CHAPTER 6


Rafael Núñez
University of California, San Diego

Virtually all of basic calculus achieves its primary meaning through an absolutely essential collection of motion metaphors. These metaphors control the notation. Hence we write limit statements using arrows and use image-laden words such as "diverges," "converge," "increasing," "constant," and "transform." However, the formal mathematical definitions associated with these notations, being elemental, are not connected to motion (Jim Kaput, 1979).

One of the main goals of education is to prepare individuals, usually young ones, for a hopefully productive and enriching immersion in the community. This process is normally instantiated through an explicit and systematic effort orchestrated by the society through specific institutions and organizations. The general term "education" is often used along with a noun that refers to a specific domain of human knowledge (which we could call X) such as "music," "language," or "mathematics," thus designating the corresponding subfields of the form: "X education": "music education," "language education," and "mathematics education." Interestingly, when confronted to a question such as: what is the nature of X? or, what is the essence of that X you are teaching? music and language education tend to provide an essentially different answer from the one given by mathematics education. The former two take their subject matter X to be inherently human, while the latter doesn’t. In this chapter I’ll argue that this situation—seeing mathematics as an essentially pre-given dehumanized body of knowledge—is deeper and more widespread than what we may think, having deep negative implications for mathematics education. I shall defend my position by showing how the study of the cognitive science of mathematics can provide a new understanding of the nature of mathematics education’s X, and help students, teachers, and researchers in the field. My arguments will go along the lines of some deep insights the extraordinary mathematician educator Jim Kaput had more than 25 years ago on the nature of mathematics and the relationship between formalization and mathematical concepts (Kaput, 1979). This chapter is a tribute to him and his well-ahead-of-the-times ideas.

In our society there is a clear folk and academic conception of "music" and "language" as being inherently human. Both, music and language are seen as
This claim is explicitly made by the hero, Dr. Ellis Arrowsmith (played by Rod Taylor), in a pivotal scene of the film "Operation Crossbow." He expresses his belief that teaching something as complex as mathematics can inspire a child to explore the grandeur of the universe, understand the language of the universe, and perhaps even determine its existence. Arrowsmith's discourse is a powerful depiction of the profound impact that education, particularly in the realm of mathematics, can have on human understanding and potential for discovery.

Arrowsmith's view is supported by the philosophical backdrop of the film, which portrays mathematics as a fundamental language through which humanity can communicate with the universe. The film suggests that mathematics is not just a tool for understanding the physical world but a conduit for exploring the abstract and infinite dimensions of existence.

This perspective aligns with the broader cultural and scientific sentiments of the time, where the exploration of mathematics was seen as a bridge to understanding the cosmos and the nature of reality itself. The film's portrayal of mathematics as a universal language highlights its role in human intellectual and spiritual development, emphasizing its capacity to transcend the boundaries of physical observation and reach into the realm of pure thought and discovery.

References:

Further Reading:
more evident when the concerned entities invoke infinity, where, because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself. Yet, infinity is at the core of mathematics. It lies at the very basis of many fundamental concepts such as limits, joint upper bounds, point-set topology, mathematics. As I'll claim in this chapter, those answers can be found via the scientific study of how the human mind, with the conceptual systems it creates, makes mathematics possible.

The romance of mathematics, despite its immediate intuitiveness, and despite being supported by many outstanding physicists and mathematicians, is, (nowadays) what mathematicians do, is a myth, and as such, arising from the fact it is a matter of faith, not a matter of scientific discussion. In this chapter, I will argue that Cognitive Science, the contemporary scientific study of the human mind, gathers interdisciplinary efforts from neuroscience to linguistics to cognitive psychology, has important things to say about the nature of mathematics, which in turn can provide positive insight into the enterprise of mathematics education. I shall defend the idea that the answer to the question of what is the nature of mathematics deeply affects what mathematics education is taken to be; by shaping not only what it is to be taught, but in what order, through what pedagogical activities and using what methods, but also by prescribing how generations of mathematics teachers should be formed. A scientifically informed view of the nature of mathematics, I'll claim, one that explains the human conceptual mechanisms that make the development of edifice of mathematics possible, should help prepare better new generations of mathematics teachers. Teachers who would, not only have a better sense of how mathematics is acquired but also of what it is that makes it engaging. A sufficient amount of pedagogical software and classrooms activities in an informed cognitive-friendly manner are compatible with fiction or poetry. Any pedagogues who follow in this chapter, therefore, will not be to simply provide suggestions of how to better teach X (mathematics), leaving X untouchable. My goal will rather focus on giving an informed account (from the perspective of cognitive science) of what is the nature of X such that a cognitive-friendly and meaningful form of mathematics education can follow. In order to illustrate my arguments I will focus on a case study involving links between mathematics and the biological social psyche at the college level. The accent will not be put on student’s individual performance. On the semantic organization of the very concepts themselves.

