

CHAPTER 11

GESTURE, INSCRIPTIONS, AND ABSTRACTION

The Embodied Nature of Mathematics or Why Mathematics Education Shouldn't Leave the Math Untouched

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An essential question in mathematics education is how to improve the teaching and learning of mathematics. A tremendous amount of efforts and resources are dedicated to provide answers to this question, from curriculum planning and teacher development, to textbook design, software development, evaluation methods, and classroom dynamics. In the quest for providing answers, mathematics education, unlike other domains of teaching and learning, usually proceeds leaving the very subject matter—mathematics—untouched. Whereas other domains in education such as music, language, and literature, implicitly and naturally see the human nature of the subject matter involved in the teaching, mathematics education does not. An important factor is the widely spread view in our culture that mathematics is a transcendently objective body of knowledge,

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which exists independently of human beings, or, at best, that it is the “only truly universal language.”¹ This view is endorsed—unquestioned—by many in the academic world as well as in pop culture (e.g., film, advertising, and video game industries). Thus, mathematical facts, theorems, definitions, proofs, notations, and so on, are largely taken as pre-given disembodied facts, external to human beings (e.g., Núñez, 2005). Consequently, most school mathematics is taught, generation after generation, in a relatively dogmatic form, where the very mathematical facts are rarely (or never) questioned. And I am not exaggerating. Simply think of the simple “rule” we all learn at school that “negative times negative yields positive.” We all become more or less pretty good at using the rule. But are we able to explain what is the *meaning* of such statement? Or, *why* is this rule true? Or, *what* makes it true? What do you, as a reader interested in education, have to say about this question? *Why* is the above statement true, and what does it *mean*?

The fact is that most educated people I have talked to, have no clear idea of what such rule really means, or why is this rule “true.” Moreover, most people I have talked to say that teachers never told them the why’s. And it appears that the educational system has teachers teaching this “truth” by simply following curriculum-driven textbooks that do not mention the underlying meaningful *why*’s. This is hardly an isolated example. It is the rule rather than the exception. So, why is it that mathematics education does not get *at* the mathematics *itself*, and teaches its contents in a way that is friendly to known forms of human sense making?

In this chapter I argue that contemporary mathematics education would take an important step forward if, in the process of searching how to improve the teaching and learning of mathematics, it addressed *meaning* in a more fundamental way, and if it actually got into the mathematics itself (see also Davis’ chapter 9). I suggest that one way of doing so is by informing its goals and procedures with relatively recent developments in the cognitive science *of* mathematics, that is, the scientific study of what mathematics ideas and facts are and what mechanisms of human imagination make them possible. Taking a case study from foundational ideas in calculus, such as continuity of functions, I focus on embodied cognitive mechanisms such as image schemas, conceptual metaphor, notation systems, and co-speech gesture production.

WHAT MAKES THE TEACHING OF CONTINUITY DIFFICULT?

To illustrate my arguments let us take a look at a specific mathematical content that in mathematics education is known for being elusive, difficult for teachers to teach and for students to learn: limits and continuity of

functions (e.g., Freudenthal, 1973; Núñez, Edwards, & Matos, 1996). Are these concepts intrinsically difficult to learn? And if yes, why are they difficult? Is it that the methods used to teach them are not appropriate? Or is there something cognitively unfriendly about the very mathematical ideas themselves that needs to be understood?

Let us start by asking what is continuity. The usual definition of continuity we find in textbooks reads as follows:

- A function f is continuous at a number a if the following three conditions are satisfied:
 1. f is defined on an open interval containing a ,
 2. $\lim_{x \rightarrow a} f(x)$ exists, and
 3. $\lim_{x \rightarrow a} f(x) = f(a)$.

Where by $\lim_{x \rightarrow a} f(x)$ what is meant is the following:

Let a function f be defined on an open interval containing a , except possibly at a itself, and let L be a real number. The statement

$$\lim_{x \rightarrow a} f(x) = L$$

means that $\forall \varepsilon > 0, \exists \delta > 0$,

such that if $0 < |x - a| < \delta$,

then $|f(x) - L| < \varepsilon$.

We can see that formal mathematics defines continuity in terms of limits, and limits in terms of expressions that use universal and existential quantifiers over real numbers (e.g., $\forall \varepsilon > 0, \exists \delta > 0$), and the satisfaction of certain conditions which are described in terms of arithmetic difference (e.g., $|f(x) - L|$) and smaller than relations (e.g., $0 < |x - a| < \delta$). The statements are clear and unambiguous. But, why do they create difficulties in students' minds?

An important part of the mathematics education community attributes the difficulties of teaching and learning the concept of continuity to problems related to the use of existential and universal quantifiers. Nowhere is this position clearer than in the work of the famous mathematics educator Hans Freudenthal (1973), who writes:

The difficulties implicit in the continuity concept are quantifiers and the order of quantifiers of different kinds.... Continuity of f means intuitively: small changes of x correspond with small changes of $f(x)$. Or: if x changes little, $f(x)$, also changes little. Words like "small," "big," "little," "much," "short," "long," may hide a quantifier, but formal linguistic criteria are often

insufficient to know which kind. Always, sometimes, everywhere, somewhere—exhibit clearly the universal or existential quantifier, but the linguistic formulation does not unveil that in the continuity definition the second small (or little) hides a universal, and the first an existential, quantifier. To grasp it, a logical analysis is badly needed. The meaning of the quantifiers in “small” or “little” is better indicated in the more exact formulation: to sufficiently small changes of x correspond arbitrarily small ones of $f(x)$. Or: if x changes sufficiently little, $f(x)$ changes arbitrarily little. Still from this formulation it is a long step to understand that first the “arbitrarily little” must be prescribed before the “sufficiently little” is to be determined.... The intuitive continuity definition involves two difficulties of formalizing—first, decoding of hidden quantifiers, second, settling the order of quantifiers of different kinds. Good didactics should at least separate these difficulties from each other. (p. 561)

