Source-Path-Goal schema, while "from" profiles the source. With "into" (a), the interior is the goal and the exterior is the source. With "out of" (b), the interior is the source and the exterior is the goal.

non has been studied scientifically for more than two decades and is in general as well established as any result in cognitive science (though particular details of analyses are open to further investigation). One of the major results is that metaphorical mappings are systematic and not arbitrary.

Affection, for example, is understood in terms of physical warmth, as in sentences like "She *warmed* up to me," "You've been *cold* to me all day," "He gave me an *icy* stare," "They haven't yet *broken the ice*." As can be seen by this example, the metaphor is not a matter of words, but of conceptual structure. The words are all different (*warm, cold, icy, ice*), but the conceptual relationship is the same in all cases: Affection is conceptualized in terms of warmth and disaffection in terms of cold.

This is hardly an isolated example:

- Importance is conceptualized in terms of size, as in "This is a big issue," "He's a giant in the meatpacking business," and "It's a small matter; we can ignore it."
- Similarity is conceptualized in terms of physical closeness, as in "These colors are very close," "Our opinions on politics are light-years apart," "We may not agree, but our views are in the same ballpark," and "Over the years, our tastes have diverged."
- Difficulties are conceptualized as burdens, as in "I'm weighed down by responsibilities," "I've got a light load this semester," and "He's overburdened."
- Organizational structure is conceptualized as physical structure, as in "The theory is full of holes," "The fabric of this society is unraveling," "His proposed plan is really tight; everything fits together very well."

Hundreds of such conceptual metaphors have been studied in detail. They are extremely common in everyday thought and language (see Lakoff & Johnson, 1980, 1999; Grady, 1998; Núñez, 1999). On the whole, they are used unconsciously, effortlessly, and automatically in everyday discourse; that is, they are part of the cognitive unconscious. Many arise naturally from correlations in our commonplace experience, especially our experience as children. Affection correlates with warmth in the experience of most children. The things that are important in their lives tend to be big, like their parents, their homes, and so on. Things that are similar tend to occur close together: trees, flowers, dishes, clouds. Carrying something heavy makes it difficult to move and to perform other activities. When we examine a complex physical object with an internal

structure, we can perceive an organization in it. Not all conceptual metaphors arise in this way, but most of the basic ones do.

Such correlations in experience are special cases of the phenomenon of *conflation* (see C. Johnson, 1997). Conflation is part of embodied cognition. It is the simultaneous activation of two distinct areas of our brains, each concerned with distinct aspects of our experience, like the physical experience of warmth and the emotional experience of affection. In a conflation, the two kinds of experience occur inseparably. The coactivation of two or more parts of the brain generates a single complex experience—an experience of affection-with-warmth, say, or an experience of difficulty-with-a-physical-burden. It is via such conflations that neural links across domains are developed—links that often result in conceptual metaphor, in which one domain is conceptualized in terms of the other.

Each such conceptual metaphor has the same structure. Each is a unidirectional mapping from entities in one conceptual domain to corresponding entities in another conceptual domain. As such, conceptual metaphors are part of our system of thought. Their primary function is to allow us to reason about relatively abstract domains using the inferential structure of relatively concrete domains. The structure of image schemas is preserved by conceptual metaphorical mappings. In metaphor, conceptual cross-domain mapping is primary; metaphorical language is secondary, deriving from the conceptual mapping. Many words for source-domain concepts also apply to corresponding target-domain concepts. When words for source-domain concepts do apply to corresponding target concepts, they do so systematically, not haphazardly.

To see how the inferential structure of a concrete source domain gives structure to an abstract target domain, consider the common conceptual metaphor that States Are Locations, as in such expressions as "I'm in a depression," "He's close to hysteria; don't push him over the edge," and "I finally came out of my funk." The source domain concerns bounded regions in physical space. The target domain is about the subjective experience of being in a state.

STATES ARE LOCATIONS

| *Source Domain*<br>SPACE | | *Target Domain*<br>STATES |
|---|---|---|
| Bounded Regions in Space | → | States |

Here is an example of how the patterns of inference of the source domain are carried over to the target domain.

| | | |
|---|---|---|
| If you're in a *bounded region*, you're not out of that *bounded region*. | → | If you're in a *state*, you're not out of that *state*. |
| If you're out of a *bounded region*, you're not in that *bounded region*. | → | If you're out of a *state*, you're not in that *state*. |
| If you're deep in a *bounded region*, you are far from being out of that *bounded region*. | → | If you're deep in a *state*, you are far from being out of that *state*. |
| If you are on the edge of a *bounded region*, you are close to being in that *bounded region*. | → | If you are on the edge of a *state*, you are close to being in that *state*. |

Throughout this book we will use the common convention that *names* of metaphorical mappings are given in the form "A Is B," as in "States Are Bounded Regions in Space." It is important to distinguish between such names for metaphorical mappings and the metaphorical mappings themselves, which are given in the form "B → A," as in "Bounded Regions in Space → States." Here the source domain is to the left of the arrow and the target domain is to the right.

An enormous amount of our everyday abstract reasoning arises through such metaphorical cross-domain mappings. Indeed, much of what is often called logical inference is in fact spatial inference mapped onto an abstract logical domain. Consider the logic of the Container schema. There is a commonplace metaphor, Categories Are Containers, through which we understand a category as being a bounded region in space and members of the category as being objects inside that bounded region. The metaphorical mapping is stated as follows:

CATEGORIES ARE CONTAINERS

| *Source Domain* CONTAINERS | | *Target Domain* CATEGORIES |
|---|---|---|
| Bounded regions in space | → | Categories |
| Objects inside the bounded regions | → | Category members |
| One bounded region inside another | → | A subcategory of a larger category |

Suppose we apply this mapping to the two inference patterns mentioned above that characterize the spatial logic of the Container schema, as follows:

| *Source Domain*<br>CONTAINER SCHEMA INFERENCES | | *Target Domain*<br>CATEGORY INFERENCES |
|---|---|---|
| *Excluded Middle* | | *Excluded Middle* |
| Every object $X$ is either in *Container schema* $A$ or out of *Container schema* $A$. | → | Every entity $X$ is either in *category* $A$ or out of *category* $A$. |
| *Modus Ponens* | | *Modus Ponens* |
| Given two *Container schemas* $A$ and $B$ and an object $X$, if $A$ is in $B$ and $X$ is in $A$, then $X$ is in $B$. | → | Given two *categories* $A$ and $B$ and an entity $X$, if $A$ is in $B$ and $X$ is in $A$, then $X$ is in $B$. |
| *Hypothetical Syllogism* | | *Hypothetical Syllogism* |
| Given three *Container schemas* $A$, $B$, and $C$, if $A$ is in $B$ and $B$ is in $C$, then $A$ is in $C$. | → | Given three *categories* $A$, $B$ and $C$, if $A$ is in $B$ and $B$ is in $C$, then $A$ is in $C$. |
| *Modus Tollens* | | *Modus Tollens* |
| Given two *Container schemas* $A$ and $B$ and an object $Y$, if $A$ is in $B$ and $Y$ is outside $B$, then $Y$ is outside $A$. | → | Given two *categories* $A$ and $B$ and an entity $Y$, if $A$ is in $B$ and $Y$ is outside $B$, then $Y$ is outside $A$. |

The point here is that the logic of Container schemas is an embodied spatial logic that arises from the neural characterization of Container schemas. The excluded middle, modus ponens, hypothetical syllogism, and modus tollens of classical categories are metaphorical applications of that spatial logic, since the Categories Are Containers metaphor, like conceptual metaphors in general, preserves the inferential structure of the source domain.

Moreover, there are important entailments of the Categories Are Containers metaphor:

| | | |
|---|---|---|
| The overlap of the interiors of two bounded regions | → | The conjunction of two categories |
| The totality of the interiors of two bounded regions | → | The disjunction of two categories |

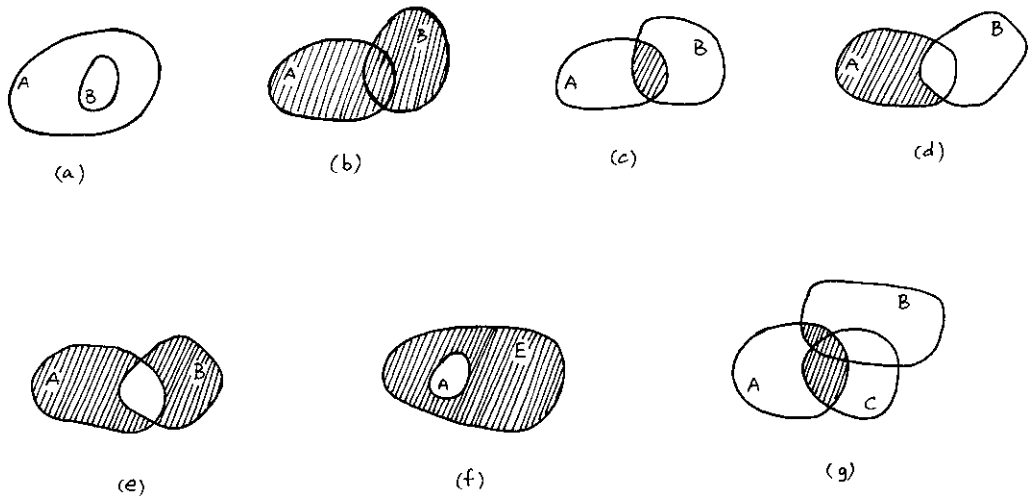


FIGURE 2.4 Venn diagrams. Here is a common set of Venn diagrams of the sort one finds in texts on classes and sets, which are typically conceptualized metaphorically as containers and derive their logics from the logic of conceptual Container schemas. When one "visualizes" classes and sets in this way, one is using cognitive Container schemas in the visualization. The diagrams depict various mathematical ideas: (a) the relation $B \subseteq A$; (b) $A \cup B$; (c) $A \cap B$; (d) the difference $A \backslash B$; (e) the symmetric difference $A \Delta B$; (f) the complement $C_E A$; and (g) $A \cap (B \cup C)$, which equals $(A \cap B) \cup (A \cap C)$.

In short, given the spatial logic of Container schemas, the Categories Are Containers metaphor yields an everyday version of what we might call folk Boolean logic, with intersections and unions. That is why the Venn diagrams of Boolean logic look so natural to us (see Figure 2.4), although there are differences between folk Boolean logic and technical Boolean logic, which will be discussed in Chapter 6. Folk Boolean logic, which is *con*ceptual, arises from a *per*ceptual mechanism—the capacity for perceiving the world in terms of contained structures.

From the perspective of the embodied mind, spatial logic is primary and the abstract logic of categories is secondarily derived from it via conceptual metaphor. This, of course, is the very opposite of what formal mathematical logic suggests. It should not be surprising, therefore, that embodied mathematics will look very different from disembodied formal mathematics.

## Metaphors That Introduce Elements

Conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also *introduce*

*new elements* into the target domain. Consider, for example, the concept of love. There is a common metaphor in the contemporary Western world in which Love Is a Partnership. Here is the mapping.

LOVE IS A PARTNERSHIP

| *Source Domain* BUSINESS | | *Target Domain* LOVE |
|---|---|---|
| Partners | → | Lovers |
| Partnership | → | Love relationship |
| Wealth | → | Well-being |
| Profits from the business | → | "Profits" from the love relationship |
| Work for the business | → | "Work" put into the relationship |
| Sharing of work for the business | → | Sharing of "work" put into the relationship |
| Sharing of profits from the business | → | Sharing of "profits" from the relationship |

Love need not always be conceptualized via this metaphor as a partnership. Romeo and Juliet's love was not a partnership, nor was Tristan and Isolde's. Similarly, love in many cultures around the world is not conceptualized in terms of business—and it need not be so conceptualized for individual cases in the Western world. But this is a common metaphorical way of understanding love—so common that it is sometimes taken as literal. For example, sentences like "I'm putting all the work into this relationship and you're getting everything out of it," "It was hard work, but worth it," and "The relationship was so unrewarding that it wasn't worth the effort" are so commonplace in discussions of love relationships that they are rarely noticed as metaphorical at all.

From the perspective of this book, there is an extremely important feature of this metaphor: It *introduces* elements into the target domain that are not inherent to the target domain. It is not inherent in love-in-itself that there be "work in the relationship," "profits (increases in well-being) from the relationship," and a "sharing of relationship work and profits." Romeo and Juliet would have been aghast at such ideas. These ideas are elements introduced into the target domain by the Love Is a Partnership metaphor, and they don't exist there without it.

The fact that metaphors can introduce elements into a target domain is extremely important for mathematics, as we shall see later.

## Evidence

Over the past two decades, an enormous range of empirical evidence has been collected that supports this view of conceptual metaphor. The evidence comes from various sources:

- generalizations over polysemy (cases where the same word has multiple systematically related meanings)
- generalizations over inference patterns (cases where source and target domains have corresponding inference patterns)
- novel cases (new examples of conventional mappings, as in poetry, song, advertisements, and so on) (see Lakoff & Turner, 1989)
- psychological experiments (see Gibbs, 1994)
- historical semantic change (see Sweetser, 1990)
- spontaneous gesture (see McNeill, 1992)
- American Sign Language (see Taub, 1997)
- child language development (see C. Johnson, 1997)
- discourse coherence (see Narayanan, 1997)
- cross-linguistic studies.

For a thorough discussion of such evidence, see Lakoff and Johnson, 1999.

## Sophisticated Mathematical Ideas

Sophisticated mathematics, as we have pointed out, is a lot more than just basic arithmetic. Mathematics extends the use of numbers to many other ideas, for example, the numerical study of angles (trigonometry), the numerical study of change (calculus), the numerical study of geometrical forms (analytic geometry), and so on. We will argue, in our discussion of all these topics and more, that conceptual metaphor is the central cognitive mechanism of extension from basic arithmetic to such sophisticated applications of number. Moreover, we will argue that a sophisticated understanding of arithmetic itself requires conceptual metaphors using nonnumerical mathematical source domains (e.g., geometry and set theory). We will argue further that conceptual metaphor is also the principal cognitive mechanism in the attempt to provide set-theoretical foundations for mathematics and in the understanding of set theory itself.

Finally, it should become clear in the course of this discussion that much of the "abstraction" of higher mathematics is a consequence of the systematic layering of metaphor upon metaphor, often over the course of centuries.