**SO, WHAT IS THE NATURE OF WHAT WE TEACH IN MATHEMATICS EDUCATION?**

**Mathematics is unique. It is an extraordinary conceptual system characterized by the fact that the very entities that constitute it are imaginary, idealized, mental abstractions. These entities cannot be perceived directly through the senses. Consider, for instance, the concept of infinity: a Euclidean point, a line, a plane. What is it? How can we find it in the universe? A point, as defined by Euclid is a dimensionless entity, an entity that has only location but no extension. Such a thing doesn’t exist anywhere in the entire universe, as we know it. A Euclidean point cannot be actually perceived or observed through any scientific empirical method. Humans, however, can create via imagination a Euclidean point in a clear, precise, and non-ambiguously defined way, and they can create a whole world of complicated purely imaginary entities, such as segments, planes, polygons, and spheres. A Euclidean point is an idealized abstract entity realized via human cognitive mechanisms. The imaginary (but precise nature) of mathematics becomes even more evident when the concerned entities invoke infinity, where, because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself. Yet, infinity is at the core of mathematics. It lies at the very basis of many fundamental concepts such as limits, joint upper bounds, point-set topology, mathematics. As I’ll claim in this chapter, those answers can be found via the scientific study of how the human mind, with the conceptual systems it creates, makes mathematics possible.**
mathematical concepts in several areas in mathematics, from set theory to infinitesimal calculus, to transfer and arithmetic, and showed how, via everyday human embodied mechanisms such as conceptual metaphor and conceptual blending, the inferential patterns drawn from direct bodily experience in the real world get extend- ed to very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains.

It is important to keep in mind that Mathematical Idea Analysis focuses on inferential organization and semantic structure rather than on individual cognizing abilities and learning patterns. In order to follow the approach developed in the remainder of this chapter, it is thus advisable to de-emphasize for a moment the actual performances, learning processes, and behaviors of single individuals, which usually define the level of analyses in cognitive psychology and those of many important areas of mathematics education, and to focus on the meaning and inferential organization of mathematical ideas themselves (as accepted within the professional community of contemporary mathematicians). Let us now have a closer look into the study of everyday conceptual mappings and inferential organization.

Conceptual Mappings and Inferential Organization

Consider the following two everyday linguistic expressions: "The spring is ahead of us" and "the presidential election is now behind us." Literally, these expressions don't make any sense—"ahead of us" in any measurable or observable way, and "election is behind us" is nothing that can be physically "behind us." Hundreds of thousands of these expressions, whose meaning is not literal but metaphorical, can be part of human everyday language. They are the product of the human imagination, they convey precise meanings, and allow speakers to make precise inferences about them. A branch of semantics (and more specifically, cognitive semantics), has studied this phenomenon in detail and has shown that the semantics of these hundreds of thousands metaphorical linguistic expressions can be modeled by a relatively small number of conceptual metaphors (Lakoff & Johnson, 1980; Lakoff, 1993). These conceptual metaphors, which are inference-preserving cross-domain mappings, are cognitive mechanisms that allow us to project the inferential structure of one domain onto another domain (Lakoff & Johnson, 1980; Núñez, 1999; Lakoff, 1993; Lakoff & Johnson, 1999). For cross-linguistic and gestural studies, Núñez & Sweetser, 2000; Porodinsky, 2000; Núñez, Motz, & Tuschek, 2006). For the purposes of this chapter, there are three very important goals to keep in mind:

1. At this level of the cognitive idea analysis, the primary focus is not how single individuals learn how to use these conceptual metaphors, or what difficulties they encounter when they learn them, or how they may lose the ability to use them after a brain injury, and so on. The focus is to characterize (i.e., to model), across hundreds of linguistic expressions, the structure of the inferences that can be drawn from them. For example, if "the spring is ahead of us," we can infer that the summer is not just "behind us," but rather "in front of us." Similarly, if "the presidential election is behind us," we can infer that the various effects immediately following the election are not only "behind us," but also much closer to us than the election itself.
2. Truth, when imaginary entities are concerned, is always relative to the inferential organization of the mappings involved in the underlying conceptual metaphors. For instance, "last summer" can be conceptualized as being behind us as long as we operate with the general conceptual metaphor THAT EVENTS ARE TRAVEL IN UNDIMENSIONAL SPACE mentioned above, which determines a specific bodily orientation respect to metaphorically conceived events in time, namely, the future as being "in front of" us and the past as being "behind" us. Núñez and Sweetser (2001, 2006), however, have shown that the details of that mapping are not universal. Through ethnographic field work, as well as cross-cultural, gestural and lexical analysis of the Aymara language of the Andes' highlands, they provided the first well-documented case violating the postulated universality of the metaphorical orientation future-in-front of ego and past behind ego. In Aymara, for instance, "last summer" is conceptualized as being in front of ego and past behind ego. Moreover, Aymara speakers not only utter these words when referring to time, but also produce co-timed corresponding gestures, strongly sustaining the metaphorical orientation. The moral is that there is no ultimate truth regarding these imagistic structures. In this case, there is no ultimate truth about where, really, is the ultimate metaphorical location of the future (or the past). Truth will depend on the details of the mappings of the underlying conceptual metaphor. As we will see, this will turn out to be of paramount importance when mathematical concepts are concerned. Their ultimate truth is not hidden in the structure of the universe, but it will be encoded in the underlying conceptual mappings (e.g., metaphors) used to create them.

3. It is crucial to keep in mind that the abstract conceptual systems we deal with are possible because we are biological beings with specific morphological and anatomical features. In this sense, human abstraction is embedded in nature. It is because we are living creatures with a salient and unambiguous front at the back that we can build on these properties and the related bodily experiences we have to bring forth stable and solid concepts such as "the future in front of us." This wouldn't be possible if we had the body of a jellyfish or of an amoeba.

4. Finally abstract conceptual systems are not "simply" socially constructed, as a matter of convention. Biological properties and specificities of human body-grounded experience impose very strong constraints on what concepts can be created. While social conventions usually have a huge number of degrees of freedom, many human abstract concepts don't. For example, the color pattern of the Euro bills was socially constructed via convention (and so were the design patterns they have). But virtually any color ordering would have done the job. Metaphorical construals of time, on the contrary, are only based on a spatial source domain. And this is in an empirical observation, not an arbitrary or speculative statement: as far as we know, there is no language or culture on earth where time is conceived in terms of thematic or chromatic source domains. And there is more: not just any spatial domain does the job. Spatial conceptualizations of time are, as far as we know, always based on unidimensional space. Human abstraction is thus not "merely" socially constructed, it is constructed through strong non-arbitrary biological and cognitive constraints that play an essential role in constituting what human abstraction is. Human cognition is embodied, shaped by species-specific non-arbitrary constraints. Again, this property will turn out to be very important when mathematical concepts are concerned.

With these morals in mind, we are now in a position to analyze our case study.