In the pages following this citation, Freudenthal provides an insightful analysis of these difficulties and gives helpful recommendations for achieving good didactics regarding the use of quantifiers. Whereas his analysis is precise, deep, and clear, it only focuses on the formal aspects of the ε - δ definition, missing the fundamental dimensions of everyday human cognition that may be interfering with the conceptualization and understanding of the very formalization. Freudenthal (like many in mathematics education) takes the ε - δ definition for granted, thus perpetuating the common belief that you can teach mathematics while leaving it somehow untouched or unquestioned. His analysis *starts* at the level of the formalization. This, to the point that for him “Continuity of f means intuitively: small changes of x correspond with small changes of $f(x)$.” The statement is indeed a very good linguistic formulation of what is intuitive in the ε - δ definition. It has the right concepts relative to *that* definition, and it provides the right relations between them. What is missing, however, is precisely the need of questioning the very mathematics itself. That is, questioning the very mathematization of the everyday notion of continuity—*natural continuity* (Núñez & Lakoff, 1998), which underlie the technical notion expressed in ε - δ terms. As we shall see, a detailed cognitive analysis of the semantic organization of the ideas of *natural continuity* (present in everyday cognition) and ε - δ continuity (the technical definition stated above) reveals that, beyond the problem with quantifiers, these two ideas are in fact *different*, with different semantic structure and inferential organization. Not only that, these two ideas are in fact cognitive opposites: natural continuity is intrinsically holistic and dynamic, while ε - δ continuity is atomistic and static. This suggests that learning, and getting good at mastering the ε - δ technical characterization of continuity requires a considerable extra effort that goes beyond the use of quantifiers, which mathematics education should be able to address and account for. But for

this, mathematics education must get at the very mathematics itself, and not take the mathematics and its formalizations for granted.

MOTION, STASIS, AND FORMALIZATION

A close inspection of mathematics textbooks reveals that usually right before giving the above formal ϵ - δ definition of continuity, a paragraph or two are dedicated to the so-called “informal” characterization of the idea of continuity, one that appeals to an “intuitive” description. For instance, the classic Russian book *Matematika, ee sodержanie metody i znachenie* [Mathematics, its Contents, Methods and Meaning] by A. Aleksandrov, A. N., Kolmogorov, and M. A. Lavrent’ev (1956/1999) says: “The general idea of a continuous function may be obtained from the fact that its graph is *continuous*: that is, its curve may be drawn without lifting the pencil from the paper” (p. 88, emphasis added). And a standard calculus textbook, while discussing the same topic, says: “In everyday speech a ‘continuous’ process is one that *proceeds without gaps* or interruptions or sudden changes. Roughly speaking, a function $y = f(x)$ is continuous if it displays similar behavior” (Simmons, 1985, p. 58, emphasis added).

In both texts, we observe a characterization of continuous functions given in *dynamic* terms. In both cases there is something *moving*: the pencil drawing a curve on the paper in the former, and something unfolding without gaps in the latter. In both cases we have something moving from some position in space towards some other location in an uninterrupted manner. In both books these dynamic descriptions are given as a way of helping the reader by providing some immediate intuitive idea of what a continuous function *means*. The Russian book even characterizes the meaning of a “continuous function” *in terms of* something that is “continuous,” whose meaning corresponds to what Simmons’ Calculus textbook characterize as “everyday speech.” At this point, right after setting this introductory presentation of continuous functions in dynamic terms, textbooks usually make a radical move. In a somewhat downgrading tone they make clear that the “intuitive” examples given so far are “merely illustrative,” that they are not precise enough, and that a rigorous formal definition is required. Simmons’ (1985) textbook, for instance, says: “Up to this stage our remarks about continuity have been rather loose and intuitive, and intended more to *explain* than to *define*” (p. 58, emphasis added).

This is a remarkable and profoundly informative passage. The choice of the words “explain” and “define” is not random. It characterizes the widespread idea in mathematics education that “explaining” may be a good thing, but what mathematics teaching really is about, is in “defining” entities and properties in a rigorous and precise way (in this

case, presumably through the use of existential and universal quantifiers over real numbers and by establishing precise inequalities). From the perspective of cognitive science, teaching focusing on defining rather than on explaining goes against most of what is known about how humans learn and make sense of things, from perception, attention, and memory, to categorization and problem solving.

Natural Continuity, the Source-Path-Goal Schema, and ε - δ Continuity

From textbooks such as those cited above, we can see that mathematics education makes the following claims:

1. the ε - δ definition of continuity makes the so-called “informal” conception of continuity (i.e., natural continuity) *rigorous* and *precise*.
2. the ε - δ definition of continuity *generalizes* the so-called “informal” conception of continuity (i.e., natural continuity).

