Reply to Bonnie Gold's Review

Reply to Bonnie Gold's review of
"Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being"

by George Lakoff and Rafael E. Núñez.

With great interest we have read the review of our book by Bonnie Gold that appeared in the MAA online book review column. It is a serious and professional review and we are glad that she found our book "unique and fascinating" and "a significant contribution to our understanding to mathematics' relation to people." We also deeply appreciate her efforts to understand our work (which is not within the field of mathematics itself) and to try to describe it accurately to an audience of mathematicians. That is certainly not an easy task for someone outside the field of cognitive science. We think she has done a good job in trying to explain some of the main ideas of the "new discipline of intellectual inquiry" we introduce. We find, however, that important misunderstandings of the cognitive science occur in the review, misunderstandings referred to as "flaws," "mathematical errors," and "defects." We think that these are serious misunderstandings and they need to be clarified - all the more so because of the serious purpose and mathematical competence of the reviewer and amount of effort put into the review (all of which we praise highly).

First, there is a major misunderstanding from which many misinterpretations follow. We have the impression that Gold didn't really get the main thesis of our book: It is the human embodied mind that brings mathematics into being. This is precisely what the subtitle of the book explicitly indicates. "Embodiment" thus, with its strong biological and cognitive constraints, is a fundamental theoretical component that gives shape and continuity to the entire book. It is under this view that the cognitive mechanisms we describe, among which conceptual metaphor is one of the most important ones, make sense at all: Conceptual metaphor has explanatory power precisely because they are empirically observable embodied mechanisms that satisfy strong biological and cognitive constraints. The term "embodiment" and its related concepts appear all over the book, to the point that both summarizing theoretical chapters at the end are entitled "The Theory of Embodied Mathematics" (Chapter 15) and "The Philosophy of Embodied Mathematics" (Chapter 16). In Gold's review, however, the term embodiment is not even mentioned once!

This is not just picky terminology. Understanding the role of embodiment and its biological and cognitive constraints is extremely important to get an idea of what our book is about. So when Gold refers to conceptual metaphors as being "essentially isomorphisms" she presents an inaccurate picture, which leads her to miss some fundamental points of the book, as well as the continuity and overall structure of it. First, isomorphisms are mathematical entities, while conceptual metaphors are not. Second, as mathematical entities within a mathematical subject matter, isomorphisms don't have to satisfy any scientific (and therefore empirical) constraints. Of course, within certain scientific applications, they can be applied to satisfy constraints, but they don't have to.

Conceptual metaphors, on the contrary, are empirically observable mechanisms of the human mind, which as explanatory constructs must satisfy strong biological and cognitive constraints. With a conceptual metaphor you have to explain data observed empirically, while with isomorphisms you don't necessarily have to. It is simply a confusion between disciplines to refer to conceptual metaphors as "isomorphisms." Our book is within the discipline of cognitive science and its subject matter is the cognitive science of mathematical ideas. To refer to conceptual metaphors as isomorphisms is to assume that the book is within the discipline of mathematics, which it is not. Our book is an attempt to give an account of mathematical ideas and inferences in terms of biologically and cognitively plausible mechanisms of the human mind, such as conceptual metaphors.

For example, the conceptual metaphor Arithmetic As Object Collection to which Gold refers is not a mere descriptive isomorphism. It is the embodied cognitive mechanism that gives an account of why the empirically observed expressions that exist in human (even technical) communication such as "three is bigger than two" or "four is smaller than eight" have the precise meaning they have, despite the fact that numbers in themselves don't have size. We believe that, because Gold misses the deep role of embodiment in our theoretical account throughout the book, she sees our characterization of "human mathematics" as "brief" and "not eloquent" enough, and having "little direct connection with the rest of the book."

We would also like to clarify a couple of technical details regarding how conceptual metaphors work. Gold says, correctly, that the Basic Metaphor of Infinity (BMI) is the most important metaphor in the book, and she points out (also correctly) that the BMI is not an isomorphism (that's right!). Gold accurately observes that the Basic Metaphor of Infinity characterizes "something genuinely new" (i.e., an end to an unending process: actual infinity). Unfortunately, she seems not to understand what conceptual metaphors are and how they differ in kind from disembodied mathematical isomorphisms (which are literal, not metaphorical). As a result, she mistakenly claims that the BMI introduces an "ambiguity of how to go from the intermediate states to the final state", leaving "a gap that needs more explanation." It is incorrectly taking conceptual metaphors to be mathematical isomorphisms that generate that gap. Conceptual metaphors, being human cognitive mechanisms have many properties not captured by isomorphisms. As we say it explicitly in pages 45 and 46, "conceptual metaphors do not just map preexisting elements of the source domain onto preexisting elements of the target domain. They can also introduce new elements into the target domain" (italics in the book). These elements are not inherent to the target domain. In the BMI case, an end is not inherent to the target domain constituted by an unending process that goes on and on. It is the BMI that brings forth this new metaphorical entity: an ending to an unending process. Asking the question of how (exactly) to go from the intermediate states to the final state is a question that belongs to the realm of literal, not metaphorical processes. We give a simple example on page 46.

The moral here is this: It is totally consistent with what we know about human cognitive mechanisms that actual infinity could be a metaphorical idea. Via a specific conceptual metaphor (the BMI), an unending iterative process that goes on and on can be conceptualized as a process with an actual end and an actual final resultant state (which are precisely the elements not inherent to the target domain of unending processes).