A CASE STUDY: LIMITS, INFINITE SERIES, AND CONTINUOUS FUNCTIONS

In the spirit of how the semantics of everyday human language is studied, let's analyze how mathematical ideas are conceived and expressed in the community of professional mathematicians. Keep in mind that the goal is to focus on how mathematici- ans' ideas are actually presented, described, characterized, and even formally defined in mathematics books, academic journals, and textbooks (thus de-emphasizing what students may say when they are learning these ideas). A careful analysis of mathematical books and articles in mathematics provides very good insights into the question of how the inferential organization of human everyday ideas has been used to create mathematical concepts. For the purpose of this chapter, I would like to focus on some concepts known to mathematics education as being particularly elusive, difficult for teachers to teach and for students to learn: limits, continuous functions, and infinite series (Tiebout and Vinner, 1975). Núñez, 1995, Núñez, Edwards, and Matos, 1996). But, are these concepts intrinsically difficult to learn? And if yes, why? Or is it that the methods used to teach them are not appropriate? I will claim that "X education" when X is "mathematics" has been too blind in addressing the problem of the nature of X (mathematics) and as a consequence has been framed for teaching X while leaving X itself largely untouched and unques- tioned. Mathematics, as the content to be taught, is pre-given, and it is widely taught as such. In part due to the fact that the process of mathematics education perpetuates and sustains the romance of mathematics described earlier, is that mathematics edu- cation rarely questions how specific natural everyday ideas such as change, containment, continuity, rotations, and so on, are mathematicized to create what mathematic- ics is. In this section I'll try to show that the Cognitive Science of Mathematics can accomplish this goal, and that mathematics education can benefit from it.

Let us start by looking at some classic books and textbooks:

1. While discussing limits, we read in the Russian classic Matematika, or sovremennyi uchebnoy (Mathematics, its contents, methods and meaning) by A. Aleksandrov, A.N. Kolmogorov, and M.A. Lavrent'ev (1966-1967):

   If a variable $x$ may be represented as a sum $x = a + z$,

   where $a$ is a constant and $z$ is an infinitesimal, then we say that the variable $x$, for $n$ increasing beyond all bounds, approaches the number $a$ and we write

   $\lim_{n \to \infty} x = a$ or $x = a$.

   The number $a$ is called the limit of $x$. (Vol. 1: 82; italics are ours.)

Strictly speaking, this statement refers to a sequence of discrete and motionless values (real numbers) that a variable $x$ takes corresponding to increasing discrete and motion-
less values taken by $n$. If we examine this statement closely we can see that it describes static facts about numbers. We can observe that there is no motion whatsoever involved. No entity is actually approaching anything or moving beyond anywhere. So, why then did these well-respected Russian authors (or why do mathematicians in general, for that matter) use dynamic language to express static properties of static entities? And what does it mean to say that the "variable $s_n$ approaches a number $a$" when in fact the variable can only have a fixed and distinct values given fixed and distinct values of $n$?


We describe the behavior of $s_n$, by saying that the sum $s_n$ approaches the limit $1$ as $n$ tends to infinity, and by writing

$$1 = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

where on the right we have an infinite series (p. 64, underlined italics are ours).

This statement refers to a sequence of discrete and motion-less partial sums of $s_n$ (real numbers), corresponding to increasing discrete and motion-less values taken by $n$ in the expression $1/2^n$ where $n$ is a natural number. If we examine this statement closely we can observe that it describes some facts about numbers, and about the result of discrete operations with numbers. Again, no motion whatsoever is involved. No entity is actually approaching or tending to anything. So, why then did Courant and Robbins (or why do mathematicians in general) use dynamic language to express static properties of static entities? And what does it mean to say that the "sum $s_n$ approaches," when in fact a sum is simply a fixed number, a result of an operation of addition?

3. Later in the book, Courant and Robbins analyze cases of continuity and discontinuity of trigonometric functions in the real plane. Referring to the function $f(x) = \sin 1/x$, whose graph is shown in Figure 6-3, they say:

"... since the denominators of these fractions increase without limit, the values of $x$ for which the function $x(x)$ has the values $1, -1, 0$ will cluster nearer and nearer to the point $x = 0$. Between any such point and the origin there will be still an infinite number of oscillations of the function." (p. 280, underlined indices are ours).

Once again, if, strictly speaking, a function is a mapping between elements of a set (coordinate values on the x-axis) with one and only one of the elements of another set (coordinate values on the y-axis), all that we have is a static correspondence between points on the x-axis with points on the y-axis. How then can the authors (or mathematicians in general) speak of "oscillations of the function," let alone an infinite number of them?