Let us analyze these claims separately. First, is natural continuity rigorous and precise? The so-called “informal” conception of continuity (natural continuity) is the one conceived by the creators of calculus Leibniz and Newton in the seventeenth century, and in fact, all mathematicians up to the nineteenth century (and, no surprise here, it is this notion of continuity that is brought forth over and over by students and teachers into the classroom today). It is natural continuity that brought Euler to refer to a continuous curve as “a curve described by freely leading the hand” (cited in Stewart, 1995, p. 237), and Kepler to measure “an area swept out by the motion of a (celestial) point on a physical ‘continuous curve’ ” (Kramer, 1970, p. 528). Natural continuity—continuity as we normally conceive it outside of mathematics—is based on a *source-path-goal schema*, a fundamental cognitive schema concerned with simple motion along trajectories which has the following elements (Lakoff & Núñez, 2000): (a) a trajector that moves; (b) a source location (the starting point); (c) a goal—that is, an intended destination of the trajectory; (d) a route from the source to the goal; (e) the actual trajectory of motion; (f) the position of the trajector at a given time; (g) the direction of the trajector at that time; and (h) the actual final location of the trajector, which may or may not be the intended destination.

The source-path-goal schema is quite generic in nature and can be extended in many ways: the speed of motion, the trail left by the thing moving, the scale of motion, obstacles to motion, forces that move one along a trajectory, additional trajectors, and so on. The schema is topo-

logical in the sense that a path can be expanded, shrunk, or deformed and still remain a path and it has an internal spatial logic and built-in inferences (see Figure 11.1). For instance, if a trajector has traversed a path to a current location, it has been at all previous locations on that path; if the trajector moves from *A* to *B* and from *B* to *C*, then it has traveled from *A* to *C*; if there is a direct route from *A* to *B* and the trajector is moving along that path toward *B*, then it will keep getting closer to *B*; if *X* and *Y* are traveling along a direct path from *A* to *B* and *X* passes *Y*, then *X* is further from *A* and closer to *B* than *Y* is; and so on. As we can see, the ensemble of entailments involving the source-path-goal schema is very precise (i.e., the entailments themselves are not ambiguous).

Natural continuity builds on the source-path-goal schema, and as a result has the following essential features in its inferential organization (Núñez & Lakoff, 1998): (a) continuity, traced by motion, takes place over time; and (b) the trace of the motion is a static holistic line with no “jumps.” If we take all these properties together, we can see that in terms of inferential organization, natural continuity is quite precise. The list of entailments is rich, specific, and unambiguous. In fact the level of

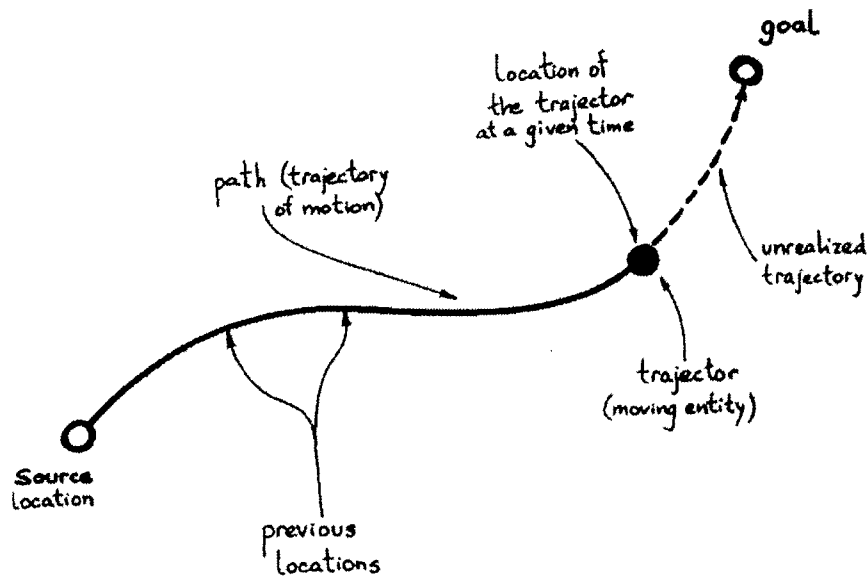


Figure 11.1. The Source-Path-Goal schema. We conceptualize linear motion using a conceptual schema in which there is a moving entity (called a trajector), a source of motion, a trajectory of motion (called a path), and a goal with an unrealized trajectory approaching that goal. There is a logic inherent in the structure of the schema. For example, if the trajector is at a given location on a path, then the trajector has been at all previous locations on that path (Lakoff & Núñez, 2000).

precision is such that it constituted an essential building block for the invention of calculus in the seventeenth century. How about rigor? In terms of rigor, natural continuity is characterizable in terms of the source-path-goal schema with all required rigor, and it is able to handle all mathematical objects existing up to the nineteenth century (for details, see Núñez & Lakoff, 1998). Now, it is true that around the mid-1800s new, more elaborated mathematical objects emerged (e.g., “pathological” functions) that pushed for new methods to handle them. As a result the work of Cauchy, Weierstrass, Dedekind and others brought forth the ε - δ method with the specific goal of banning motion (thought to be intuitive but misleading and fallible) from the foundation of analysis. But, the point is that these were *new methods* (and very effective ones!) especially concocted for handling the cases natural continuity already handled *plus* the new untractable cases. These new methods recruited different cognitive mechanisms for its realization, such as the ideas of *preservation of closeness* and *gaplessness*. The ε - δ definition thus did not add rigor and precision to natural continuity, but rather it created a new rigorous and precise method for handling a larger universe of functions and mathematical entities. This, of course, is a great accomplishment in mathematics. But, providing a new method with new underlying ideas does not necessarily mean, cognitively, that there is a generalization or an extension of the old ideas. Let us analyze this issue in more detail.

Does the ε - δ Continuity Generalize the Notion of Natural Continuity?