Each metaphorical layer, as we shall see, carries inferential structure systematically from source domains to target domains—systematic structure that gets lost in the layers unless they are revealed by detailed metaphorical analysis. A good part of this book is concerned with such metaphorical decomposition of sophisticated mathematical concepts. Because this kind of study has never been done before, we will not be able to offer the extensive forms of evidence that have been found in decades of studies of conceptual metaphor in everyday language and thought. For this reason, we will limit our study to cases that are relatively straightforward—cases where the distinctness of the source and target domains is clear, where the correspondences across the domains have been well established, and where the inferential structures are obvious.

## *Conceptual Blends*

A *conceptual blend* is the conceptual combination of two distinct cognitive structures with fixed correspondences between them. In mathematics, a simple case is the unit circle, in which a circle is superimposed on the Cartesian plane with the following fixed correspondences: (a) The center of the circle is the origin (0,0), and (b) the radius of the circle is 1. This blend has entailments that follow from these correspondences, together with the inferential structure of *both domains*. For example, the unit circle crosses the $x$-axis at (1,0) and (–1,0), and it crosses the $y$-axis at (0,1) and (0,–1). The result is more than just a circle. It is a circle that has a fixed position in the plane and whose circumference is a length commensurate with the numbers on the $x$- and $y$-axes. A circle in the Euclidean plane, where there are no axes and no numbers, would not have these properties.

When the fixed correspondences in a conceptual blend are given by a metaphor, we call it a *metaphorical blend*. An example we will discuss extensively below is the Number-Line Blend, which uses the correspondences established by the metaphor Numbers Are Points on a Line. In the blend, new entities are created—namely, *number-points*, entities that are at once numbers and points on a line (see Fauconnier 1997; Turner & Fauconnier, 1995; Fauconnier & Turner, 1998). Blends, metaphorical and nonmetaphorical, occur throughout mathematics.

Many of the most important ideas in mathematics are metaphorical conceptual blends. As will become clear in the case-study chapters, understanding mathematics requires the mastering of extensive networks of metaphorical blends.

## *Symbolization*

As we have noted, there is a critical distinction to be made among mathematical concepts, the written mathematical symbols for those concepts, and the

words for the concepts. The words (e.g., "eighty-five" or "quatre-vingt-cinq") are part of some natural language, not mathematics proper.

In embodied mathematics, mathematical symbols, like 27, π, or $e^{\pi i}$, are meaningful by virtue of the mathematical concepts that they attach to. Those mathematical concepts are given in cognitive terms (e.g., image schemas; imagined geometrical shapes; metaphorical structures, like the number line; and so on), and those cognitive structures will ultimately require a neural account of how the brain creates them on the basis of neural structure and bodily and social experience. To understand a mathematical symbol is to associate it with a concept—something meaningful in human cognition that is ultimately grounded in experience and created via neural mechanisms.

As Stanislas Dehaene observed in the case of Mr. M—and as many of us experienced in grade school—numerical calculation may be performed with or without genuine understanding. Mr. M could remember his multiplication tables, but they were essentially meaningless to him.

The meaning of mathematical symbols is not in the symbols alone and how they can be manipulated by rule. Nor is the meaning of symbols in the interpretation of the symbols in terms of set-theoretical models that are themselves uninterpreted. Ultimately, mathematical meaning is like everyday meaning. It is part of embodied cognition.

This has important consequences for the teaching of mathematics. Rote learning and drill is not enough. It leaves out understanding. Similarly, deriving theorems from formal axioms via purely formal rules of proof is not enough. It, too, can leave out understanding. The point is not to be able to prove *that* $e^{\pi i} = -1$ but, rather, to be able to prove it knowing what $e^{\pi i}$ means, and knowing *why* $e^{\pi i} = -1$ on the basis of what $e^{\pi i}$ means, not just on the basis of the formal proof. In short, what is required is an adequate mathematical idea analysis to show *why* $e^{\pi i} = -1$ given our understanding of the ideas involved.

Euler's equation, $e^{\pi i} + 1 = 0$, ties together many of the most central ideas in classical mathematics. Yet on the surface it involves only numbers: $e$, π, $i$, 1, and 0. To show how this equation ties together *ideas*, we must have a theory of mathematical ideas and a theory of how they are mathematicized in terms of numbers.

Our interest, of course, goes beyond just $e^{\pi i}$ as such. Indeed, we are also interested in the all-too-common conception that mathematics is about calculation and about formal proofs from formal axioms and definitions and not about ideas and understanding. From the perspective of embodied mathematics, ideas and understanding are what mathematics is centrally about.

# 3

# Embodied Arithmetic: The Grounding Metaphors

ARITHMETIC IS A LOT MORE THAN SUBITIZING and the elementary numerical capacities of monkeys and newborn babies. To understand what arithmetic is from a cognitive perspective, we need to know much more. Why does arithmetic have the properties it has? Where do the laws of arithmetic come from? What cognitive mechanisms are needed to go from what we are born with to full-blown arithmetic? Arithmetic may seem easy once you've learned it, but there is an awful lot to it from the perspective of the embodied mind.

## What Is Special About Mathematics?

As subsystems of the human conceptual system, arithmetic in particular and mathematics in general are special in several ways. They are:

- precise,
- consistent,
- stable across time and communities,
- understandable across cultures,
- symbolizable,
- calculable,
- generalizable, and
- effective as general tools for description, explanation, and prediction in a vast number of everyday activities, from business to building to sports to science and technology.

Any cognitive theory of mathematics must take these special properties into account, showing how they are possible given ordinary human cognitive capacities. That is the goal of this chapter.

## The Cognitive Capacities Needed for Arithmetic

We are born with a minimal innate arithmetic, part of which we share with other animals. It is not much, but we do come equipped with it. Innate arithmetic includes at least two capacities: (1) a capacity for subitizing—instantly recognizing small numbers of items—and (2) a capacity for the simplest forms of adding and subtracting small numbers. (By "number" here, we mean a *cardinal* number, a number that specifies how many objects there are in a collection.) When we subitize, we have already limited ourselves to a grouping of objects in our visual field and we are distinguishing how many objects there are in that grouping.

In addition, we and many animals (pigeons, parrots, raccoons, rats, chimpanzees) have an innate capacity for "numerosity"—the ability to make consistent rough estimates of the number of objects in a group.

But arithmetic involves more than a capacity to subitize and estimate. Subitizing is certain and precise within its range. But we have additional capacities that allow us to extend this certainty and precision. To do this, we must count. Here are the cognitive capacities needed in order to count, say, on our fingers:

- *Grouping capacity:* To distinguish what we are counting, we have to be able to group discrete elements visually, mentally, or by touch.
- *Ordering capacity:* Fingers come in a natural order on our hands. But the objects to be counted typically do not come in any natural order in the world. They have to be ordered—that is, placed in a sequence, as if they corresponded to our fingers or were spread out along a path.
- *Pairing capacity:* We need a cognitive mechanism that enables us to sequentially pair individual fingers with individual objects, following the sequence of objects in order.
- *Memory capacity:* We need to keep track of which fingers have been used in counting and which objects have been counted.
- *Exhaustion-detection capacity:* We need to be able to tell when there are "no more" objects left to be counted.
- *Cardinal-number assignment:* The last number in the count is an ordinal number, a number in a sequence. We need to be able to assign that ordinal number as the size—the cardinal number—of the group counted. That cardinal number, the size of the group, has no notion of sequence in it.

- *Independent-order capacity:* We need to realize that the cardinal number assigned to the counted group is independent of the order in which the elements have been counted. This capacity allows us to see that the result is always the same.

When these capacities are used within the subitizing range between 1 and 4, we get stable results because cardinal-number assignment is done by subitizing, say, subitizing the fingers used for counting.

To count beyond four—the range of the subitizing capacity—we need not only the cognitive mechanisms listed above but the following additional capacities:

- *Combinatorial-grouping capacity:* You need a cognitive mechanism that allows you to put together perceived or imagined groups to form larger groups.
- *Symbolizing capacity:* You need to be able to associate physical symbols (or words) with numbers (which are conceptual entities).

But subitizing and counting are the bare beginnings of arithmetic. To go beyond them, to characterize arithmetic operations and their properties, you need much richer cognitive capacities:

- *Metaphorizing capacity:* You need to be able to conceptualize cardinal numbers and arithmetic operations in terms of your experiences of various kinds—experiences with groups of objects, with the part-whole structure of objects, with distances, with movement and locations, and so on.
- *Conceptual-blending capacity*. You need to be able to form correspondences across conceptual domains (e.g., combining *subitizing* with *counting*) and put together different conceptual metaphors to form complex metaphors.

Conceptual metaphor and conceptual blending are among the most basic cognitive mechanisms that take us beyond minimal innate arithmetic and simple counting to the elementary arithmetic of natural numbers. What we have found is that there are two types of conceptual metaphor used in projecting from subitizing, counting, and the simplest arithmetic of newborns to an arithmetic of natural numbers.

The first are what we call *grounding metaphors*—metaphors that allow you to project from everyday experiences (like putting things into piles) onto abstract

concepts (like addition). The second are what we call *linking metaphors,* which link arithmetic to other branches of mathematics—for example, metaphors that allow you to conceptualize arithmetic in spatial terms, linking, say, geometry to arithmetic, as when you conceive of numbers as points on a line.

### *Two Kinds of Metaphorical Mathematical Ideas*

Since conceptual metaphors play a major role in characterizing mathematical ideas, grounding and linking metaphors provide for two types of metaphorical mathematical ideas:

1. Grounding metaphors yield *basic, directly grounded ideas*. Examples: addition as adding objects to a collection, subtraction as taking objects away from a collection, sets as containers, members of a set as objects in a container. These usually require little instruction.
2. Linking metaphors yield *sophisticated ideas*, sometimes called *abstract ideas*. Examples: numbers as points on a line, geometrical figures as algebraic equations, operations on classes as algebraic operations. These require a significant amount of explicit instruction.

This chapter is devoted to *grounding* metaphors. The rest of the book is devoted primarily to *linking* metaphors.

Incidentally, there is another type of metaphor that this book is not about at all: what we will call *extraneous* metaphors, or metaphors that have nothing whatever to do with either the grounding of mathematics or the structure of mathematics itself. Unfortunately, the term "metaphor," when applied to mathematics, has mostly referred to such extraneous metaphors. A good example of an extraneous metaphor is the idea of a "step function," which can be drawn to look like a staircase. The staircase image, though helpful for visualization, has nothing whatever to do with either the inherent content or the grounding of the mathematics. Extraneous metaphors can be eliminated without any substantive change in the conceptual structure of mathematics, whereas eliminating grounding or linking metaphors would make much of the conceptual content of mathematics disappear.

### *Preserving Inferences About Everyday Activities*

Since conceptual metaphors preserve inference structure, such metaphors allow us to ground our understanding of arithmetic in our prior understanding of ex-

tremely commonplace physical activities. Our understanding of elementary arithmetic is based on a correlation between (1) the most basic literal aspects of arithmetic, such as subitizing and counting, and (2) everyday activities, such as collecting objects into groups or piles, taking objects apart and putting them together, taking steps, and so on. Such correlations allow us to form metaphors by which we greatly extend our subitizing and counting capacities.

One of the major ways in which metaphor preserves inference is via the preservation of image-schema structure. For example, the formation of a collection or pile of objects requires conceptualizing that collection as a container—that is, a bounded region of space with an interior, an exterior, and a boundary—either physical or imagined. When we conceptualize numbers as collections, we project the logic of collections onto numbers. In this way, experiences like grouping that correlate with simple numbers give further logical structure to an expanded notion of number.

### *The Metaphorizing Capacity*

The metaphorizing capacity is central to the extension of arithmetic beyond mere subitizing, counting, and the simplest adding and subtracting. Because of its centrality, we will look at it in considerable detail, starting with the Arithmetic Is Object Collection metaphor. This is a grounding metaphor, in that it grounds our conception of arithmetic directly in an everyday activity.

No metaphor is more basic to the extension of our concept of number from the innate cardinal numbers to the natural numbers (the positive integers). The reason is that the correlation of grouping with subitizing and counting the elements in a group is pervasive in our experience from earliest childhood.

Let us now begin an extensive guided tour of everything involved in this apparently simple metaphor. As we shall see, even the simplest and most intuitive of mathematical metaphors is incredibly rich, and so the tour will be extensive.

## Arithmetic As Object Collection

If a child is given a group of three blocks, she will naturally subitize them automatically and unconsciously as being three in number. If one is taken away, she will subitize the resulting group as two in number. Such everyday experiences of subitizing, addition, and subtraction with small collections of objects involve correlations between addition and adding objects to a collection and between subtraction and taking objects away from a collection. Such regular correlations, we hypothesize, result in neural connections between sensory-motor physical opera-

tions like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe, *constitute a conceptual metaphor* at the neural level—in this case, the metaphor that Arithmetic Is Object Collection. This metaphor, we hypothesize, is learned at an early age, prior to any formal arithmetic training. Indeed, arithmetic training assumes this unconscious conceptual (not linguistic!) metaphor: In teaching arithmetic, we all take it for granted that the adding and subtracting of numbers can be understood in terms of adding and taking away objects from collections. Of course, at this stage all of these are mental operations *with no symbols!* Calculating with symbols requires additional capacities.

The Arithmetic Is Object Collection metaphor is a precise mapping from the domain of physical objects to the domain of numbers. The metaphorical mapping consists of

1. the source domain of object collection (based on our commonest experiences with grouping objects);
2. the target domain of arithmetic (structured nonmetaphorically by subitizing and counting); and
3. a mapping across the domains (based on our experience subitizing and counting objects in groups). The metaphor can be stated as follows:

ARITHMETIC IS OBJECT COLLECTION

| *Source Domain* OBJECT COLLECTION | | *Target Domain* ARITHMETIC |
|---|---|---|
| Collections of objects of the same size | → | Numbers |
| The size of the collection | → | The size of the number |
| Bigger | → | Greater |
| Smaller | → | Less |
| The smallest collection | → | The unit (One) |
| Putting collections together | → | Addition |
| Taking a smaller collection from a larger collection | → | Subtraction |

### *Linguistic Examples of the Metaphor*

We can see evidence of this conceptual metaphor in our everyday language. The word *add* has the physical meaning of physically placing a substance or a num-

ber of objects into a container (or group of objects), as in "*Add* sugar to my coffee," "*Add* some logs to the fire," and "*Add* onions and carrots to the soup." Similarly, *take . . . from, take . . . out of,* and *take . . . away* have the physical meaning of removing a substance, an object, or a number of objects from some container or collection. Examples include "*Take* some books *out of* the box," "*Take* some water *from* this pot," "*Take away* some of these logs." By virtue of the Arithmetic Is Object Collection metaphor, these expressions are used for the corresponding arithmetic operations of addition and subtraction.