These three simple examples illustrate some deep and important issues regarding the semantic structure of mathematical ideas. They show how mathematical ideas and concepts are described, defined, illustrated, and analyzed in mathematics books. You can pick your favorite mathematics books and you will see similar patterns. In all three examples above, static, numerical structures are involved, such as partial sums and mappings between coordinates on one axis with coordinates on another. Strictly speaking, absolutely no motion or dynamic entities are involved in the formal definitions of these terms. So, if no entities are really moving, why do authors continue to speak of "approaching," "tending to," and "oscillating"? If mathematical definitions are indeed so precise, why is there still dynamic language when purely static entities are concerned? Where is this motion coming from? What does dynamism mean in these cases? What role is it playing (if any) in the meaning of these statements about mathematics facts?

In order to answer to these questions we will first look at how pure mathematics characterizes real numbers, limits and continuity of functions. We will eventually find that in these cases the logic of formal mathematics of set-theoretic entities and of universal and existential quantifiers is intrinsically static, and that the presence of dynamic content along with its inferential structure is a manifestation of human meaningful cognition that is not captured by mathematics formalisms.

### Pure Mathematics and Real Numbers

In pure mathematics, entities are brought to existence via formal definitions, formal proofs (theorems) and axiomatic methods (i.e., by declaring the existence of some entity without the need of proof. For example, in set theory the axiom of infinity assures the existence of infinite sets. Without that axiom, there are no infinite sets). In the case of real numbers, ten axioms taken together, fully characterize this number system and its inferential organization (i.e., theorems about real numbers). The following are the axioms of the real numbers.

1. **Commutative laws for addition and multiplication.**
2. **Associative laws for addition and multiplication.**
3. **The distributive law.**
4. **The existence of identity elements for both addition and multiplication.**
5. **The existence of additive inverses (i.e., negatives).**
6. **The existence of multiplicative inverses (i.e., reciprocals).**
7. **Total ordering.**
8. **If $x$ and $y$ are positive, so is $x + y$.**
9. **If $x$ and $y$ are positive, so is $x \cdot y$.**
10. **The Least Upper Bound Axiom.**
The first 6 axioms provide the structure of what is called a field for a set of numbers and two binary operations. Axioms 7 through 9. assure ordering concepts. The first nine axioms fully characterize ordered fields, such as the rational numbers with the operations of addition and multiplication. Up to here we already have a lot of structure and complexity. For instance we can characterize and prove theorems about all possible numbers that can be expressed as the division of two whole numbers (i.e., rational numbers). Along a line we can also locate (accord- ing to their magnitude) any two different rational numbers and be sure (via proof) that there will always be (infinitely many) more rational numbers between them (a property referred to as density). With the rational numbers we can describe with any given (finite) degree of precision the proportion given by the perimeter of a cir- cle and its diameter (e.g., 3.143; 3.1415, etc.). With the rational numbers, however, we cannot "complete" the points on the line, and we cannot express with infinite exact- tude the magnitude of the proportion mentioned above (e 3.14159...). For this we need the full extension of the real numbers. In axiomatic terms, this is accomplished by the tenth axiom: the least upper bound axiom. All ten axioms characterize a complete ordered field.

Nothing in the first nine axioms of real numbers helps us understanding the origin of motion in the above mathematical statements about infinite series and continuity. All nine axioms simply specify the existence of static properties regarding binary operations and their results, and properties regarding ordering. There is no explicit or implicit reference to motion in these axioms. Since what makes a real number a real number (with its infinite precision) is the Least Upper Bound axiom, it is perhaps this very axiom that hides the dynamic secret we are looking for. Let's see what this axiom says:

10. Least Upper Bound axiom: every nonempty set that has an upper bound has a least upper bound.

And what exactly is an upper bound and a least upper bound? This is what pure mathematicians say:

Upper Bound
If b is an upper bound for S if
x ≤ b, for every x ∈ S.

Least Upper Bound
b is a least upper bound for S if
• b is an upper bound for S, and
• b ≤ S for every upper bound b ≤ S.