The ε - δ continuity does not have any of the essential properties of natural continuity described above. It defines continuity in terms of limits, and limits in terms of expressions that use *static* universal and existential quantifiers over *static* real numbers (e.g., $\forall \varepsilon > 0, \exists \delta > 0$), and the satisfaction of certain conditions which are described in terms of *motion-less* arithmetic difference (e.g., $|f(x) - L|$) and smaller than relations (e.g., $0 < |x - a| < \delta$). Moreover, ε - δ continuity predicates on atomic entities (i.e., discrete points) rather than on holistic ones, such as lines. The nature of these two ideas, natural continuity and ε - δ continuity, is just radically different. In fact, with a relatively simple example, we can see that the ε - δ definition of continuity *does not* characterize the intuitive meaning of natural continuity, and in this sense it does not generalize natural continuity.

Consider the function $f(x) = x \sin 1/x$ whose graph is depicted in Figure 11.2.

$$f(x) = \begin{cases} x \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

According to the ε - δ definition of continuity this function is continuous at every point. Indeed, for all x , it is always possible to find the specified ε 's and δ 's to satisfy the conditions for preservation of closeness (i.e., conditions expressed in static statements that indicate position within a range of closeness). However, according to *natural continuity* this function is *not* continuous. The inferential organization of natural continuity requires that certain conditions have to be met. For instance, in the semantics of a naturally continuous line we should be able to tell how long the line is between two points. We should also be able to describe essential properties of the motion of a point along that line. With this function it is not possible to do that. Since the function “oscillates” infinitely many times as it “approaches” the point $(0, 0)$ we cannot really tell how long the line is between two points located on the left and right sides of the plane. Moreover, as the function approaches the origin $(0, 0)$ we cannot tell whether it will cross from the right plane to the left plane “going down” or “going

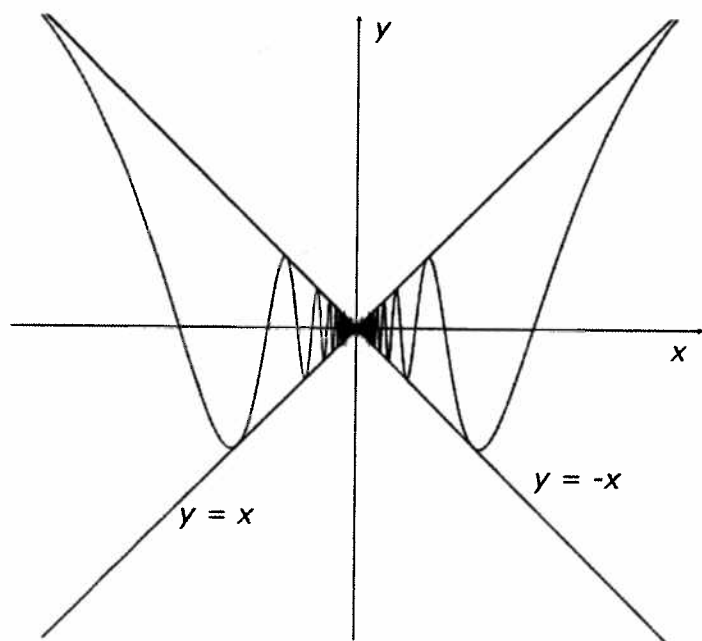


Figure 11.2. The graph of the function $f(x) = x \sin 1/x$.

up.” As a result, the function violates two essential properties of natural continuity and therefore it is not continuous.

This leads us to think that the reported difficulties students have in learning limits and continuity then may not be so much due to a lack of mastery in the use of universal and existential quantifiers as Freudenthal (1973) has suggested, but to the fact that the formal ε - δ definition of continuity (a) simply does not capture the inferential organization of the human everyday notion of continuity (natural continuity), and (b) contrary to what is claimed in most mathematics books and textbooks, it *does not generalize* the notion of continuity either. The function $f(x) = x \sin 1/x$ is ε - δ continuous but it is not naturally continuous.

The moral here is that what is characterized formally in mathematics leaves out a huge amount of precise inferential organization of the human ideas that constitute mathematics. In the next section we will see that this is indeed what happens with the dynamic aspects of natural continuity that entail precise concepts such as “approaching,” “tending to,” “oscillating,” and so on. Motion, in these cases, is a genuine and constitutive manifestation of the nature of these mathematical ideas. And, as we said, it played a key role in the work of Leibniz, Newton, Euler, Kepler, and many others. With respect to pure mathematics, however, the essential dynamic components of the inferential organization of these ideas are not captured by the ε - δ formalisms and the axiomatic system for real numbers.

THE LIMITATIONS OF FORMALIZATION AND WHAT GESTURES REVEAL ABOUT MEANING

Let us now bring some empirical dimension to this discussion. Consider the following problem²:

Prove that for a function f in R^2 $f: [0,1] \rightarrow$ (into) $[0,1]$, if f is increasing,

it implies $\exists x_0 \in [0,1], f(x_0) = x_0$.

The statement says that for an increasing function in the real plane, mapping (into) points from the unit segment to points on the unit segment (endpoints inclusive), there is a point (x_0) whose value is that of $f(x_0)$ (called fixed point). Although the problem and its proof do not directly make use of the formal definition of continuity, issues involving continuity are deeply embedded in the statement. Let us imagine that a mathematician effectively proves this statement by appropriately using, among other concepts, the notation and idea of ε - δ continuity. How would he or she explain the proof and the *meaning* of what it means to other colleagues, say, in mathematics education? If the language and concepts of the ε - δ

formal method fully expresses the meaning of the above statement, then our mathematician will most likely unfold the explanation exclusively in terms of static universal and existential quantifiers over motion-less numbers and inequalities of numerical expressions. Consequently, one would expect the explanation not to contain elements from natural continuity such as source-path-goal motion, trajectors, trajectories, entities “approaching” others, or “tending” to other locations, and so on, elements which the creators of the ϵ - δ method specifically wanted out of the realm of analysis.