> If you *add* 4 apples *to* 5 apples, how many do you have? If you *take* 2 apples *from* 5 apples, how many apples are *left*? *Add* 2 *to* 3 and you have 5. *Take* 2 *from* 5 and you have 3 *left*.

It follows from the metaphor that adding yields something *bigger* (more) and subtracting yields something *smaller* (less). Accordingly, words like *big* and *small*, which indicate size for objects and collections of objects, are also used for numbers, as in "Which is *bigger*, 5 or 7?" and "Two is *smaller* than four." This metaphor is so deeply ingrained in our unconscious minds that we have to think twice to realize that numbers are not physical objects and so do not literally have a size.

### *Entailments of the Metaphor*

The Arithmetic Is Object Collection metaphor has many entailments. Each arises in the following way: Take the basic truths about collections of physical objects. Map them onto statements about numbers, using the metaphorical mapping. The result is a set of "truths" about the natural numbers under the operations of addition and subtraction.

For example, suppose we have two collections, $A$ and $B$, of physical objects, with $A$ bigger than $B$. Now suppose we add the same collection $C$ to each. Then $A$ plus $C$ will be a bigger collection of physical objects than $B$ plus $C$. This is a fact about collections of physical objects of the same size. Using the mapping Numbers Are Collections of Objects, this physical truth that we experience in grouping objects becomes a mathematical truth about numbers: If $A$ is greater than $B$, then $A + C$ is greater than $B + C$. All of the following truths about numbers arise in this way, via the metaphor Arithmetic Is Object Collection.

### *The Laws of Arithmetic Are Metaphorical Entailments*

In each of the following cases, the metaphor Arithmetic Is Object Collection maps a property of the source domain of *object collections* (stated on the left)

to a unique corresponding property of the target domain of *numbers* (stated on the right). This metaphor extends properties of the innate subitized numbers 1 through 4 to an indefinitely large collection of natural numbers. In the cases below, you can see clearly how properties of object collections are mapped by the metaphor onto properties of natural numbers in general.

| MAGNITUDE | | |
|---|---|---|
| *Object collections* have a magnitude | → | *Numbers* have a magnitude |

| STABILITY OF RESULTS FOR ADDITION | | |
|---|---|---|
| Whenever you add a fixed *object collection* to a second fixed *object collection,* you get the same result. | → | Whenever you add a fixed *number* to another fixed *number*, you get the same result. |

| STABILITY OF RESULTS FOR SUBTRACTION | | |
|---|---|---|
| Whenever you subtract a fixed *object collection* from a second fixed *object collection*, you get the same result. | → | Whenever you subtract a fixed *number* from another fixed *number*, you get the same result. |

| INVERSE OPERATIONS | | |
|---|---|---|
| For *collections:* Whenever you subtract what you added, or add what you subtracted, you get the original *collection*. | → | For *numbers:* Whenever you subtract what you added, or add what you subtracted, you get the original *number*. |

| UNIFORM ONTOLOGY | | |
|---|---|---|
| *Object collections* play three roles in addition.<br>• what you add to something;<br>• what you add something to;<br>• the result of adding.<br>Despite their differing roles, they all have the same nature with respect to the operation of the *addition* of *object collections*. | → | *Numbers* play three roles in addition.<br>• what you add to something;<br>• what you add something to;<br>• the result of adding.<br>Despite their differing roles, they all have the same nature with respect to the operation of the *addition* of *numbers.* |

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#### CLOSURE FOR ADDITION

| | | |
|---|---|---|
| The process of *adding* an *object collection* to *another object collection* yields a *third object collection.* | → | The process of *adding* a *number* to a *number* yields a third *number.* |

#### UNLIMITED ITERATION FOR ADDITION

| | | |
|---|---|---|
| You can add *object collections* indefinitely. | → | You can add *numbers* indefinitely. |

#### LIMITED ITERATION FOR SUBTRACTION

| | | |
|---|---|---|
| You can subtract *object collections* from other *object collections* until nothing is left. | → | You can subtract *numbers* from other *numbers* until nothing is left. |

#### SEQUENTIAL OPERATIONS

| | | |
|---|---|---|
| You can do combinations of adding and subtracting *object collections*. | → | You can do combinations of adding and subtracting *numbers*. |

### *Equational Properties*

#### EQUALITY OF RESULT

| | | |
|---|---|---|
| You can obtain the same resulting *object collection* via different operations. | → | You can obtain the same resulting *number* via different operations. |

#### PRESERVATION OF EQUALITY

| | | |
|---|---|---|
| For *object collections*, adding equals to equals yields equals. | → | For *numbers*, adding equals to equals yields equals. |
| For *object collections*, subtracting equals from equals yields equals. | → | For *numbers*, subtracting equals from equals yields equals. |

#### COMMUTATIVITY

| | | |
|---|---|---|
| For *object collections*, adding $A$ to $B$ gives the same result as adding $B$ to $A$. | → | For *numbers*, adding $A$ to $B$ gives the same result as adding $B$ to $A$. |

| ASSOCIATIVITY | | |
|---|---|---|
| For *object collections*, adding $B$ to $C$ and then adding $A$ to the result is equivalent to adding $A$ to $B$ and adding $C$ to that result. | → | For *numbers*, adding $B$ to $C$ and then adding $A$ to the result is equivalent to adding $A$ to $B$ and adding $C$ to that result. |

### *Relationship Properties*

| LINEAR CONSISTENCY | | |
|---|---|---|
| For *object collections*, if $A$ is bigger than $B$, then $B$ is smaller than $A$. | → | For *numbers,* if $A$ is greater than $B$, then $B$ is less than $A$. |

| LINEARITY | | |
|---|---|---|
| If $A$ and $B$ are two *object collections*, then either $A$ is bigger than $B$, or $B$ is bigger than $A$, or $A$ and $B$ are the same size. | → | If $A$ and $B$ are two *numbers*, then either $A$ is greater than $B$, or $B$ is greater than $A$, or $A$ and $B$ are the same magnitude. |

| SYMMETRY | | |
|---|---|---|
| If *collection A* is the same size as *collection B*, then $B$ is the same size as $A$. | → | If *number A* is the same magnitude as *number B*, then $B$ is the same magnitude as $A$. |

| TRANSITIVITY | | |
|---|---|---|
| For *object collections*, if $A$ is bigger than $B$ and $B$ is bigger than $C$, then $A$ is bigger than $C$. | → | For *numbers*, if $A$ is greater than $B$ and $B$ is greater than $C$, then $A$ is greater than $C$. |

In order for there to be a metaphorical mapping from object collections to numbers, the entailments of such a mapping must be consistent with the properties of innate arithmetic. Innate arithmetic has some of these properties—for example, uniform ontology, linear consistency, linearity, symmetry, commutativity, and preservation of equality. The Arithmetic Is Object Collection metaphor will map the object-collection version of these properties onto the version of these properties in innate arithmetic (e.g., $2 + 1 = 1 + 2$).

However, this metaphor will also extend innate arithmetic, adding properties that the innate arithmetic of numbers 1 through 4 does not have, because of its limited range—namely, *closure* (e.g., under addition) and what follows from closure: unlimited iteration for addition, sequential operations, equality of result, and preservation of equality. The metaphor will map these properties from the domain of object collections to the expanded domain of number. The result is the elementary arithmetic of addition and subtraction for natural numbers, *which goes beyond innate arithmetic.*

Thus, the fact that there is an innate basis for arithmetic does not mean that all arithmetic is innate. Part of arithmetic arises from our experience in the world with object collections. The Arithmetic Is Object Collection metaphor arises naturally in our brains as a result of regularly using innate neural arithmetic while interacting with small collections of objects.

## Extending Elementary Arithmetic

The version of the Arithmetic Is Object Collection metaphor just stated is limited to conceptualizing addition and subtraction of numbers in terms of addition and subtraction of collections. Operations in one domain (using only collections) are mapped onto operations in the other domain (using only numbers). There is no single operation characterized in terms of elements from both domains—that is, no single operation that uses both numbers *and* collections simultaneously.

But with multiplication, we *do* need to refer to numbers and collections simultaneously, since understanding multiplication in terms of collections requires performing operations on *collections* a certain *number* of times. This cannot be done in a domain with collections alone or numbers alone. In this respect, multiplication is cognitively more complex than addition or subtraction.

The cognitive mechanism that allows us to extend this metaphor from addition and subtraction to multiplication and division is *metaphoric blending*. This is not a new mechanism but simply a consequence of having metaphoric mappings.

Recall that each metaphoric mapping is characterized neurally by a fixed set of connections across conceptual domains. The results of inferences in the source domain are mapped to the target domain. If both domains, together with the mapping, are activated at once (as when one is doing arithmetic on object collections), the result is a metaphoric blend: the simultaneous activation of two domains with connections across the domains.

### *Two Versions of Multiplication and Division*

Consider 3 times 5 in terms of collections of objects:

- Suppose we have 3 small collections of 5 objects each. Suppose we pool these collections. We get a single collection of 15 objects.
- Now suppose we have a big pile of objects. If we put 5 objects in a box 3 times, we get 15 objects in the box. This is repeated addition: We added 5 objects to the box repeatedly—3 times.

In the first case, we are doing multiplication by *pooling*, and in the second by *repeated addition*.

Division can also be characterized in two corresponding ways, *splitting up* and *repeated subtraction:*

- Suppose we have a single collection of 15 objects, then we can split it up into 3 collections of 5 objects each. That is, 15 divided by 3 is 5.
- Suppose again that we have a collection of 15 objects and that we repeatedly subtract 5 objects from it. Then, after 3 repeated subtractions, there will be no objects left. Again, 15 divided by 3 is 5.

In each of these cases we have used numbers with only addition and subtraction defined in order to characterize multiplication and division metaphorically in terms of object collection. From a cognitive perspective, we have used a metaphoric blend of object collections together with numbers to extend the Arithmetic Is Object Collection metaphor to multiplication and division.

We can state the pooling and iteration extensions of this metaphor precisely as follows:

THE POOLING/SPLITTING EXTENSION OF
THE ARITHMETIC IS OBJECT COLLECTION METAPHOR

| *Source Domain*<br>THE OBJECT-COLLECTION/<br>ARITHMETIC BLEND | | *Target Domain*<br>ARITHMETIC |
|---|---|---|
| The pooling of $A$ subcollections of size $B$ to form an overall collection of size $C$. | → | Multiplication ($A \cdot B = C$) |
| The splitting up of a collection of size $C$ into $A$ subcollections of size $B$. | → | Division ($C \div B = A$) |

THE ITERATION EXTENSION OF THE ARITHMETIC IS OBJECT COLLECTION METAPHOR

| *Source Domain* THE OBJECT-COLLECTION/ ARITHMETIC BLEND | | *Target Domain* ARITHMETIC |
|---|---|---|
| The repeated addition ($A$ times) of a collection of size $B$ to yield a collection of size $C$. | → | Multiplication ($A \cdot B = C$) |
| The repeated subtraction of collections of size $B$ from an initial collection of size $C$ until the initial collection is exhausted. $A$ is the number of times the subtraction occurs. | → | Division ($C \div B = A$) |

Note that in each case, the result of the operation is given in terms of the size of the collection as it is understood in the source domain of collections. Since the result of a multiplication or division is always a *collection* of a given size, multiplication and division (in this metaphor) can be combined with the addition and subtraction of collections to give further results in terms of collections.

What is interesting about these two equivalent metaphorical conceptions of multiplication and division is that they are both defined relative to the number-collection blend, but *they involve different ways of thinking about* operating on collections.

These metaphors for multiplication and division map the properties of the source domain onto the target domain, giving rise to the most basic properties of multiplication and division. Let us consider the commutative, associative, and distributive properties.

COMMUTATIVITY FOR MULTIPLICATION

| | | |
|---|---|---|
| Pooling $A$ collections of size $B$ gives a collection of the same resulting size as pooling $B$ collections of size $A$. | → | Multiplying $A$ times $B$ gives the same resulting number as multiplying $B$ times $A$. |

**ASSOCIATIVITY FOR MULTIPLICATION**

| | | |
|---|---|---|
| Pooling $A$ collections of size $B$ and pooling that number of collections of size $C$ gives a collection of the same resulting size as pooling the number of $A$ collections of the size of the collection formed by pooling $B$ collections of size $C$. | → | Multiplying $A$ times the result of multiplying $B$ times $C$ gives the same number as multiplying $A$ times $B$ and multiplying the result times $C$. |

**DISTRIBUTIVITY OF MULTIPLICATION OVER ADDITION**

| | | |
|---|---|---|
| First, pool $A$ collections of the size of the collection formed by adding a collection of size $B$ to a collection of size $C$. This gives a collection of the same size as adding a collection formed by pooling $A$ collections of size $B$ to $A$ collections of size $C$. | → | First, multiply $A$ times the sum of $B$ plus $C$. This gives the same number as adding the product of $A$ times $B$ to the product of $A$ times $C$. |

**MULTIPLICATIVE IDENTITY**

| | | |
|---|---|---|
| Pooling one collection of size $A$ results in a collection of size $A$. | → | Multiplying one times $A$ yields $A$ |
| Pooling $A$ collections of size one yields a collection of size $A$. | → | Multiplying $A$ times one yields $A$. |

**INVERSE OF MULTIPLICATION**

| | | |
|---|---|---|
| Splitting a collection of size $A$ into $A$ subcollections yields subcollections of size one. | → | Dividing $A$ by $A$ yields one. |

In each case, a true statement about collections is projected by the metaphor in the pooling/splitting extension onto the domain of numbers, yielding a true statement about arithmetic. The same will work for iterative extension (i.e., repeated addition and repeated subtraction).

Thus, the Arithmetic Is Object Collection metaphor extends our understanding of number from the subitized numbers of innate arithmetic and from sim-

ple counting to the arithmetic of the natural numbers, grounding the extension of arithmetic in our everyday experience with groups of physical objects.