But once again, all we find here are statements about motionless entities such as universal quantifiers (e.g., for every x; for every upper bound b of S), membership relations (e.g., for every x ∈ S), greater than relations (e.g., x ≤ y; b ≤ y), and so on. In other words, there is absolutely no indication of motion in the Least Upper Bound axiom, or in any of the other nine axioms. In short, the axioms of real numbers, which are supposed to completely characterize the "truths" (i.e., theorems) of real numbers don't tell us anything about a sum "approaching" a number, or a number "tending to" infinity (whatever that means!).

Now, let's take a look at the concept of continuity.

What Is Continuity?
What is, according to pure mathematics, continuity of functions? Today mathematics defines continuity for functions as follows:

A function f is continuous at a number a if the following three conditions are satisfied:
1. f is defined on an open interval containing a,
2. \lim_{x \to a} f(x) exists, and
3. \lim_{x \to a} f(x) = f(a).

Where by \lim_{x \to a} f(x) is meant is the following:

Let a function f be defined on an open interval containing a, except possibly at a itself, and let I be a real number.
The statement
\lim_{x \to a} f(x) = I
means that for every ε > 0, \exists δ > 0, such that if 0 < |x − a| < δ, then |f(x) − I| < ε.

As we can see, pure formal mathematics defines continuity in terms of limits, and limits in terms of static universal and existential quantifiers applied on static numbers (e.g., ∀ε > 0, ∃δ > 0, and the satisfaction of certain conditions which are described in terms of motionless arithmetic difference (e.g., |f(x) − L|) and static smaller than relations (e.g., 0 < |x − a| < δ). Once again, these formal definitions don't tell us anything about a sum "approaching" a number, or a number "tending to" infinity, or about a function "oscillating" between values (let alone doing it infi- nitely many times, as is the function f(x) = sin(1/x).

A close inspection of mathematics textbooks reveals that often, right before giving this formal ε-δ definition of continuity, a paragraph or two are dedicated to the "informal" characterization of the idea of continuity, one that appeals to an "intuitive" description. Here is, for instance, the famous Russian book Mathematics, Its Contents, Methods and Meaning by Aleksandrov, Kolmogorov, and Lavrent’ev (1956/1999) mentioned earlier: "The general idea of a continuous function may be obtained from the fact that its graph is continuous: that is, its curve may be drawn without lifting the pencil from the paper." (p. 88; our emphasis).

Here is a quote from the classic textbook Calculus by G. Simmons (1985), while discussing the same topic: "In everyday speech a ‘continuous’ process is one that proceeds without gaps or interruptions or sudden changes. Roughly speaking, a function y = f(x) is continuous if it displays similar behavior" (p. 58; our emphasis).

In both texts, we observe a characterization of continuous functions given in dynamic terms. In both cases there is something moving: the pencil drawing a curve on the paper in the former, and something unfolding without gaps in the latter. In both cases we have something moving from some position in space towards some other location in an uninterrupted manner. In both books these dynamic descriptions are given as a way of helping the reader by providing some immediate intuitive idea of what a continuous function means. The Russian book even characterizes the meaning of a “continuous function” in terms of something that is “continuous,” whose meaning corresponds to what Simmons’ Calculus textbook characterize as “everyday speech” (as we’ll see later, this meaning corresponds precisely to the concept of natural continuity described by...
Núñez & Lakoff, 1998). At this point, right after setting this introductory presentation of continuous functions in dynamic terms, textbooks usually make a radical move. In a somewhat downgrading tone they make clear that the "intuitive" examples given so far are merely illustrative, that they are not precise enough, and that a rigorous formal definition is required. Simmons' textbook, for instance, says: "Up to this stage our remarks about continuity have been rather loose and intuitive, and intended more to explain than to define." (Simmons, 1985, p. 58; our emphasis). This is a remarkable and profound passage. The choice of the words "explain" and "define" is not random. It characterizes the widespread idea in mathematics education that "explaining" may be a good thing, but what mathematics teaching really is about, is in "defining" entities and properties in a rigorous and precise way. From the perspective of cognitive science this statement goes against most of what we know about how humans learn and make sense of things, from perception, attention, and memory, to categorization and problem solving.

Why is Continuity Difficult to Understand?