The question of what our mathematician would do in providing the explanation is an *empirical* one, not a mathematical or philosophical question. It is of particular interest to analyze such explanation *in vivo* and in real time. That is, rather than explicitly asking him/her “how would you explain this statement and its proof,” or “how do you think we should explain the meaning of this statement to students”—which require introspection, meta-analysis and reflection on personal beliefs and behaviors—one should simply ask him/her to just explain the statement and its meaning (to, say, interlocutors in mathematics education with experience in undergraduate education). Then, one should carefully observe and analyze the gestalt of the production of such explanation, from actual notations and drawings, wordings, and verbalizations, and, most interestingly, speech-gesture co-production.

The study of gesture production and its temporal dynamics is particularly interesting because it reveals aspects of thinking and meaning that are effortless, extremely fast, and lying beyond conscious awareness (therefore not available for introspection). Rather than getting impressions of what people think about their thinking, through embodied speech-thought-gesture co-production one gets at the actual thinking in real time. Research in a large variety of areas, from child development, to neuropsychology, to linguistics, and to anthropology, has shown the intimate link between oral and gestural production. It has been shown that the phenomenon that gestures accompany speech is universal (Núñez & Sweetser, 2006) and that gestures are less monitored than speech. Speakers are often unaware that they are gesturing at all (McNeill, 1992). Besides, finding after finding has shown that gestures are produced in astonishing synchronicity with speech, that in children they develop in close relation with speech (Bates & Dick, 2002), and that gestures are co-processed with speech. For instance, stutterers stutter in gesture too (Mayberry & Jacques, 2000). Interestingly, gestures are produced without the presence of interlocutors. Studies of people gesturing while talking on the telephone, or in monologues, and studies of conversations among congenitally blind subjects have shown that there is no need of interlocutors for people to gesture (Iverson & Goldin-Meadow, 1998), which shows

how deeply these hand and body movements are engrained in the thinking process. Also, it has been shown that gestures provide complementary content to speech content in that speakers synthesize and subsequently cannot distinguish information taken from the two channels (Kendon, 2000). Finally, and crucially to this chapter, gestures are co-produced with abstract metaphorical thinking, where linguistic metaphorical mappings are paralleled systematically in gesture (e.g., Núñez, 2006). In all these studies, a careful analysis of important parameters of gestures such as hand shapes, hand and arm positions, palm orientation, type of movements, trajectories, manner, and speed, as well as a careful examination of timing, indexing properties, preservation of semantics, and the coupling with environmental features, give deep insight into human thought. Taken together, all these sources of evidence support (a) the view that speech and gesture are in reality two facets of the same cognitive linguistic reality and (b) an embodied approach for understanding language, conceptual systems, and high-level cognition.

With these tools from gesture studies and cognition, we can go back to our mathematician explaining the proof, and address the empirical question we raised at the beginning of this section: How would a mathematician explain the meaning of the above statement and its proof? The following speech-gesture-writing co-production are excerpts from a recorded session that took place at the University of California, San Diego, where the mathematician Guershon Harel provided a 12-minute explanation of such proof to fellow colleagues in mathematics education. Obviously, from such a session there are many elements that can be analyzed, but for the purposes of this chapter, we are going to focus only on a couple of passages of the video.

Harel starts by laying the main concepts underlying the problem. He begins by drawing the coordinates of the xy real plane and by specifying on the positive quadrant, the unit square. He then immediately draws—from top-right to bottom-left (and from a peripheral position, relative to the body, towards the center) the line $y = x$. He follows the drawing by saying: “so, we have a function that is increasing” (Figure 11.3).

As he says “increasing,” he gestures with his right hand (palm down), with a wavy upward and diagonal movement (slightly along the line $y = x$). This gesture is co-produced with the enactment of a source-path-goal schematic notion, where the source corresponds to a generic location $(x, f(x))$ with low but positive values of x and $f(x)$, the goal of the instantiated schema corresponds to a generic location $(x, f(x))$ with greater values for both x and $f(x)$, the trajector is indexed by the external edge of the right hand, and the trajectory (path) is the trace left by the motion of the hand that is indexing the increasing values of the function as x gets increasingly greater values. The gesture is inherently dynamic and it is coupled to the