### *Zero*

The Arithmetic Is Object Collection metaphor does, however, leave a problem. What happens when we subtract, say, seven from seven? The result cannot be understood in terms of a collection. In our everyday experience, the result of taking a collection of seven objects from a collection of seven objects is an absence of any objects at all—*not a collection of objects*. If we want the result to be a number, then in order to accommodate the Arithmetic Is Object Collection metaphor we must conceptualize the absence of a collection *as a collection*. A new conceptual metaphor is necessary. What is needed is a metaphor that creates something out of nothing: From the absence of a collection, the metaphorical mapping creates a unique collection of a particular kind—a collection with no objects in it.

THE ZERO COLLECTION METAPHOR

| | | |
|---|---|---|
| The lack of objects to form a collection | → | The empty collection |

Given this additional metaphor as input, the Arithmetic Is Object Collection metaphor will then map the empty collection onto a number—which we call "zero."

This new metaphor is of a type common in mathematics, which we will call an *entity-creating metaphor*. In the previous case, the conceptual metaphor *creates* zero as an actual number. Although zero is an extension of the object-collection metaphor, it is not a natural extension. It does not arise from a correlation between the experience of collecting and the experience of subitizing and doing innate arithmetic. It is therefore an artificial metaphor, concocted ad hoc for the purpose of extension.

Once the metaphor Arithmetic Is Object Collection is extended in this way, more properties of numbers follow as entailments of the metaphor.

ADDITIVE IDENTITY

| | | |
|---|---|---|
| Adding the empty collection to a collection of size $A$ yields a collection of size $A$. | → | Adding zero to $A$ yields $A$. |
| Adding a collection of size $A$ to the empty collection yields a collection of size $A$. | → | Adding $A$ to zero yields $A$. |

INVERSE OF ADDITION

| | | |
|---|---|---|
| Taking a collection of size *A* away from a collection of size *A* yields the empty collection. | → | Subtracting *A* from *A* yields zero. |

These metaphors ground our most basic extension of arithmetic—from the innate cardinal numbers to the natural numbers plus zero. As is well known, this understanding of number still leaves gaps: It does not give a meaningful characterization of 2 minus 5 or 2 divided by 3. To fill those gaps we need further entity-creating metaphors, e.g., metaphors for the negative numbers. We will discuss such metaphors shortly.

At this point, we have explored only one of the basic grounding metaphors for arithmetic. There are three more to go. It would be unnecessarily repetitive to go into each in the full detail given above. Instead, we will sketch only the essential features of these metaphors.

## Arithmetic As Object Construction

Consider such commonplaces of arithmetic as these: "Five is *made up of* two plus three." "You can *factor* 28 *into* 7 times 4." "If you *put* 2 and 2 *together*, you get 4."

How is it possible to understand a number, which is an abstraction, as being "made up," or "composed of," other numbers, which are "put together" using arithmetic operations? What we are doing here is conceptualizing numbers as wholes made up of parts. The parts are other numbers. And the operations of arithmetic provide the patterns by which the parts fit together to form wholes. Here is the metaphorical mapping used to conceptualize numbers in this way.

ARITHMETIC IS OBJECT CONSTRUCTION

| *Source Domain*<br>OBJECT CONSTRUCTION | | *Target Domain*<br>ARITHMETIC |
|---|---|---|
| Objects (consisting of ultimate parts of unit size) | → | Numbers |
| The smallest whole object | → | The unit (one) |
| The size of the object | → | The size of the number |
| Bigger | → | Greater |
| Smaller | → | Less |
| Acts of object construction | → | Arithmetic operations |
| A constructed object | → | The result of an arithmetic operation |

| | | |
|---|---|---|
| A whole object | → | A whole number |
| Putting objects together with other objects to form larger objects | → | Addition |
| Taking smaller objects from larger objects to form other objects | → | Subtraction |

As in the case of Arithmetic Is Object Collection, this metaphor can be extended in two ways via metaphorical blending: fitting together/splitting up and iterated addition and subtraction.

### THE FITTING TOGETHER/SPLITTING UP EXTENSION

| | | |
|---|---|---|
| The fitting together of $A$ parts of size $B$ to form a whole object of size $C$ | → | Multiplication ($A \cdot B = C$) |
| The splitting up of a whole object of size $C$ into $A$ parts of size $B$, a number that corresponds in the blend to an object of size $A$, which is the result | → | Division ($C \div B = A$) |

### THE ITERATION EXTENSION

| | | |
|---|---|---|
| The repeated addition ($A$ times) of $A$ parts of size $B$ to yield a whole object of size $C$ | → | Multiplication ($A \cdot B = C$) |
| The repeated subtraction of parts of size $B$ from an initial object of size $C$ until the initial object is exhausted. The result, $A$, is the number of times the subtraction occurs. | → | Division ($C \div B = A$) |

Fractions are understood metaphorically in terms of the characterizations of division (as splitting) and multiplication (as fitting together).

| FRACTIONS | | |
|---|---|---|
| A part of a unit object (made by splitting a unit object into $n$ parts) | → | A simple fraction ($1/n$) |
| An object made by fitting together $m$ parts of size $1/n$ | → | A complex fraction ($m/n$) |

These additional metaphorical mappings yield an important entailment about number based on a truth about objects.

| | | |
|---|---|---|
| If you split a unit object into $n$ parts and then you fit the $n$ parts together again, you get the unit object back. | → | If you divide 1 by $n$ and multiply the result by $n$, you get 1. That is, $1/n \cdot n = 1$. |

In other words, $1/n$ is the multiplicative inverse of $n$.

As in the case of the object-collection metaphor, a special additional metaphor is needed to conceptualize zero. Since the lack of an object is not an object, it should not, strictly speaking, correspond to a number. The zero object metaphor is thus an artificial metaphor.

| THE ZERO OBJECT METAPHOR | | |
|---|---|---|
| The Lack of a Whole Object | → | Zero |

The object-construction metaphor is intimately related to the object-collection metaphor. The reason is that constructing an object necessarily requires collecting the parts of the object together. Every whole made up of parts is a collection of the parts, with the added condition that the parts are assembled according to a certain pattern. Since object construction is a more specific version of object collection, the metaphor of Arithmetic As Object Construction is a more specific version of the metaphor of Arithmetic As Object Collection. Accordingly, the object-construction metaphor has all the inferences of the object-collection metaphor—the inferences we stated in the previous section. It differs in that it is extended to characterize fractions and so has additional inferences—for example, $(1/n) \cdot n = 1$.

It also has metaphorical entailments that characterize the decomposition of numbers into parts.

| | | |
|---|---|---|
| Whole objects are composites of their parts, put together by certain operations. | → | Whole numbers are composites of their parts, put together by certain operations. |

It is this metaphorical entailment that gives rise to the field of number theory, the study of which numbers can be decomposed into other numbers and operations on them.

## The Measuring Stick Metaphor

The oldest (and still often used) method for designing buildings or physically laying out dimensions on the ground is to use a measuring stick or string—a stick or string taken as a unit. These are physical versions of what in geometry are called *line segments*. We will refer to them as "physical segments." A distance can be measured by placing physical segments of unit length end-to-end and counting them. In the simplest case, the physical segments are body parts: fingers, hands, forearms, arms, feet, and so on. When we put physical segments end-to-end, the result is another physical segment, which may be a real or envisioned tracing of a line in space.

In a wide range of languages throughout the world, this concept is represented by a classifier morpheme. In Japanese, for example, the word *hon* (literally, "a long, thin thing") is used for counting such long, thin objects as sticks, canes, pencils, candles, trees, ropes, baseball bats, and so on—including, of course, rulers and measuring tapes. Even though English does not have a single word for the idea, it is a natural human concept.

THE MEASURING STICK METAPHOR

| *Source Domain* THE USE OF A MEASURING STICK | | *Target Domain* ARITHMETIC |
|---|---|---|
| Physical segments (consisting of ultimate parts of unit length) | → | Numbers |
| The basic physical segment | → | One |
| The length of the physical segment | → | The size of the number |
| Longer | → | Greater |
| Shorter | → | Less |
| Acts of physical segment placement | → | Arithmetic operations |
| A physical segment | → | The result of an arithmetic operation |

| | | |
|---|---|---|
| Putting physical segments together end-to-end with other physical segments to form longer physical segments | → | Addition |
| Taking shorter physical segments from larger physical segments to form other physical segments | → | Subtraction |

As in the previous two metaphors, there are two ways of characterizing multiplication and division: fitting together/dividing up and iterated addition and subtraction.

THE FITTING TOGETHER/DIVIDING UP EXTENSION

| | | |
|---|---|---|
| The fitting together of $A$ physical segments of length $B$ to form a line segment of length $C$ | → | Multiplication ($A \cdot B = C$) |
| The splitting up of a physical segment $C$ into $A$ parts of length $B$. $A$ is a number that corresponds in the blend to a physical segment of length $A$, which is the result. | → | Division ($C \div B = A$) |

THE ITERATION EXTENSION

| | | |
|---|---|---|
| The repeated addition ($A$ times) of $A$ physical segments of length $B$ to form a physical segment of length $C$. | → | Multiplication ($A \cdot B = C$) |
| The repeated subtraction of physical segments of length $B$ from an initial physical segment of length $C$ until nothing is left of the initial physical segment. The result, $A$, is the number of times the subtraction occurs. | → | Division ($C \div B = A$) |

As in the case of the object-construction metaphor, the physical segment metaphor can be extended to define fractions.

FRACTIONS

| | | |
|---|---|---|
| A part of a physical segment (made by splitting a single physical segment into $n$ equal parts) | → | A simple fraction ($1/n$) |
| A physical segment made by fitting together (end-to-end) $m$ parts of size $1/n$ | → | A complex fraction ($m/n$) |

Just as in the object-construction metaphor, this metaphor needs to be extended in order to get a conceptualization of zero.

| | | |
|---|---|---|
| The lack of any physical segment | → | Zero |

Up to this point, the measuring stick metaphor looks very much like the object-construction metaphor: A physical segment can be seen as a physical object, even if it is an imagined line in space. But physical segments are very special "constructed objects." They are unidimensional and they are continuous. In their abstract version they correspond to the line segments of Euclidean geometry. As a result, the blend of the source and target domains of this metaphor has a very special status. It is a blend of line (physical) segments with numbers specifying their length, which we will call the Number/Physical Segment blend.

Moreover, once you form the blend, a fateful entailment arises. Recall that the metaphor states that Numbers Are Physical Segments, and that given this metaphor you can characterize natural numbers, zero, and positive complex fractions (the rational numbers) in terms of physical segments. That is, for every positive rational number, this metaphor (given a unit length) provides a unique physical segment. The metaphorical mapping is unidirectional. It does not say that for any line segment at all, there is a corresponding number.

But the blend of source and target domains goes beyond the metaphor itself and has new entailments. When you form a blend of physical segments and numbers, constrained by the measuring stick metaphor, then within the blend there is a one-to-one correspondence between physical segments and numbers. The fateful entailment is this: Given a fixed unit length, it follows that for every physical segment there is a number.

Now consider the Pythagorean theorem: In $A^2 + B^2 = C^2$, let $C$ be the hypotenuse of a right triangle and $A$ and $B$ be the lengths of the other sides. Let $A = 1$ and $B = 1$. Then $C^2 = 2$. The Pythagoreans had already proved that $C$ could not be expressed as a fraction—that is, that it could not be a rational number—

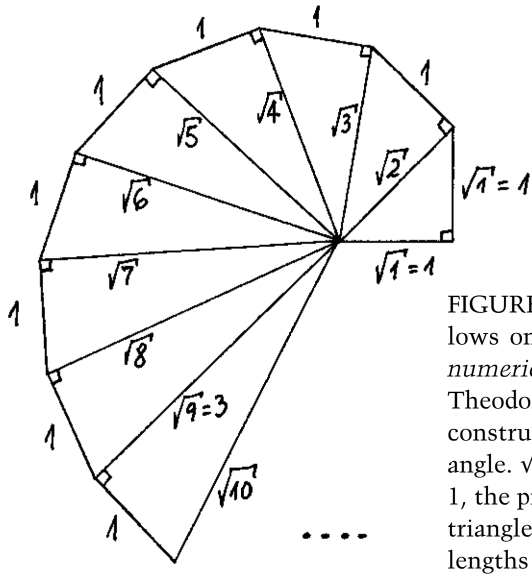


FIGURE 3.1 The measuring stick metaphor allows one to form physical segments of particular *numerical* lengths. In the diagram, taken from Theodorus of Cyrene (fourth century B.C.), √2 is constructed from the unit length 1 and a right triangle. √3 is then constructed from the unit length 1, the previously constructed length √2, and a right triangle. And so on. Without the metaphor, the lengths are just lengths, not numbers.

a ratio of physical lengths corresponding to integers. They assumed that only natural numbers and their ratios (the rational numbers) existed and that the length *C* was not a number at all; they called it an *incommensurable*—without ratio, that is, without a common measure.

But Eudoxus (c. 370 B.C.) observed, implicitly using the Number/Physical Segment blend, that corresponding to the hypotenuse in this triangle there must be a number: $C = \sqrt{2}$! This conclusion could not have been reached using numbers by themselves, taken literally. If you assume that only rational numbers exist and you prove that √2 cannot be a rational number, then it could just as well follow (as it did initially for the Pythagoreans) that 2 does not exist—that is, that 2 does not have any square root. But if, according to the Number/Physical Segment blend, there must exist a number corresponding to the length of every physical segment, then and only then must √2 exist as a number!

It was the measuring stick metaphor and the Number/Physical Segment blend that gave birth to the irrational numbers.

## Arithmetic As Motion Along a Path

When we move in a straight line from one place to another, the path of our motion forms a physical segment—an imagined line tracing our trajectory. There is a simple relationship between a path of motion and a physical segment. The origin of the motion corresponds to one end of a physical segment; the endpoint

of the motion corresponds to the other end of the physical segment; and the path of motion corresponds to the rest of the physical segment.

Given this correspondence between motions and physical segments, there is a natural metaphorical correlate to the measuring stick metaphor for arithmetic, namely, the metaphor that Arithmetic Is Motion Along a Path. Here is how that metaphor is stated.