An important part of the mathematics education community attributes the difficulties of teaching and learning the concept of continuity to problems related to the use of existential and universal quantifiers. Nowhere is this position clearer than in the work of the famous mathematicians educator Hans Freudenthal:

"The difficulties implicit in the continuity concept are quantifiers and the order of quantifiers of different kinds. Continuity of means intuitively: small changes of x correspond with small changes of f(x). Or: if x changes little, f(x) also changes little. Words like "small", "negligible", "little", "much", "short", "long", may hide a quantifier, but formal linguistic criteria are often insufficient to know which kind. Always, sometimes, everywhere, somewhere—certainly the universal or existential quantifier, but the linguistic formulation does not unambiguously reveal that in the continuity definition the second small (or little) hides a universal, and the first an existential, quantifier. To grasp it, a logical analysis is badly needed. The quantifiers in "small" or "little" is better indicated in the more exact formulation: sufficiently small changes of a corresponding arbitrarily small ones of f(x). Or: if x changes sufficiently little, f(x) changes arbitrarily little. Still from this formulation it is a long step to understand that first the "arbitrarily little" must be prescribed before the "sufficiently little" is to be determined.

The intuitive concept, two notions of formalizing—first, decoding of hidden quantifiers, second, settling the order of quantifiers of different kinds. Good didactics should at least separate these difficulties from each other."


In the pages following this citation, Freudenthal provides an insightful analysis of these difficulties and gives helpful recommendations for achieving good didactics regarding the use of quantifiers. While his analysis is precise, deep, and clear, he only focuses on the formal aspects of the ε-δ definition, completely missing the fundamental dimensions of everyday human cognition that may be interfering in understanding such formalization. Perpetuating the common belief that in "X education" where X is mathematics, you can teach X (mathematics) leaving X untouched or unquestioned, Freudenthal (like many in mathematics education) takes the ε-δ definition for granted. This, to the point that for him "Continuity of f means intuitively: small changes of x correspond with small changes of f(x)." The statement is indeed a very good linguistic

formulation of what is intuitive in the ε-δ definition. It has the right static concepts, and it provides the right relations between them. What is missing, however, is precisely the need of questioning X, that is, the very mathematicalization of the everyday notion of continuity—natural continuity (Núñez & Lakoff, 1998)—evoked in both our textbook examples. What is missing is the fact that cognitively speaking, the statement does not characterize the intuitive meaning of natural continuity—the continuity conceived usually by the creators of calculus, Leibniz and Newton, in the 17th century (and in fact, all mathematicians up to the 19th century), as well as by the students and teachers who naturally bring it into the classroom today. It is natural continuity that brought Euler to refer to a continuous curve as "a curve described by freely leading the hand" (cited in Stewart, 1995, p. 237), and the great Kepler to measure "an area swept out by the motion of a (celestial) point on a physical "continuous curve" (Kramer, 1970, p. 528). Natural continuity—continuity as we normally conceive it outside of mathematics—is based on a source-path-goal schema, a fundamental cognitive scheme concerned with motion which has the following elements:

A trajectory that moves
A source location (the starting point)
A goal—that is, an intended destination of the trajectory
A route from the source to the goal
The actual trajectory of motion
The position of the trajectory at a given time
The direction of the trajectory at that time

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The graph of the function \( f(x) = \sin(x) \) is shown. The area under the curve from 0 to \( \pi \) is 
\[
\int_0^\pi \sin(x) \, dx = 2.
\]

According to the definition of the definite integral, the area under the curve is the limit of the sum of the areas of the rectangles that approximate the region under the curve as the number of rectangles approaches infinity. 

For this function, the area can also be calculated using the fundamental theorem of calculus, which states that the definite integral of a function over an interval can be found by evaluating the antiderivative at the limits of integration and subtracting the values. 