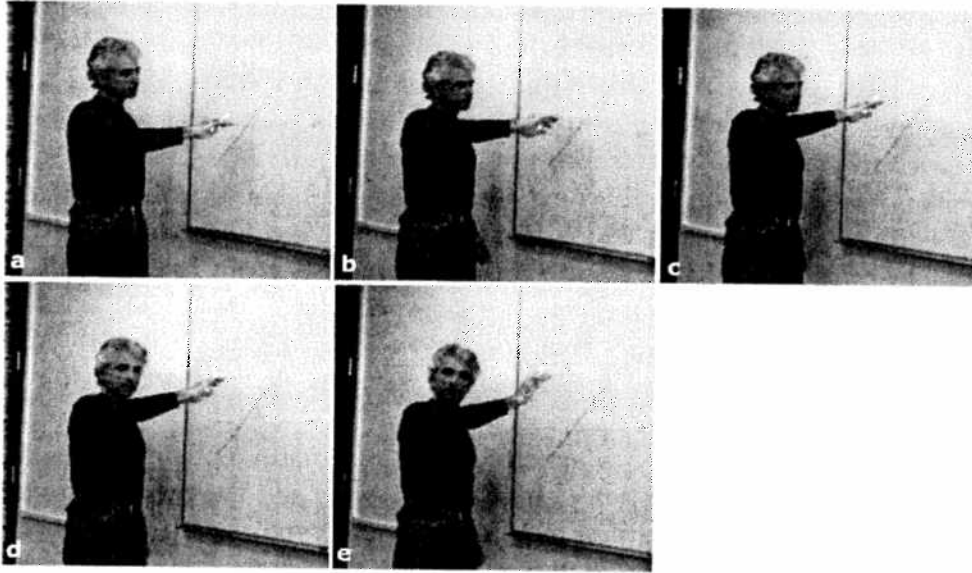


Figure 11.3. “So we have a function that is increasing.”

inscription on the board (the line $y = x$). About 45 seconds later (after discarding the trivial cases when $x = 0$ and $x = 1$, and after evoking the condition that the function is increasing) he stresses the previous idea by specifically drawing the path (or sequential paths) on the board. He says:

But the function is increasing, so intuitively, if you think about this, it's going up (draws a slightly curved increasing line segment), it's going up (draws another curved increasing line segment following the previous one), it is going up (draws another segment, getting progressively closer to $y = x$), it is going up (draws yet another even smaller segment closer to $y = x$) ... well, at some point is going to touch here (the $y = x$ line) ... okay? ... so, something like that (and he draws another line segment on the other side of $y = x$) and then (he draws another segment) it is going to end here (corresponding to a point $(1, f(1))$ which is “below” the diagonal $y = x$) ... because it is only into. And the problem is how to capture this point [the point at which the function “touches” the line $y = x$].

The trace of the trajectory, which was drawn from the bottom left to the top right, can be seen in Figure 11.4. Although the actual tracing of the path on the board is composed of several physically disconnect segments (possibly denoting several arbitrary regions of the path), we know from the gesture production that, conceptually, the path is a holistic integrated one. Indeed, when he completes his long utterance he says “because it [the function] is only into.” And he says this while producing a fast

frontward and smooth gesture with his left hand (palm down), extending his arm in a quick and uninterrupted way, and along the extension of the drawn path (Figure 11.4). From this we know that the conceptualized path is a single interrupted one whose source is located where x is small to the goal when x is near 1, going through (“touching”) the line $y = x$.

Then, he moves on to the next step of the explanation which is to try to “capture” the point at which the path crosses the line $y = x$. For the present purposes, what is important is that in laying out the fundamental meaning of the statement to be proved, Harel evokes an implicit understanding of continuity that corresponds to natural continuity as it is sustained by cognitive mechanisms such as the source-path-goal schema.

The remaining question is, whether once our mathematician states the general idea of the problem and moves on to work out the proof using ϵ - δ like formalizations, is he going to abandon these dynamic schemas or not. The answer is that those dynamic schemas will continue to co-habit the conceptualizations that are driven by ϵ - δ formalisms, indicating that—when continuity is implicit—there is something fundamental about the meaning of space, motion, traces, paths, and trajectories that continues to be present in the technical conception of continuity which are not grasped by the ϵ - δ formalisms.

For instance, after completing the formal proof of the statement, Harel stresses the crucial role played by the point at which the previously drawn path of the function “touches” the line $y = x$. (which he called x_0). When invoking the crucial least upper bound axiom of real numbers (which formally is defined in static terms only, see Núñez, 2006) with respect to x_0 he says:

What do I mean by the least upper bound? What do I mean by this (pointing to the location $(x_0, f(x_0))$) is the edge of that set [the set $A = \{x \mid \forall x \in [0, 1] \wedge f(x) \geq x\}$], ... well it is the edge of that set, because if I enter the set (gestures from the right towards the left), okay? Then if I move a little bit to the left then they (points for $x < x_0$) are going to be in the set (points in the set

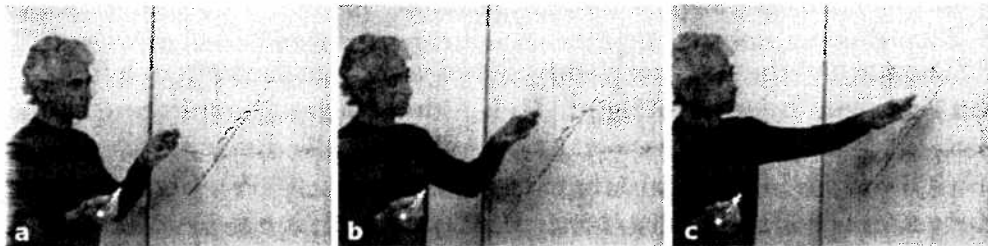


Figure 11.4. Trace of the trajectory.

A) because they (the points in A) have this property [the constraints that define the set A], but if they go out of the set they are going to not to have this property anymore.

When he says “if I enter the set” he extends his right arm and at the level of the value x_0 he bends his wrist so the tip of the fingers are pointing towards him, indicating motion from the high end of the set A towards locations with lower values (see Figure 11.5), all of which have the properties specified by the set A. He closed this sentence by saying “but if they go out of the set they are going to not have this property anymore.” And as he said “they go out” he points with his right index finger outwards away from his body, indicating motion towards greater values of x , values that determine points that are not members of set A (Figure 11.6). Despite the static formalization (e.g., universal quantifier and inequality signs in the definition of set A), the underlying conceptualization is carried out via the use of linear motion as characterized by the source-path-goal schema. The evidence for this comes from both, specific motor action in real time (gestures) and the properties of the co-produced linguistic expressions, which specifically recruited verbs of motion such as “to enter” and “to go out.”