ARITHMETIC IS MOTION ALONG A PATH

| *Source Domain*<br>MOTION ALONG A PATH | | *Target Domain*<br>ARITHMETIC |
|---|---|---|
| Acts of moving along the path | → | Arithmetic operations |
| A point-location on the path | → | The result of an arithmetic operation |
| The origin, the beginning of the path | → | Zero |
| Point-locations on a path | → | Numbers |
| A point-location | → | One |
| Further from the origin than | → | Greater than |
| Closer to the origin than | → | Less than |
| Moving from a point-location $A$ away from the origin, a distance that is the same as the distance from the origin to a point-location $B$ | → | Addition of $B$ to $A$ |
| Moving toward the origin from $A$, a distance that is the same as the distance from the origin to $B$ | → | Subtraction of $B$ from $A$ |

This metaphor can be extended to multiplication and division by means of iteration over addition and subtraction.

THE ITERATION EXTENSION

| | | |
|---|---|---|
| Starting at the origin, move $A$ times in the direction away from the origin a distance that is the same as the distance from the origin to $B$. | → | Multiplication ($A \cdot B = C$) |

| | | |
|---|---|---|
| Starting at $C$, move toward the origin distances of length $B$ repeatedly $A$ times. | → | Division ($C \div B = A$) |

FRACTIONS

| | | |
|---|---|---|
| Starting at 1, find a distance $d$ such that by moving distance $d$ toward the origin repeatedly $n$ times, you will reach the origin. $1/n$ is the point-location at distance $d$ from the origin. | → | A simple fraction ($1/n$) |
| Point-location reached moving from the origin a distance $1/n$ repeatedly $m$ times. | → | A complex fraction ($m/n$) |

As we mentioned, the Arithmetic Is Motion metaphor corresponds in many ways to the measuring stick metaphor. But there is one major difference. In all the other metaphors that we have looked at so far, including the measuring stick metaphor, there had to be some entity-creating metaphor added to get zero. However, when numbers are point-locations on a line, the origin is by its very nature a point-location. When we designate zero as the origin, it is already a point-location.

Moreover, this metaphor provides a natural extension to negative numbers—let the origin be somewhere on a pathway extending indefinitely in both directions. The negative numbers will be the point-locations on the other side of zero from the positive numbers along the same path. This extension was explicitly made by Rafael Bombelli in the second half of the sixteenth century. In Bombelli's extension of the point-location metaphor for numbers, positive numbers, zero, and negative numbers are all point-locations on a line. This made it commonplace for European mathematicians to think and speak of the concept of a number *lying between* two other numbers—as in *zero lies between minus one and one*. Conceptualizing all (real) numbers metaphorically as point-locations on the same line was crucial to providing a uniform understanding of number. These days, it is hard to imagine that there was ever a time when such a metaphor was not commonly accepted by mathematicians!

The understanding of numbers as point-locations has come into our language in the following expressions:

How *close* are these two numbers?
37 is *far away from* 189,712.

4.9 is *near* 5.
The result is *around* 40.
Count up *to* 20, without *skipping* any numbers.
Count *backward* from 20.
Count *to* 100, *starting at* 20.
Name all the numbers *from* 2 *to* 10.

The linguistic examples are important here in a number of respects. First, they illustrate how the language of motion can be recruited in a systematic way to talk about arithmetic. The conceptual mappings characterize what is systematic about this use of language. Second, these usages of language provide evidence for the existence of the conceptual mapping—evidence that comes not only from the words but also from what the words mean. The metaphors can be seen as stating generalizations not only over the use of the words but also over the inference patterns that these words supply from the source domain of motion, which are then used in reasoning about arithmetic.

We have now completed our initial description of the four basic grounding metaphors for arithmetic. Let us now turn to the relation between arithmetic and elementary algebra.

## The Fundamental Metonymy of Algebra

Consider how we understand the sentence "When the pizza delivery boy comes, give him a good tip." The conceptual frame is Ordering a Pizza for Delivery. Within this frame, there is a role for the Pizza Delivery Boy, who delivers the pizza to the customer. In the situation, we do not know which *individual* will be delivering the pizza. But we need to conceptualize, make inferences about, and talk about that individual, whoever he is. Via the Role-for-Individual metonymy, the role "pizza delivery boy" comes to stand metonymically for the particular individual who fills the role—that is, who happens to deliver the pizza today. "Give him a good tip" is an instruction that applies to the individual, whoever he is.

This everyday conceptual metonymy, which exists outside mathematics, plays a major role in mathematical thinking: It allows us to go from concrete (case by case) arithmetic to general algebraic thinking. When we write "$x + 2 = 7$," $x$ is our notation for a role, Number, standing for an individual number. "$x + 2 = 7$" says that whatever number $x$ happens to be, adding 2 to it will yield 7.

This everyday cognitive mechanism allows us to state general laws like "$x + y = y + x$," which says that adding a number $y$ to another number $x$ yields

the same result as adding $x$ to $y$. It is this metonymic mechanism that makes the discipline of algebra possible, by allowing us to reason about numbers or other entities without knowing which particular entities we are talking about.

Clear examples of how we unconsciously use and master the Fundamental Metonymy of Algebra are provided by many passages in this very chapter. In fact, every time we have written (and every time you have read and understood) an expression such as "If collection $A$ is the same size as collection $B$," or "adding zero to $A$ yields $A$," we have been implicitly making use of the Fundamental Metonymy of Algebra. It is this cognitive mechanism that permits general proofs in mathematics—for example, proofs about any number, whatever it is.

## The Metaphorical Meanings of One and Zero

The four grounding metaphors mentioned so far—Object Collection, Object Construction, the Measuring Stick, and Motion Along a Line—contain metaphorical characterizations of zero and one. Jointly, these metaphors characterize the symbolic meanings of zero and one. In the collection metaphor, zero is the empty collection. Thus, zero can connote *emptiness*. In the object-construction metaphor, zero is either the lack of an object, the absence of an object or, as a result of an operation, the destruction of an object. Thus, zero can mean *lack*, *absence*, or *destruction*. In the measuring stick metaphor, zero stands for the *ultimate in smallness*, the lack of any physical segment at all. In the motion metaphor, zero is the origin of motion; hence, zero can designate an *origin*. Hence, zero, in everyday language, can symbolically denote emptiness, nothingness, lack, absence, destruction, ultimate smallness, and origin.

In the collection metaphor, one is the collection with a lone member and, hence, symbolizes *individuality* and *separateness* from others. In the object-construction metaphor, one is a whole number and, by virtue of this, signifies *wholeness*, *unity*, and *integrity*. In the measuring stick metaphor, one is the length specifying the unit of measure. In this case, one signifies a *standard*. And in the motion metaphor, one indicates the first step in a movement. Hence, it symbolizes a *beginning*. Taken together, these metaphors give one the symbolic values of individuality, separateness, wholeness, unity, integrity, a standard, and a beginning. Here are some examples.

- *Beginning:* One small step for a man; one great step for mankind.
- *Unity:* E pluribus unum ("From many, one").
- *Integrity:* Fred and Ginger danced as one.
- *Origin:* Let's start again from zero.

- *Emptiness:* There's zero in the refrigerator.
- *Nothingness:* I started with zero and made it to the top.
- *Destruction:* This nullifies all that we have done.
- *Lack (of ability):* That new quarterback is a big zero.

These grounding metaphors therefore explain why zero and one, which are literally numbers, have the symbolic values that they have. But that is not the real importance of these metaphors for mathematics. The real importance is that they explain how innate arithmetic gets extended systematically to give arithmetic distinctive properties that innate arithmetic does not have. Because of their importance we will give the four grounding metaphors for arithmetic a name: *the 4Gs*. We now turn to their implications.

# 4

# Where Do the Laws of Arithmetic Come From?

## The Significance of the 4Gs

Innate arithmetic, as we saw, is extremely limited: It includes only subitizing, addition, and subtraction up to the number 4 at most. The 4Gs each arise via a conflation in everyday experience. Take object collection. Young children form small collections, subitize them, and add and take away objects from them, automatically forming additions and subtractions within the subitizable range. The same is true when they make objects, take steps, and later use sticks, fingers, and arms to estimate size. These correlations in everyday experience between innate arithmetic and the source domains of the 4Gs give rise to the 4Gs. The metaphors—at least in an automatic, unconscious form—arise naturally from such conflations in experience.

The significance of the 4Gs is that they allow human beings, who have an innate capacity to form metaphors, to extend arithmetic beyond the small amount that we are born with, while preserving the basic properties of innate arithmetic. The mechanism is as follows: In each conflation of innate arithmetic with a source domain, the inferences of innate arithmetic fit those of the source domain (say, object collection). Just as $3 - 1 = 2$ abstractly, if you take one object from a collection of three objects, you get a collection of two objects. In other words, the inferences of abstract innate arithmetic hold when it is conceptually blended with object collection.

This may seem so obvious as to hardly be worth mentioning, but it is the basis for the extension of innate arithmetic way beyond its inherent limits. Because innate arithmetic "fits" object collection and construction, motion, and manipulation of physical segments, those four domains of concrete experience are suitable for metaphorical extensions of innate arithmetic that preserve its properties. Taking one step after taking two steps gets you to the same place as taking three steps, just as adding one object to a collection of two objects yields a collection of three objects.

Thus, the properties of innate arithmetic can be seen as "picking out" these four domains for the metaphorical extension of basic arithmetic capacities beyond the number 4. Indeed, the reason that these four domains all fit innate arithmetic is that there are structural relationships across the domains. Thus, object construction always involves object collection; you can't build an object without gathering the parts together. The two experiences are conflated and thereby neurally linked. Putting physical segments end-to-end is similar to object construction (think of legos here). When you use a measuring stick to mark off a distance, you are mentally constructing a line segment out of parts—a "path" from the beginning of the measurement to the end. A path of motion from point to point corresponds to such an imagined line segment. In short, there are structural correspondences between

- object collection and object construction
- the construction of a linear object and the use of a measuring stick to mark off a line segment of certain length
- using a measuring stick to mark off a line segment, or "path," and moving from location to location along a path.

As a result of these structural correspondences, there are *isomorphisms* across the 4G metaphors—namely, the correlations just described between the source domains of those metaphors.

That isomorphism defines a one-to-one correlation between metaphoric definitions of arithmetic operations—addition and multiplication—in the four metaphors. For there to be such an isomorphism, the following three conditions must hold:

- There is a one-to-one mapping, *M*, between elements in one source domain and elements in the other source domain—that is, the "images" under the "mapping."

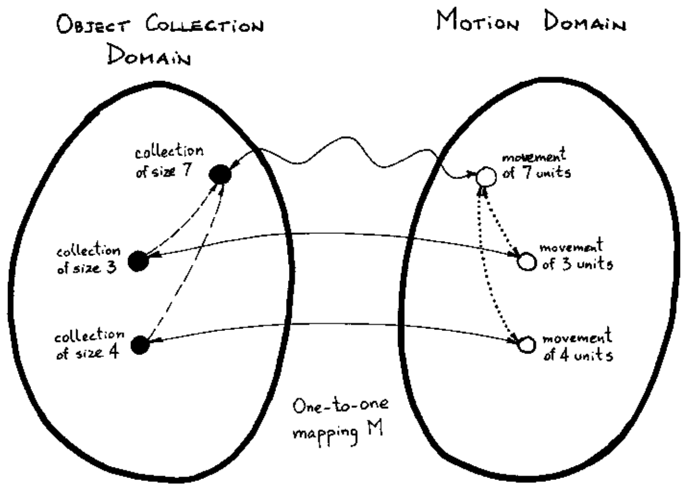


FIGURE 4.1 There are structural correspondences (isomorphisms) across the source domains of the four grounding metaphors for arithmetic (4Gs). The diagram depicts the isomorphism between the source domains of Object Collection and Motion Along a Path. There is a one-to-one mapping between the elements of the two source domains. Moreover, the images of sums correspond to the sums of images, and the images of products correspond to the products of images. For example, for a collection of sizes 3 and 4 there are unique movements of 3 and 4 units, respectively (solid lines). Besides, the result of "adding collections of size 3 and 4" (segmented lines) has a unique image, that is, the image of sums: "a movement of 7 units" (wavy line). This is equivalent to the sum of images: Adding the image of "a collection of size 3" (i.e., "a movement of 3 units") and the image of "a collection of size 4" (i.e., "a movement of 4 units") yields "a movement of 7 units" (dotted lines).

- *M* preserves sums: $M(x + y) = M(x) + M(y)$; that is, the images of sums correspond to the sums of images.
- *M* preserves products: $M(x \cdot y) = M(x) \cdot M(y)$; that is, the images of products correspond to the products of images.

Consider, for example, the source domains of object collection and motion, which appear quite dissimilar. There is such an isomorphism between those two source domains. First, there is a one-to-one correspondence, *M*, between sizes of collections and distances moved. For example, a collection of size three is uniquely mapped to a movement of three units of length, and conversely. It should be obvious from Figure 4.1 that the last two conditions are also met.

This is something very special about the conceptual system of mathematics: The source domains of all four basic grounding metaphors for the arithmetic of

natural numbers are isomorphic in this way! Note that there are no numbers in these source domains; there are only object collections, motions, and so on. But given how they are mapped onto the natural numbers, the relevant inferential structures of all these domains are isomorphic.

Aside from the way they are mapped onto the natural numbers, these four source domains are not isomorphic: Object construction characterizes fractions but not zero or negative numbers, whereas motion along a path characterizes zero and negative numbers. In other words, if you look at the complete domains in isolation, you will not see an isomorphism across the source domains. What creates the isomorphism is the collection of mappings from these source domains of the 4Gs onto natural numbers. And what grounds the mappings onto natural numbers are the experiences we have, across the four domains, with innate arithmetic—with subitizing and counting in such early experiences as forming collections, putting things together, moving from place to place, and so on.

## Numbers Are Things

In each of the 4Gs, numbers are things that exist in the world: collections of objects, complex objects with parts, physical segments, and locations. These four metaphors thus induce a more general metaphor, that *Numbers Are Things in the World*. Though they can function as quantifiers in everyday language (e.g., "five apples"), numbers in arithmetic statements function like things; the names of numbers (e.g., "five") are proper nouns and go in noun positions, as in "Two is less than four," "Divide 125 by 5," and so on.