Now, regarding the "mathematical errors" mentioned by Gold, she is in some extent right. There are some errors in the text of the first printing. But there are several things to say about them. First, several of these mistakes are editorial errors (not ours) that unfortunately affect mathematical content, which may mislead a careful reader, especially those who are mathematically trained (e.g., the definition of limits in p. 199). These editorial mistakes (and others) have been corrected in the errata section of the website for our book, which we set up soon after the publication of the book (www.unifr.ch/perso/nunezr). Second, there are indeed some passages in which we stated things in a sloppy way and we apologize (e.g., the characterization of the infinitesimals through the BMI in page 228, correctly pointed out by Gold). We agree with Gold, that these passages "can be fixed, of course, by being a bit more detailed." We will fix them in the next printing and place a corrected version on our website as soon as possible. Some corrections are already there. In any case, to our knowledge, none of these errors affect the substance of our arguments.

Finally, there is a whole group of concepts and analyses we present that are incorrectly called "mathematical errors" because they are taken as mathematical analyses and not as cognitive analyses. We think that, because of this important misunderstanding, Gold has made the judgment that "the book deteriorates from the BMI onward." For example she says that we "decide" to use a sequential definition of limit in which we don't "follow a standard treatment of this topic." The problem is that the kind of standard treatment Gold refers to is mathematical in nature, not cognitive. Our job is not to improve mathematics (how could we possibly do that?). Nor is it our job to duplicate what occurs in standard math texts (that isn't cognitive science!). And it is certainly not our job to tell mathematicians how to do mathematics (just as it not the job of a zoologist to tell a bird how to fly!). As cognitive scientists our job is to give cognitive accounts of largely unconscious mechanisms of mind used to characterize ideas - ideas of all sorts, both mathematical and otherwise.

Therefore, when it comes to limits, we ask different questions than mathematicians do. It is part of our job to understand ideas such as "the limit of f(x) as x approaches a." As cognitive scientists, we have to answer questions like: From a cognitive perspective, what is approaching what? What is moving? From where to where? In what (cognitive) space? And so on. These are questions about a human understanding of a humanly created idea: approaching a limit. This is inherently a dynamic idea, not captured by the standard mathematical treatment using a static logical expression with epsilons and deltas. Thus, characterizing limits in terms of dynamic sequences is not a "decision" we make; rather it is the reality of how people think, a cognitive reality that we must explain in cognitively plausible terms.

Gold's misunderstanding of our goals and intentions can be more clearly seen through her interpretation of what we have called "granular numbers." Nowhere in our book do we say that "the granular numbers are a new mathematical object that we have discovered" (how could we possibly make such a statement!). To begin with, "discovering" such a thing would be inconsistent with the non-platonic nature of embodied mathematics we endorse. What we say instead is that we have "invented the granulars by applying the BMI" (p. 254). This is overtly not a process within formal mathematics. It is the use of a cognitive process for creating mathematical ideas. In this case, we use everyday cognitive mechanisms such as the BMI together with the inferential structure of the idea of "speck" (which we hypothesize as the everyday idea that was the inspiration for Leibniz's idea of infinitesimal).

It is in the realm of this cognitive exercise that we can affirm that the BMI produces "the first infinitesimal" (p. 235). This is not a mathematical result in the classic sense (i.e., it is not a result obtained by proving a mathematical theorem). Therefore it is not a "mathematical error" to say such a thing. "The first infinitesimal" is a consequence of the inferential structure of the BMI when applied to the particular case being discussed. The fact that this infinitesimal is the "first infinitesimal," is an entailment of the metaphor, which generates a unique final resultant state, with no prior resultant state of that kind. That is why it is "first." For all these reasons, Gold is incorrect when she says on page 254 that we "discuss why mathematicians could have been so blind as not to have found them [the granulars] sooner." In that passage we ask an entirely different question: Why didn't the remarkable mathematicians who have worked on the hyperreals develop such a number system when they could have done it easily? On the same page we give our answer, which doesn't have anything to do with considering these mathematicians "so blind as not to have found them [the granulars] sooner." That view simply does not fit with the respect bordering on awe that we have for those extraordinary mathematicians.

We hope these clarifications help mathematicians to understand (and to enjoy!) our book better. We are aware that it is not a simple task to try to follow a cognitive analysis of mathematical ideas when one has been trained as a professional mathematician, which is probably the case of most readers of this MAA column. We are also aware, and here we agree with Gold, that "most first attempts to introduce a new discipline involve some important insights but also some stumbling in the dark." So, it is true that we are just starting this enterprise, and therefore there are still many unclear components. We see the cognitive science of mathematics as a multidisciplinary field, and we certainly need all the help we can get from professional mathematicians. We have to keep in mind, however, that our goal is to characterize mathematics in terms of cognitive mechanisms, not in terms of mathematics itself, e.g., formal definitions, axioms, and so on. Indeed, part of our job is to characterize how such formal definitions and axioms are themselves understood in embodied cognitive terms.

We simply have a different job than professional mathematicians have. We have to answer such questions as: How can a number express a concept? How can mathematical formulas and equations express general ideas that occur outside of mathematics, ideas like recurrence, change, proportions, self-regulating processes, and so on? How do ideas within mathematics differ from similar (but not identical) ideas outside mathematics (e.g., the idea of "space" or "continuity")? How can "abstract" mathematics be understood? What cognitive mechanisms are used in mathematical understanding?

We hope that such questions asked and answered from outside of mathematics proper will interest mathematicians. And we hope that they will not be mistaken for questions and answers within mathematics.

Finally, we want to thank Bonnie Gold once more for her review, and to thank the MAA online book review column for finding a reviewer of such mathematical competence and with the openness and energy required for such an undertaking. We have the greatest respect for her. We are all too aware of how much effort goes into such a review and how misunderstandings can arise naturally across disciplines. It is only through a forum such as this that such issues can be aired in a spirit of cooperation and honest inquiry.

George Lakoff and Rafael E. Núñez