At this point the reader may be asking the following obvious question: Since in order to produce gestures a speaker has to move hands and arms in space, wouldn't in that case *all* gestures imply motion in one way or another? Well, the issue is that although hands and arms are moved in space they are not necessarily co-produced with a *dynamic construal*. For example, at some point in his explanation Harel uttered the expression “every set that is bounded has an edge” (Figure 11.7). At the very moment he said “bounded” he produced a bi-manual gesture (with palms towards

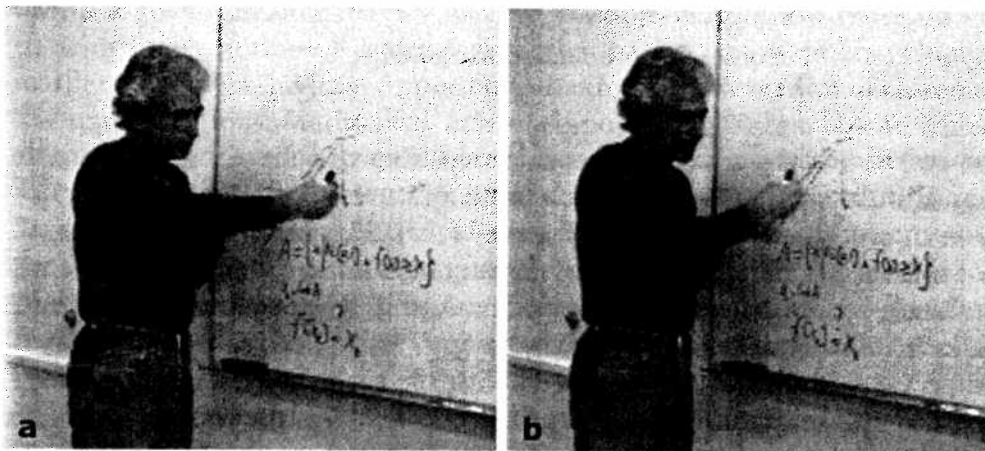


Figure 11.5. “If I enter the set ...”

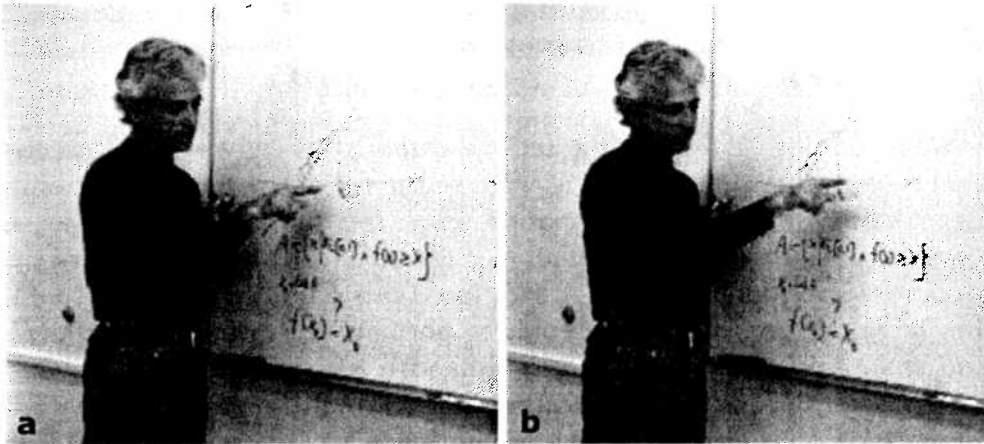


Figure 11.6. "... they go out ..."

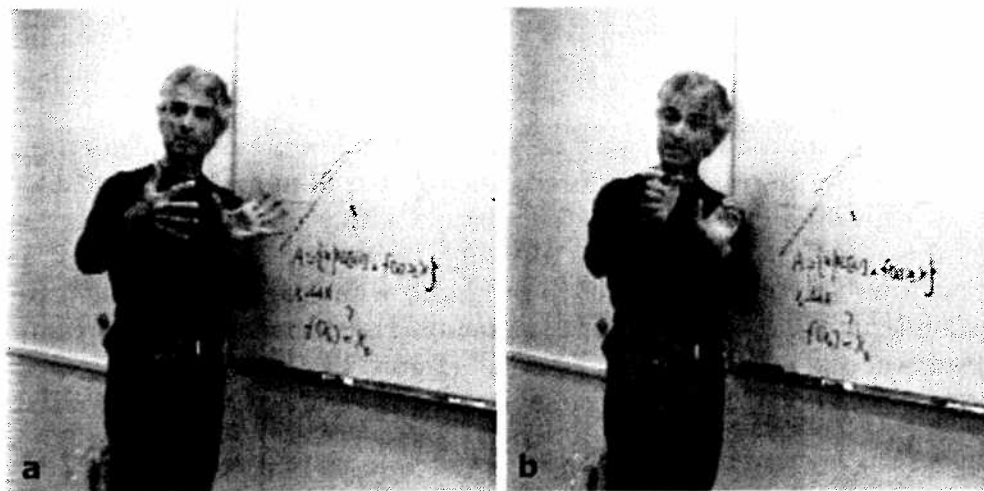


Figure 11.7. "Every set that is bounded has an edge."

center) indicating a fixed and static one-dimensional extension where the hands indexed the bounds. Neither the gesture nor the linguistic expression uttered indicates motion (Figure 11.7a). The verbs used in this case, for instance, are the verb "to be" (existence) and the verb "to have" (possession), which do not imply motion. And a few hundred milliseconds later he completed the sentence by saying "has an edge." At the moment when he said "edge" he produced with his right hand a gesture with a hand shape resembling a horizontal letter "C," with the thumb and index finger indexing a smaller fixed and static extension (the "edge"). Again, no motion is conveyed by either the gesture or the linguistic expressions

used (Figure 11.7b). So, conceptual construals can be, among others, dynamic or static. Specific properties of the linguistic expressions co-produced with specific gestures will provide evidence for one or the other.