The metaphor Numbers Are Things in the World has deep consequences. The first is the widespread view of mathematical Platonism. If objects are real entities out there in the universe, then understanding Numbers metaphorically as Things in the World leads to the metaphorical conclusion that numbers have an objective existence as real entities out there as part of the universe. This is a metaphorical inference from one of our most basic unconscious metaphors. As such, it seems natural. We barely notice it. Given this metaphorical inference, other equally metaphorical inferences follow, shaping the intuitive core of the philosophy of mathematical Platonism:

- Since real objects in the world have unique properties that distinguish them from all other entities, so there should be a uniquely true mathematics. Every mathematical statement about numbers should be absolutely true or false. There should be no equally valid alternative forms of mathematics.

- Numbers should not be products of minds, any more than trees or rocks or stars are products of minds.
- Mathematical truths are discovered, not created.

What is particularly ironic about this is that *it follows from the empirical study of numbers as a product of mind that it is natural for people to believe that numbers are* not *a product of mind!*

## Closure

The metaphor that Numbers Are Things in the World has a second important consequence for both the structure and practice of mathematics. In most of our everyday experience, when we operate on actual physical entities, the result is another physical entity. If we put two objects together, we get another object. If we combine two collections, we get another collection. If we start moving at one location, we wind up at another location. Over much of our experience, a general principle holds:

- An operation on physical things yields a physical thing of the same kind.

The metaphor that Numbers Are Things yields a corresponding principle:

- An operation on numbers yields a number of the same kind.

The name for this metaphorical principle in mathematics is *closure*. Closure is not a property of innate arithmetic. "Subitizable 3" plus "subitizable 4" does not produce a subitizable number; we don't normally subitize 7. But closure does arise naturally from the grounding metaphors.

Closure is a central idea in mathematics. It has led mathematicians to extend number systems further and further until closure is achieved—and to stop with closure. Thus, the natural numbers had to be extended in various ways to achieve closure relative to the basic arithmetic operations (addition, subtraction, multiplication, division, raising to powers, and taking roots):

- Because, say, 5 – 5 is not a natural number, zero had to be added.
- Because, say, 3 – 5 is not a natural number, negative numbers had to be added.

- Because, say, $3 \div 5$ is not a natural number, rational numbers (fractions) had to be added.
- Because, say, $\sqrt{2}$ is not a rational number, irrational numbers had to be added to form the "real numbers."
- Because, say, $\sqrt{-1}$ is not a real number, the "imaginary numbers" had to be added to form the "complex numbers."

Extending the natural numbers to the complex numbers finally achieved closure relative to the basic operations of arithmetic. As the fundamental theorem of algebra implies, any arithmetic operation on any complex numbers yields a complex number.

The notion of closure is central to all branches of mathematics and is an engine for creating new mathematics. Given any set of mathematical elements and a set of operations on them, we can ask whether that set is "closed" under those operations—that is, whether those operations always yield members of that set. If not, we can ask what other elements need to be added to that set in order to achieve closure, and whether closure is even achievable. A very large and significant part of mathematics arises from issues surrounding closure.

## Numbers and Numerals

### *Numbers*

Via the Arithmetic Is Object Construction metaphor, we conceptualize numbers as wholes put together out of parts. The operations of arithmetic provide the patterns by which the parts are arranged within the wholes. For example, every natural number can be conceptualized uniquely as a product of prime numbers. Thus, 70 equals 2 times 5 times 7, and so can be conceptualized as the unique sequence of primes (2, 5, 7), which uniquely picks out the number 70.

Similarly, every natural number can be conceptualized as a polynomial—that is, a sum of integers represented by simple numerals times powers of some integer $B$. $B$ is called the base of the given number system. In the binary system, $B$ is two. In the octal system, $B$ is eight. In the system most of the world now uses, $B$ is ten. Thus, 8,307 is eight times ten to the third power plus three times ten to the second power, plus zero times ten to the first power, plus seven times ten to the zero$^{\text{th}}$ power.

$$8{,}307 = (8 \cdot 10^3) + (3 \cdot 10^2) + (0 \cdot 10^1) + (7 \cdot 10^0)$$

When the natural numbers are extended to the reals, this metaphorical representation of numbers is extended to include infinite decimals—sums of the

same sort, where the powers of ten can be negative numbers and products of all negative powers of ten are included. Thus, $\pi$ is understood as an infinite sum:

$$\pi = (3 \cdot 10^0) + (1 \cdot 10^{-1}) + (4 \cdot 10^{-2}) + (1 \cdot 10^{-3}) + \ldots = 3.141\ldots$$

## *Numerals*

There is a big difference between numbers, which are concepts, and numerals, which are written symbols for numbers. In innate arithmetic, there are numbers but no numerals, since newborn children have not learned to symbolize numbers using numerals. The difference can be seen in Roman numerals versus Arabic numerals, where the same *number* is represented by different *numerals*; for example, the number fourteen is represented by XIV in Roman numerals and by 14 in Arabic numerals with base ten. The Arabic numeral system, now used throughout the world, is based on the metaphorical conceptualization of numbers as sums of products of small numbers times powers of ten.

Suppose we replaced Arabic numerals with Roman numerals, where, for example, 3 = III, 5 = V, 4 = IV, 6 = VI, 50 = L, 78 = LXXVIII, and 1998 = MCMXCVIII. With Roman numerals instead of Arabic numerals, nothing would be changed about *numbers*. Only the *symbols* for numbers would be changed. Every property of numbers would remain the same.

The Roman notation is also based on the Arithmetic Is Object Construction metaphor, and the Roman notation uses not only addition (e.g., VI + I = VII) but subtraction (e.g., X – I = IX) for certain cases. Is arithmetic the same with both notations, even when the properties of numbers are all the same?

Yes and no. Conceptually it would remain the same. But *doing* arithmetic by calculating with numerals would be very different. The notation for numbers is part of the mathematics of calculation. To see this, consider systems of notations with different bases. The binary number system uses a different version of the metaphor that numbers are sums of products of other numbers. In the binary version, the base is two and every number is a sum of products of powers of two. Thus, fourteen in the binary system is one times two to the third power plus one times two to the second power, plus one times two to the first power plus zero times two to the zero$^{th}$ power. In binary numerals, that sum would be represented as 1110.

The decimal, binary, octal, and other base-defined notations are all built on various versions of the metaphor that numbers are sums of products of small numbers times powers of some base. Thus, cognitively, it is important to make a three-way distinction:

- *The number* (e.g., thirteen)
- *The conceptual representation of the number:* the sum of products of powers adding up to that number (e.g., one times ten to the first power plus three times ten to the zero$^{th}$ power)
- *The numeral that symbolizes the number* by, in turn, symbolizing the sum of products of powers (e.g., 13).

From a cognitive perspective, bidirectional conceptual mappings are used to link conceptual representations to numerals. Here are the mappings for decimal and binary systems (where $n$ is an integer):

THE DECIMAL NUMERAL-NUMBER MAPPING

| Numeral Sequences | | Sums of Products of Powers of Numbers |
|---|---|---|
| $x_i$ ($i$ = '0',. . . ,'9') | $\leftrightarrow$ | . . . $(x_n \cdot 10^n) + (x_{n-1} \cdot 10^{n-1})$. . . |

THE BINARY NUMERAL-NUMBER MAPPING

| Numeral Sequences | | Sums of Products of Powers of Numbers |
|---|---|---|
| $x_i$ ($i$ = '0', '1') | $\leftrightarrow$ | . . . $(x_n \cdot 2^n) + (x_{n-1} \cdot 2^{n-1})$. . . |

Any facts about arithmetic can be expressed in any base, because all conceptual representations of numbers in terms of sums of products of powers and all symbolic representations of numbers in terms of sequences of digits refer to the same system of numbers. Thirteen is a prime number whether you conceptualize and write it 1101 in binary, 111 in ternary, 15 in octal, 13 in decimal, 10 in base thirteen, or even XIII in Roman numerals.

Though the numbers are the same, the numerical representations of them are different. The tables for numbers are the same, but the numerical representations of the tables are different. For example, here are the addition tables for numbers between zero and five in binary and decimal notations:

ADDITION TABLE IN BINARY NOTATION

| + | 0 | 1 | 10 | 11 | 100 | 101 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 10 | 11 | 100 | 101 |
| 1 | 1 | 10 | 11 | 100 | 101 | 110 |
| 10 | 10 | 11 | 100 | 101 | 110 | 111 |
| 11 | 11 | 100 | 101 | 110 | 111 | 1000 |
| 100 | 100 | 101 | 110 | 111 | 1000 | 1001 |
| 101 | 101 | 110 | 111 | 1000 | 1001 | 1010 |

ADDITION TABLE IN DECIMAL NOTATION

| + | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | 3 | 4 | 5 | 6 | 7 |
| 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 4 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5 | 5 | 6 | 7 | 8 | 9 | 10 |

The *numbers* represented in these tables are the same. 1010 in the lower right-hand cell of the binary table represents the number ten, just as 10 does in the decimal table. The numerals are different. And the way of conceptualizing the numbers using the numerals are different. Thus, in the binary table, numbers are conceptualized as well as numerically represented in terms of sums of products of powers of two.

## *Calculation*

A system of calculation based on the Roman numerals would be very different from any system we presently have. From a cognitive perspective, it would be prohibitively difficult. Parsing a long Roman numeral is simply harder than parsing an equivalently long decimal notation, since the notation is not strictly *positional*; that is, it is not the case that one symbol stands for one multiple of ten to some power. For example, in XI (eleven), you are adding I to X (one to ten). But in XIX (nineteen), the I is not being added to the first X; rather, it is being subtracted from the second (one taken from ten), with the result added to the first (ten plus nine). This demands more cognitive activity for human beings than using decimal notation. For beings like us, positional notation requires less cognitive effort, not just in recognition but especially in calculation. Procedures for adding, subtracting, multiplying, and dividing require less cognitive effort in positional notations than in nonpositional notations like Roman numerals. Imagine doing long division with Roman numerals!

Not that it isn't doable. One could program a computer to do arithmetic using Roman numerals, and given the speed of contemporary computers, we probably wouldn't notice the difference in computation time, for any normal computation. But we don't use Roman numerals, and we will never go back to them because of

the cognitive load they place on us. Moreover, we don't use binary notation, even though computers do, because our ten fingers make it easier for us to use base 10.

Our mathematics of calculation and the notation we do it in is chosen for bodily reasons—for ease of cognitive processing and because we have ten fingers and learn to count on them. But our bodies enter into the very idea of a linearly ordered symbolic notation for mathematics. Our writing systems are linear partly because of the linear sweep of our arms and partly because of the linear sweep of our gaze. The very idea of a linear symbol system arises from the peculiar properties of our bodies. And linear symbol systems are at the heart of mathematics. Our linear, positional, polynomial-based notational system is an optimal solution to the constraints placed on us by our bodies (our arms and our gaze), our cognitive limitations (visual perception and attention, memory, parsing ability), and possibilities given by conceptual metaphor.

## *Calculation Without Understanding*

As we have seen, the mathematics of calculation, including the tables and algorithms for arithmetic operations, is all defined in terms of numerals, not numbers. Using the algorithms, we can manipulate the numerals correctly without having contact with numbers and without necessarily knowing much about numbers as opposed to numerals. The algorithms have been explicitly created for such efficient calculation, not for understanding. And we can know how to use the algorithms without much understanding of what they mean.

When we learn procedures for adding, subtracting, multiplying, and dividing, we are learning algorithms for manipulating symbols—numerals, not numbers. What is taught in grade school as arithmetic is, for the most part, not ideas about numbers but automatic procedures for performing operations on numerals—procedures that give consistent and stable results. Being able to carry out such operations does not mean that you have learned meaningful content about the nature of numbers, even if you always get the right answers!

There is a lot that is important about this. Such algorithms minimize cognitive activity while allowing you to get the right answers. Moreover, these algorithms work generally—for all numbers, regardless of their size. And one of the best things about mathematical calculation—extended from simple arithmetic to higher forms of mathematics—is that the algorithm, being freed from meaning and understanding, can be implemented in a physical machine called a computer, a machine that can calculate everything perfectly without understanding anything at all.

### *Equivalent Result Frames and the Laws of Arithmetic*

Part of our knowledge about voting is that there are two equivalent ways to vote. You can write away for an absentee ballot, fill it in at home, and send it in before the election. Or you can go to your polling place, show your identification, get a ballot, fill it in, and leave it at the polling place on the day of the election. That is, given an election, an election day, a ballot, and the procedure of filling in a ballot, there are two ways to achieve the result of voting. Similarly, we have knowledge about lots of other equivalent ways to achieve desired results. You can buy a product by shopping at a store, using a mail-order catalogue and telephoning your order, or placing your order over the Internet. Familiarity with the various processes that achieve an identical result is an important part of our overall knowledge. From a cognitive perspective, such knowledge is represented in a conceptual frame within Charles Fillmore's theory of frame semantics (Fillmore, 1982, 1985). An *Equivalent Result Frame* (hereafter, ERF) includes

- a desired result,
- essential actions and entities, and
- a list of alternative ways of performing those actions with those entities to achieve the result.

For example, an important property of collections of objects can be stated in terms of the following ERF:

THE *Associative ERF* FOR COLLECTIONS

| | |
|---|---|
| Desired result: | A collection $N$ |
| Entities: | Collections $A$, $B$, and $C$ |
| Operation: | "add to" |

Equivalent alternatives:

- $A$ added to [the collection resulting from adding $B$ to $C$] yields $N$
- [the collection resulting from adding $A$ to $B$] added to $C$ yields $N$

The metaphor Arithmetic Is Object Collection maps this ERF onto a corresponding ERF for arithmetic:

THE *Associative ERF* FOR ARITHMETIC

| | |
|---|---|
| Desired result: | A number $N$ |
| Entities: | Numbers $A$, $B$, and $C$ |
| Operation: | "+" |

Equivalent alternatives:

- $A + (B + C) = N$
- $(A + B) + C = N$

This ERF expresses what we understand the associative law for arithmetic to mean. From a cognitive perspective, this ERF *is* the cognitive content of the associative law for arithmetic: $A + (B + C) = (A + B) + C$. Here we can see a clear example of how the grounding metaphors for arithmetic yield the basic laws of arithmetic, when applied to the ERFs for the source domains of collections, construction, motion, and so on.

Note, incidentally, that the associative law does not hold for innate arithmetic, where all numbers must be subitizable—that is, less than 4. The reason, of course, is that closure does not hold for innate arithmetic. Thus, if we let $A = 1$, $B = 2$, and $C =3$, then $A + B = 3$, $B + C = 5$, and $A + B + C = 6$. Since 6 is beyond the usual range of what is subitizable, this assignment of these subitizable results to $A$, $B$, and $C$ yields a result outside of innate arithmetic. This means that the associative law cannot arise in innate arithmetic and must arise elsewhere. As we have just seen, it arises in the source domains of the four grounding metaphors.