In the case of continuity, as we saw with the previous examples, dynamic construals (linguistic and gestural) are pervasive. The omnipresent enactment of dynamic source-path-goal schemas when evoking the idea of continuity strongly suggests that dynamism is an essential part of the mathematical idea of continuity even though the so-called “generalized” concept of continuity based on the ϵ - δ formalisms and the static axioms for real numbers would predict that no motion should exist in such conceptualizations. What this simple case study shows is that formal languages have inherent limitations that need to be taken into account when addressing the question of the nature of mathematics and the nature of teaching and learning mathematics. The universe of human ideas is richer than the world that can be characterized with formalisms, and therefore in its efforts of improving and understanding teaching and learning, mathematics education should not take formalisms for granted, and should get at the very mathematics without leaving it untouched.

DISCUSSION

When “continuity” is taught in classrooms around the world, what is really taught? Natural continuity? ϵ - δ continuity? An unfortunate mixture of the two? And if ϵ - δ continuity is taught, is it introduced as a generalization of natural continuity? Our analysis showed that the two ideas—natural and ϵ - δ continuity—are different human ideas. The ϵ - δ continuity is also grounded on human everyday meaning, but it is based on to the static ordinary notion of *preservation of closeness near a location*: that is, being within a given distance from a specific location. And it also relies on the static ordinary notion of *gaplessness* applied to the real numbers (which is mathematized via the least upper bound axiom). Preservation of closeness has static locations, landmarks, reference-points, distances, but no trajectories, no paths, no directionalities, no motion, and therefore no “jumps.” As I have argued elsewhere (e.g., Núñez & Lakoff, 1998) “preservation of closeness” and “gaplessness” are everyday human concepts with a very precise inferential organization, recruited by Cauchy and Weierstrass in the nineteenth century to carry out the program of arithmetizing analysis (Lakoff & Núñez, 2000). The cognitive science of mathematics shows that the inferential organization of the idea of preservation of closeness (plus gaplessness) is not the same as the one of natural continuity. The two concepts—natural continuity and ϵ - δ continuity—simply have, cognitively, two radically different logic.

As we saw above, many mathematicians and mathematics educators, thinking that mathematics education can teach mathematics leaving the very mathematics untouched, believe that the problem of natural continuity is that it is not “precise” enough. They believe that what the ϵ - δ definition does is to (a) make *precise* and (b) to *generalize* the idea of natural continuity. But this is not true. The idea of natural continuity is indeed very precise, and as we saw earlier, it has a precise inferential organization. The issue is that the inferential organization of natural continuity, which is dynamic and holistic in nature, did not serve the purposes of the arithmetization program which required a reduction of mathematical ideas into static and discrete numeric structures and concepts that later became compatible with static set-theoretic concepts (based on static concepts such as membership relation seen as presence inside a container, and lack of membership relations as presence outside the container, etc.). The inferential structure of the idea of preservation of closeness did fit the goals of the arithmetization program, and it is *this* idea (along with others such as *gaplessness*) that was made precise by the ϵ - δ definition.

The moral is that mathematics education, as an endeavor directly dealing with humans and with the question of how they think and learn, should not assume that formal definitions in mathematics (a) make intuitive ideas more “precise” and that (b) they generalize those intuitive ideas. From this it follows that contrary to what Freudenthal (and many mathematics educators) say, “the difficulties implicit in the continuity concept” are not just the quantifiers and their order. The problems are deeper. They start with the false belief (from both, the teacher and the student) that formalization necessarily generalizes and makes intuitive ideas more precise. Moreover, the problems multiply if the subject matter of the teaching—mathematics—is taken for granted, thus hiding the human nature of the discipline. Because the cognitive science of mathematics does indeed question the nature of the very mathematics, it is in a good position to show exactly why mathematical formalizations neither make more precise, nor generalize, everyday intuitive ideas.

Today, there are important methodological and theoretical advancements in research that help with the investigation of the cognitive science of mathematics and its embodied nature: gesture studies, cognitive semantics, eye-tracking studies, discourse analysis, ethnographic observation, neuroimaging, to mention a few. Mathematics education would benefit tremendously by building on these kinds of developments and by acknowledging, in a deep way, that mathematics is indeed the product of human imagination, and that it can be taught with *meaning* as one of its pillars.

NOTES

1. As I write these lines IBM is running a TV commercial which opens with a relatively young mathematician looking at the camera, who, while standing in front of a white board full of mathematical notations, utters unambiguously what seems to be an undisputable fact: "Math is the only language all humans being share." The clip is available at the URL http://www.youtube.com/watch?v=-udGE8POcZk&feature=channel_page
2. This problem was suggested by Guershon Harel in the context of a study on the nature of proof conducted in collaboration with Laurie Edwards. I thank both of them for stimulating conversations about this problem and the nature of proof, and for allowing me to use the video data included in this article.

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