## *Why Calculation with Numerals Works*

Why does calculation using symbolic numerals work? There is nothing magical about it. Suppose you have to add eighty-three to the sum of seventeen and thirty-nine. The numeral-number mapping, which underlies our understanding of the numeral-number relationship, maps this problem into the symbolization:

$$83 + (17 + 39) = ?$$

By the associative law, this is equivalent to

$$(83 + 17) + 39 = ?$$

Since we know that $83 + 17 = 100$, it is clear that the right answer is 139. But why does this work? Is the associative law god-given? Not at all. It works for the following reason:

- The 4Gs ground our understanding of arithmetic and extend it from innate arithmetic.
- The source domains of the 4Gs are object collection, object construction, physical segmentation, and motion. Each of these is part of our understanding of the real world.
- The associative equivalent result frame is true of each physical source domain.

- The 4Gs map those equivalent result frames onto the conceptual content of the associative law.
- The numeral-number mapping maps associative ERF for arithmetic onto the symbolized form of the associative law: $A + (B + C) = (A + B) + C$. This is used in calculation to replace an occurrence of "$A + (B + C)$" by "$(A + B) + C$" in the calculation.
- Because the symbolized equation corresponds to the cognitive content of the equivalent result frame, the symbolic substitution yields an equivalent conceptual result.

Note what a cognitive account such as this does *not* say: It does not say that the reason the calculation works is that the associative law is an *axiom*. Rather, it says that there is a reason following from our embodied understanding of what arithmetic is. It is our embodied understanding that allows blind calculation with numerals to work in arithmetic.

Up to this point we have looked at the 4Gs and how they extend innate arithmetic and enrich it with properties like closure and the basic laws of arithmetic. We have also considered the cognitive mechanism for the symbolization of numbers, which makes arithmetic calculable. We now need to flesh out the grounding metaphors discussed so far.

## Stretching the 4Gs

The 4Gs, as we have stated them so far, are grounding metaphors that arise naturally from experience that conflates innate arithmetic with one of the domains. These natural metaphors extend innate arithmetic considerably. They also allow for extensions from natural numbers to other numbers. For example, the Arithmetic Is Motion Along a Path metaphor allows the path to be extended indefinitely from both sides of the origin, permitting zero and negative numbers to be conceptualized as point-locations on the path and therefore to be seen as numbers just like any other numbers. Fractions can then be conceptualized as point-locations between the integers, at distances determined by division.

Given that such numbers have natural interpretations in this metaphor, one might think that this metaphor as it stands would be sufficient to ground arithmetic for zero, negative numbers, and fractions. It is not. No single natural metaphor permits closure in arithmetic. To achieve closure, the metaphors must be extended, or "stretched."

Let us begin with addition and multiplication for negative numbers. Let negative numbers be point-locations on the path on the side opposite the origin from positive numbers. The result in the source domain of the path is symme-

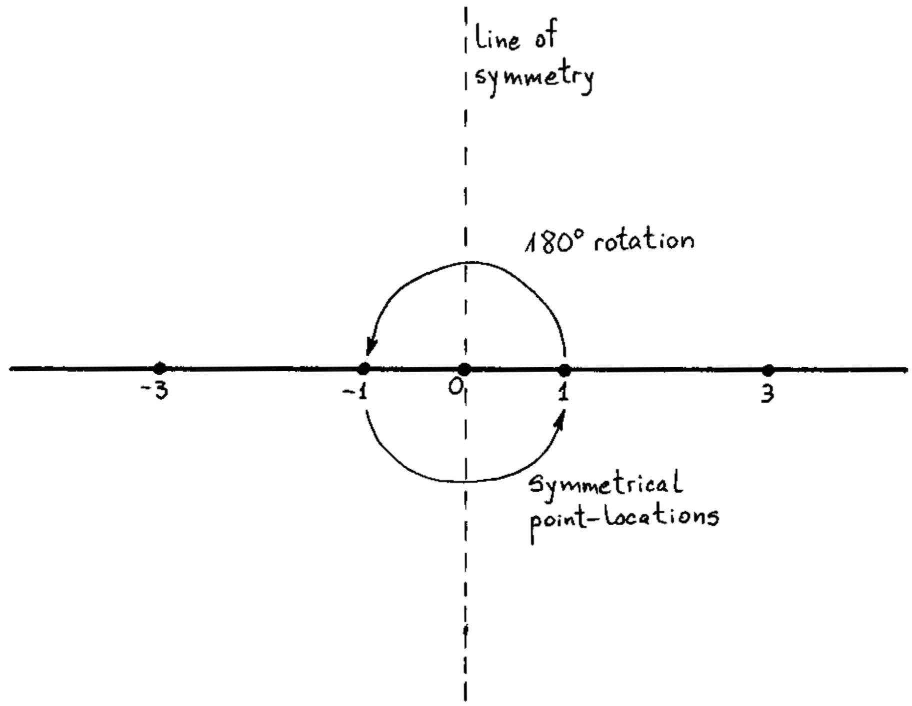


FIGURE 4.2 Mental rotation is a natural cognitive operation. Given that numbers are metaphorically conceptualized as point-locations on a line, rotation gives us a natural way of relating positive and negative numbers: A 180° rotation around zero maps the positive numbers to the corresponding negative numbers, and vice versa. This cognitive operation provides grounding for a metaphor for multiplication by negative numbers: Rotation by 180° Is Multiplication by –1.

try: For every point-location at a given distance on one side of the origin, there is a unique point-location at the same distance on the other side; let us call it the "symmetrical point." Thus, –5 is the symmetrical point of +5, and +5 is the symmetrical point of –5.

Mapping point-locations to numbers in the target domain, we get a symmetry in the number system: For every positive number, there is a unique negative number, and conversely. Since positive and negative numbers are symmetric, we need to distinguish them by picking an orientation: The usual choice is that positive numbers are on the right of the origin and negative numbers are on the left (see Figure 4.2).

The metaphorical mapping for addition must now be changed slightly. Addition of positive numbers will now be conceptualized as moving toward the

right, whereas addition of negative numbers will be moving toward the left. Thus, $3 + (-5)$ will straightforwardly be $-2$. The metaphor for subtraction must be changed accordingly. Subtraction of positive numbers will be moving toward the left, while subtraction of negative numbers will be moving toward the right. Thus, $(-4) - (-6)$ will be $+2$, straightforwardly. It follows from these metaphors that the result of adding a negative number is the same as the result of subtracting a positive number:

$$A + (-B) = A - B.$$

The converse is also entailed:

$$A - (-B) = A + B.$$

Multiplication, however, is not quite so straightforward. Multiplication by positive numbers requires performing an action—moving—a certain number of times. Multiplication of a negative number $-B$ by a positive number $A$ is no problem: You perform repeated addition of $-B$ $A$ times; that is, starting from the Origin, you move $B$ units to the left $A$ times.

But multiplying *by* a negative number is not a simple conceptual extension of multiplying by a positive number. In the source domain of motion, doing something a negative number of times makes no sense. A different metaphor is needed for multiplication by negative numbers. That metaphor must fit the laws of arithmetic—the most basic entailments of the four most basic grounding metaphors. Otherwise, the extension will not be consistent. For example, $5 \cdot (-2)$ does have a metaphorical meaning, as we just saw, and it gives the result that $5 \cdot (-2) = -10$. The commutative law therefore constrains any metaphor for multiplication by negative numbers to the following result: $(-2) \cdot 5 = -10$.

The symmetry between positive and negative numbers motivates a straightforward metaphor for multiplication by $-n$: First, do multiplication by the positive number $n$ and then move (or "rotate" via a mental rotation) to the symmetrical point—the point on the other side of the line at the same distance from the origin. This meets all the requirements imposed by all the laws. Thus, $(-2) \cdot 5 = -10$, because $2 \cdot 5 = 10$ and the symmetrical point of 10 is $-10$. Similarly, $(-2) \cdot (-5) = 10$, because $2 \cdot (-5) = -10$ and the symmetrical point of $-10$ is 10. Moreover, $(-1) \cdot (-1) = 1$, because $1 \cdot (-1) = -1$ and the symmetrical point of $-1$ is 1.

The process we have just described is, from a cognitive perspective, another metaphorical blend. Given the metaphor for multiplication by positive numbers, and given the metaphors for negative numbers and for addition, we form a

blend in which we have both positive and negative numbers, addition for both, and multiplication for only positive numbers. To this conceptual blend we add the new metaphor for multiplication by negative numbers, *which is formulated in terms of the blend!* That is, to state the new metaphor, we must use

- negative numbers as point-locations to the left of the origin,
- addition for positive and negative numbers in terms of movement, and
- multiplication by positive numbers in terms of repeated addition a positive number of times, which results in a point-location.

Only then can we formulate the new metaphor for multiplication by negative numbers using the concept of moving (or rotating) to the symmetrical point-location.

MULTIPLICATION BY –1 IS ROTATION

| *Source Domain* SPACE | | *Target Domain* ARITHMETIC |
|---|---|---|
| Rotation to the symmetry point of $n$ | → | $-1 \cdot n$ |

Given this metaphor, $-n \cdot a = (-1 \cdot n) \cdot a = -1 \cdot (n \cdot a)$, which is conceptualized as a mental rotation to the symmetrical point of $(n \cdot a)$.

### *What Is Stretched*

The metaphor Arithmetic Is Motion Along a Path has a source domain of natural motions: movement in one direction or the other, and iterated movements. Mental rotations around a center are also cognitively natural. Thus the metaphor for multiplication of negative numbers uses a natural cognitive mechanism. However, it does not arise from subitizing and counting as part of some natural activity. Instead, it is added to fit the needs of closure, which are given higher priority than consistency with subitizing or counting.

What is important to understand is the difference between the four basic grounding metaphors and extensions like this one. The 4Gs do arise naturally for natural activities correlated with subitizing and counting. *They are what make the arithmetic of natural numbers natural!* The laws of arithmetic for natural numbers are entailments of those metaphors. None of this is true of the extensions of those metaphors.

### *Further Stretching*

We can characterize division by negative numbers by stretching the 4Gs even further. Here's how.

Consider three cases. Case 1 involves dividing a negative by a negative. This is just like division of a positive by a positive, but done on the negative side of the line. It is simply repeated subtraction, moving in the opposite direction. And the result, which is a number of times, is a positive number. Thus, division of a negative by a negative gives the same result as the corresponding division of a positive by a positive.

Now consider case 2: $(-A) \div B$, where $A$ and $B$ are positive. What answer do we need to get? From the entailments of the four basic metaphors, we know that division is the inverse of multiplication: Whereas multiplication is repeated addition, division is repeated subtraction. Therefore, what we need to get is a $C$ such that $C \cdot B = -A$. From the metaphors for multiplication by negatives, we know that $C$ must be negative. We can now stretch the motion metaphor further and metaphorically define $(-A) \div B$: Perform the simple division of positive by positive for $A \div B$. Then rotate to the symmetrical point. This will give us a consistent result, which is a negative number.

Finally consider case 3: $A \div (-B)$. For the same reason, the answer has to be a $C$ which is a negative number. The stretched metaphor for division will therefore be the same in this case as in the last: Perform the simple division of positive by positive for $A \div B$. Then rotate to the symmetrical point. This will give us a consistent result, which is a negative number.

It is interesting that such metaphorical stretching yields consistent results. Compare this with the situation of division by zero. There is no possible consistent stretching of either the motion metaphor or any of the other grounding metaphors to allow division by zero. The requirements of closure say that the operations of arithmetic should give determinate results for any operation on numbers. But there is no way to metaphorically conceptualize division by zero to get a determinate answer.

Let's look at this in, say, the object-collection metaphor. Here is the metaphor for division (see Chapter 3):

- Division ($C \div B = A$) Is the Splitting Up of a Collection of Size $C$ into $A$ Subcollections of Size $B$.

Take a collection of a particular size $C$. To divide $C$ by zero, you would have to split up the collection into subcollections of size zero—that is, into a unique,

determinate number of empty subcollections exhausting the collection of size *C*. But there is no such unique, determinate number. Moreover, there is no consistent way to extend this metaphor to get such a unique, determinate number. The same situation holds in all the other grounding metaphors.

This *is* an interesting result. The lack of divisibility by zero is a consequence of the lack of any extensions of the four basic grounding metaphors consistent with the laws of arithmetic and that would meet the requirement of closure that the result be a unique, determinate number.

In short, the four basic grounding metaphors are natural and are constitutive of our fundamental understanding of arithmetic. Stretched versions of the 4Gs get more and more contrived as one stretches more and more. But sometimes it is impossible to stretch in a sensible way to fit the requirements of closure and consistency reflected in the notational system.

## Metaphoric Blends and the Effectiveness of Arithmetic in the World

We have now identified the four grounding metaphors that allow us to extend innate arithmetic. We have seen how they lead to the requirements of closure, and we have seen how much of that closure (not including the reals and the imaginaries) is realized via the stretching of those grounding metaphors. Given this conceptual structure, metaphoric blends in which source and target domain are both activated arise naturally. These are blends of arithmetic with object collections, object construction, measuring, and motion. We use such blends when we employ arithmetic in each of those activities. It should not be surprising, therefore, that arithmetic "works" for each of those activities.

*What makes the arithmetic of natural numbers effective in the world are the four basic grounding metaphors used to extend innate arithmetic and the metaphoric blends that arise naturally from those metaphors.*

## Summary

Let us review how arithmetic is "put together" cognitively. We can get an overall picture in terms of answers to a series of questions of the form "Where does *X* come from cognitively?" where *X* includes all the following:

- The natural numbers
- The concept of closure—the idea that the operations of arithmetic always work on numbers to produce other numbers

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- The laws of arithmetic
- Fractions, zero, and negative numbers
- The notation of arithmetic
- Generalizations: Laws of arithmetic work for numbers in general, not just in specific cases. For example, $a + b = b + a$, no matter what $a$ and $b$ are.
- Symbolization and calculation
- The special properties of arithmetic: precision, consistency, and stability across time and communities
- The fact that arithmetic "works" in so much of our experience.

In showing where each of these properties comes from, we answer the big question—namely, Where does arithmetic as a whole come from? Here is a brief summary of the answers we have given.

### *Where Do the Natural Numbers and the Laws of Arithmetic Come From?*

The natural numbers and the laws governing them arise from the following sources:

1. *Subitizing Plus*. We have innate capacities, instantiated neurally in our brains, for subitizing and innate arithmetic—limited versions of addition and subtraction (up to three). We have basic capacities—grouping, ordering, focusing attention, pairing, and so on—that can be combined to do primitive counting.
2. *Primary Experiences.* We have primary experiences with object collection, object construction, physical segmentation, and moving along a path. In functioning in the world, each of these primary experiences is correlated with subitizing, innate arithmetic, and simple counting, as when we automatically and largely unconsciously subitize the number of objects in a collection or automatically count a small number of steps taken.
3. *The Conflation of Subitizing Plus with Primary Experiences*. Such correlations form the experiential basis of the four basic grounding metaphors for arithmetic, in which the structure of each domain (e.g., object collection) is mapped onto the domain of number as structured by subitizing, innate arithmetic, and basic counting experiences. This is what makes the arithmetic of natural numbers natural.

4. *Conflation Among the Primary Experiences.* Object construction (putting objects together) always involves object collection. Placing physical segments end to end is a form of object construction and, hence, of object collection. Trajectories from one point to another are imagined physical segments. Each of these is a conflation of experiences. From a neural perspective, they involve coactivations of those areas of the brain that characterize each of the experiences. And constant coactivation presumably results in neural links. As a consequence, an isomorphic structure emerges across the source domains of the 4Gs. That isomorphic structure *is independent of numbers themselves* and lends stability to arithmetic.
5. *Subitizing and Innate Arithmetic.* Within their range, subitizing and innate arithmetic are precise, consistent, stable, and common to all normal human beings. In those cultures where subitizing, innate arithmetic, and basic counting are correlated with object collection, object construction, and so on, to form the experiential basis of the basic grounding metaphors, those metaphors bring precision, stability, consistency, and universality to arithmetic.
6. *Laws from Entailments of Grounding Metaphors*. The laws of arithmetic (commutativity, associativity, and distributivity) emerge, first, as properties of the source domains of the 4Gs, then as properties of numbers via those metaphors, since the metaphors are inference-preserving conceptual mappings. For example, associativity is a property of the physical addition of the contents of one collection to another. The same collection results from either order. The Numbers Are Object Collections metaphor maps that knowledge onto the associative law for arithmetic.

### *Why Does Arithmetic Fit The World?*

The metaphoric blends of the source and target domains of the 4Gs associate the arithmetic of natural numbers with a huge range of experiences in the world: experiences of collections, structurings of objects, the manipulation of physical segments, and motion. This is the basis of the link between arithmetic and the world as we experience it and function in it. It forms the basis of an explanation for why mathematics "works" in the world.

### *Where Do Fractions, Zero, and Negative Numbers Come From?*

The nonisomorphic portions of the 4Gs provide groundings for other basic concepts in arithmetic: Fractions make sense as parts of wholes in the object-

construction metaphor. Zero and negative numbers make sense as point-locations in the motion metaphor.

### *Why Do We Think of Numbers As Things in the World?*

The Numbers Are Things in the World metaphor arises as a generalization over the 4Gs, since in each one, numbers are things (collections; wholes made from parts; physical segments, like sticks; and locations). This metaphor has important entailments, many of which fit the so-called Platonic philosophy of mathematics:

- Since things in the world have an existence independent of human minds, numbers have an existence independent of human minds. Hence, they are not creations of human beings and they and their properties can be "discovered."
- There is only one true arithmetic, since things in the world have determinate properties.
- Arithmetic is consistent. It has to be, if its objects and their properties have an objective existence.
- Regarding closure, the Numbers Are Things in the World metaphor maps (a) onto (b): (a) Operations on things in the world yield unique, determinate other things in the world of the same kind. (b) Operations on numbers yield unique, determinate other numbers.

In other words, the system of numbers should be closed under arithmetic operations. If it appears not to be, then there must be numbers still to be discovered that would produce this result. Thus, given the natural numbers and their arithmetic, there should be fractions (e.g., 1 divided by 2), zero (e.g., 3 – 3), negative numbers (e.g., 3 – 5), irrational numbers (e.g., $\sqrt{2}$), and imaginary numbers (e.g., $\sqrt{-1}$).

### *Symbolization and Calculation*

Symbolization and calculation are central features of arithmetic. Embodied cognition and metaphor play important roles in both.

- Linear notation is motivated by the sweep of our gaze and our arms.
- Positional notation is motivated by memory constraints and cognitive ease of symbolic calculation, which reduces memory load.

- The numeral-number mappings build on linear and positional notation to provide a notation for the natural numbers by means of a one-to-one conceptual mapping that links Arabic notation with the conceptualization of numbers as polynomial sums.
- The ERFs characterizing the laws of arithmetic, together with the numeral-number mapping, characterize equations that can be used in defining purely symbolic algorithms to mirror rational processes.

### *Where Does Generalizability in Arithmetic Come From?*

Cognitive frames work at a general level within human conceptual systems. They specify general roles that can be filled by a member of a category. For example, in the travel frame, the role of traveler can be filled by any person physically able to travel. In mathematics this is accomplished through the Fundamental Metonymy of Algebra (see Chapter 3).

Conceptual metaphor, as it functions across the entire human conceptual system, also operates at a general level. For example, in the Love Is a Journey metaphor, where lovers are conceptualized as travelers, *any* lovers (not just particular ones) can be conceptualized as travelers. Similarly, inferences about travelers in general apply to lovers in general.

These facts about conceptual frames and metaphors have a profound consequence in arithmetic. The basic grounding metaphors for natural numbers apply to *all* natural numbers. The laws of arithmetic, which are entailments of those basic grounding metaphors, therefore apply to *all* natural numbers. It is this property of metaphor, together with the Fundamental Metonymy of Algebra, that makes arithmetic generalizable, since metaphor is used to ground arithmetic in everyday experience.

### *The Centrality of Calculation*

Calculation is the backbone of mathematics. It has been carefully designed for overall consistency. The algorithmic structure of arithmetic has been carefully put together to mirror rational processes and to be usable when those processes are disengaged. The mechanisms required are

- Equivalent Result Frames, true of the physical source domains
- the 4Gs, which map ERFs from physical source domains onto arithmetic

- the numeral-number mapping, which maps ERFs in arithmetic onto equations in symbolic form, which can be used in calculations, where one side of the equation is replaced by the other, yielding an equivalent result.

### *The Superstructure of Basic Arithmetic*

The elements required for the conceptual system of basic arithmetic are as follows:

- The innate neural structures that allow us to subitize and do basic concrete addition and subtraction up to about four items
- Basic experiences that form the basis of metaphors, like subitizing while (a) forming collections, (b) putting objects together, (c) manipulating physical segments (fingers, arms, feet, sticks, strings), (d) moving in space, (e) stretching things
- Grounding metaphors, like the 4Gs, and their inferences, such as the basic laws of arithmetic and closure
- Extensions of metaphors to satisfy the requirements of closure
- Basic arithmetic concepts, like number; the operations of addition, subtraction, multiplication, and division; identities and inverses under operations; decomposition of numbers into wholes and parts (factoring); Equivalent Result Frames (equations) characterizing which combinations of operations on the same elements yield the same results
- Symbolization: The mapping of symbols to concepts. For example, 0 maps to the additive identity, 1 maps to the multiplicative identity, $-n$ maps to the additive inverse of $n$, $1/n$ maps to the multiplicative inverse of $n$, + maps to addition, and so on.
- Numerals: Particular symbols for particular numbers
- Calculations: Algorithms using symbols and numerical ERF equations for performing calculations.

So far we have discussed just basic arithmetic. When we start discussing more advanced forms of mathematics, additional cognitive mechanisms will enter the picture. The most prominent are:

- Linking metaphors, which conceptualize one domain of mathematics in terms of another

- Arithmetization metaphors—a special case of linking metaphors, those that conceptualize ideas in other domains of mathematics in terms of arithmetic
- Foundational metaphors, which choose one domain of mathematics (say, set theory or category theory) as fundamental and, via metaphor, "reduce" other branches of mathematics to that branch.

The ideas discussed here will recur throughout the book. But before we move on, we need to bring up an important issue.

## How Do We Know?

To those unfamiliar with the methodology of cognitive linguistics, it will not be obvious how we arrived at the metaphorical mappings given throughout the last chapter and this one, and how we know that they work.

The various branches of cognitive science use a wide range of methodologies. In cognitive neuroscience and neuropsychology, there are PET scans and fMRIs, the study of the deficiencies of brain-damaged patients, animal experimentation, and so on. In developmental psychology, there is, first of all, careful observation, but also experiments where such things as sucking rates and staring are measured. In cognitive psychology, there is model building plus a relatively small number of experimental paradigms (e.g., priming studies) to gather data. In cognitive linguistics, the main technique is building models that generalize over the data. In the study of one's native language, the data may be clear and overwhelmingly extensive, although common techniques of data gathering (e.g., texts, recordings of discourse) may supplement the data. In the study of other languages, there are field techniques and techniques of text gathering and analysis. As in other sciences, one feels safe in one's conclusions when the methodologies give convergent results.

In this study, we are building on results about innate mathematics from neuroscience, cognitive psychology, and developmental psychology. The analyses given in this book must also mesh with what is generally known about the human brain and mind. In particular, these results must mesh with findings about human conceptual systems in general. As we discussed in Chapter 2, the human conceptual system is known to be embodied in certain ways. The general mechanisms found in the study of human conceptual systems are radial categories, image schemas, frames, conceptual metaphors, conceptual blends and so on.

Much of this book is concerned with conceptual metaphors. In Chapter 2, we cited ten sources of convergent evidence for the existence of conceptual metaphor

in everyday thought and language. These metaphor studies mesh with studies showing that the conceptual system is embodied—that is, shaped by the structure of our brains, our bodies, and everyday interactions in the world. In particular they show that abstract concepts are grounded, via metaphorical mappings, in the sensory-motor system and that abstract inferences, for the most part, are metaphorical projections of sensory-motor inferences.

Our job in this chapter and throughout the book is to make the case that human mathematical reason works in roughly the same way as other forms of abstract reason—that is, via sensory-motor grounding and metaphorical projection. This is not an easy job. We must propose plausible ultimate embodied groundings for mathematics together with plausible metaphorical mappings. The hypothesized groundings must have just the right inferential structure and the hypothesized metaphors must have just the right mapping structure to account for *all* the relevant mathematical inferences and all the properties of the branch of mathematics studied. Those are the data to be accounted for. For example, in this chapter, the data to be accounted for include the basic properties of arithmetic (given by the laws) and all the computational inferences of arithmetic. That is a huge, complex, and extremely precise body of data.

We made use of the model-building methodology to account for this data. The constraints on the models used were as follows:

1. The grounding metaphors must be plausible; that is, they arise via conflation in everyday experience (see Chapter 2).
2. Given the sensory-motor source domain and the mappings, all the properties and computational inferences about the mathematical target domain must follow.
3. In the case of arithmetic, the analysis must fit and extend what is known about innate arithmetic.
4. The models must be maximally general.
5. The models must accord with what is generally known about embodied cognition; that is, they must be able to fit with what is known about human brains and minds.

These constraints are so demanding that, at first, we found it difficult to come up with any models at all that fit within them.

Indeed, in studying arithmetic, for example, we depended on the prior research of Ming Ming Chiu (1996). Chiu's dissertation set out some first approximations that met a number of these constraints. Starting there, we made many successive revisions until the constraints were met.

To the novice in metaphor theory it may not be obvious that not just any metaphorical mapping will meet these constraints. Indeed, hardly any do. For example, arithmetic is not an apple. The structure of an apple—growing on trees; having a skin, flesh, and a core with seeds; being about the size to fit in your hand, having colors ranging from green to red; and so on—does not map via any plausible metaphor (or any at all) onto arithmetic. The inferences about apples just do not map onto the inferences about numbers. Apples won't do as a source domain, nor will couches, churches, clouds, or most things we experience in the everyday physical world.

As we have seen, there are four plausible grounding domains—forming collections, putting objects together, using measuring sticks, and moving through space. Each of them forms just the right kind of conflation with innate arithmetic to give rise to just the right metaphorical mappings so that the inferences of the source domains will map correctly onto arithmetic—almost. Only two of them have an equivalent to zero.

But that isn't good enough. The constraints on an analysis are so stringent that an additional metaphor is necessary to extend the system to negative numbers. That metaphor makes use of a known cognitive mechanism—mental rotation. Moreover, these metaphors must have just the right properties to account for the properties of arithmetic—its ability to fit experience, its stability over time, its universal accessibility, its combinatorial possibilities, and so on.

In short, mathematics provides a formidable challenge to the model-building methodology of cognitive science, because the constraints on the possible models are so severe. The biggest surprise in our research to date is that we have been able to get as far as we have.

The metaphors given so far are called grounding metaphors because they directly link a domain of sensory-motor experience to a mathematical domain. But as we shall see in the chapters to come, abstract mathematics goes beyond *direct* grounding. The most basic forms of mathematics are directly grounded. Mathematics then uses other conceptual metaphors and conceptual blends to link one branch of mathematics to another. By means of linking metaphors, branches of mathematics that have direct grounding are extended to branches that have only indirect grounding. The more indirect the grounding in experience, the more "abstract" the mathematics is. Yet ultimately the entire edifice of mathematics does appear to have a bodily grounding, and the mechanisms linking abstract mathematics to that experiential grounding are conceptual metaphor and conceptual blending.

In addition, a certain aspect of our linguistic capacities is used in mathematics—namely, the capacity for symbolizing. By this we mean the capacity for as-

sociating written symbols (and their phonological representations) with mathematical ideas. This is just one aspect of our linguistic capacities, but it is anything but trivial. As in natural languages, mathematical symbols can be polysemous; that is, they can have multiple, systematically associated meanings. For example, + is used not only for addition but for set union and other algebraic operations with the properties of addition. 1 and 0 are sometimes used to mean true and false—and as we shall see, it is no accident that 1 is used for true and 0 for false and not the reverse. Similarly, it is no accident that + rather than, say, √ is used for set union. As we shall also see, these uses of symbols accord with the metaphorical structure of our system of mathematical ideas.

We are now in a position to move from basic arithmetic to more sophisticated mathematics.