In this case, the antiderivative of \( \sin(x) \) is \( -\cos(x) \), so the area is 
\[
\left[ -\cos(x) \right]_0^\pi = -\cos(\pi) - (-\cos(0)) = 2.
\]
BY TO SATISFY THE CONDITIONS FOR PRESERVATION OF CLARITY. HOWEVER, ACCORDING TO NATURAL CONTINUITY, THIS FUNCTION IS NOT CONTINUOUS. THE INFERENTIAL ORGANIZATION OF MATHEMATICS OF A NATURALLY CONTINUOUS LINE WE SHOULD BE ABLE TO TELL HOW LONG THE LINE BETWEEN TWO POINTS. WE SHOULD ALSO BE ABLE TO DESCRIBE ESSENTIAL PROPERTIES OF THE MATHS OF A POINT ALONG THE LINE. THIS FUNCTION WE CANNOT DO THAT. IF THE FUNCTION "oscillates" INFINITELY MANY TIMES AS IT "APPROACHES" THE POINT 0, (0, 0) WE CANNOT TELL WHETHER IT WILL CROSS FROM THE RIGHT OR LEFT. IN THIS CASE, MOVING TOWARDS THE ORIGIN, THERE ARE INFINITE VALUES OF X WHICH ARE PROGRESSIVELY SMALLER IN ABSOLUTE TERMS. THIS MEANS THAT MOVING TOWARDS THE ORIGIN FROM TWO OPPOSITE SIDES (i.E., FOR POSITIVE AND NEGATIVE VALUES OF X) AND ALWAYS BETWEEN THE VALUES X = 1 AND X = -1. AS WATSON, A VARIATION OF THIS FUNCTION, f(x) = x sin(1/x), REVEALS DEEP CONCEPTUAL INCOMPATIBILITIES BETWEEN THE TWO OPTIONS AND THE MATHEMATICAL DEFINITION OF CONTINUITY.

FICTIVE MOTION

NOW THAT WE ARE AWARE OF THE METAPHORICAL (AND METRICAL) NATURE OF THE MATHEMATICAL IDEAS MENTIONED EARLIER, WE CAN ANALYZE MORE IN DETAIL THE DYNAMIC COMPONENT OF THESE IDEAS. FROM WHERE DO THESE IDEAS GET MOTION? WHAT COGNITIVE MECHANISMS ARE ALLOWING US TO COHERENTLY IDENTIFY ENTITIES IN DYNAMIC TERMS? THE ANSWER IS FICTIVE MOTION. FICTIVE MOTION IS A FUNDAMENTAL EMBODIED COGNITIVE MECHANISM THROUGH WHICH WE UNCONSCIOUSLY (AND EFFORTLESSLY) CONCEPTUALIZE STATIONARY ENTITIES IN DYNAMIC TERMS, AS WHEN WE SAY THE ROAD GOES ALONG THE COAST. THE ROAD ITSELF DOESN’T ACTUALLY MOVE ANYWHERE. IT IS SIMPLY STANDING STILL. BUT WE MAY CONCEPTUALIZE IT AS MOVING "ALONG THE COAST." FICTIVE MOTION WAS FIRST STUDIED BY LEN TALMY (1991), VIA THE ANALYSIS OF LINGUISTIC EXPRESSIONS TAKEN FROM EVERYDAY LANGUAGE IN WHICH STATIONARY SCENES ARE DESCRIBED IN DYNAMIC TERMS. THE FOLLOWING ARE LANGUAGEx EXAMPLES OF FICTIVE MOTION:

- The Equator passes through many countries.
- The border runs along the river.
- The US west coast goes all the way down to San Diego.
- After crossing the bridge the path goes through the forest and then it reaches the main house.
- The fence stops right after the tree.
- Unlike Tokyo, in Paris there is no subway line that goes around the city.

Motion, in all these cases, is fictive, imaginary, and not real in any literal sense. Not only these expressions use verbs of action, but they also provide precise descriptions of the quality, manner, and force of motion. In all cases of fictive motion there is a "trajectory" (the moving agent) and a "landscape" (the space in which the trajectory moves). Sometimes the trajectory may be a real object (e.g., the road goes; the fence stops), and sometimes it is metaphorical (e.g., the Equator passes through; the boarder runs). In fictive motion, real world trajectories don’t move, but they have the potential to move or the potential to enact or enable movement (e.g., a car moving along that road). In mathematics proper, however, the trajectory has always a metaphorical component. That is, the trajectory as such...


In the study of the human mind, gestures have been less of a focus than the other aspects of the mind. The cognitive conceptions of the human mind have been primarily in terms of the work of philosophers such as Descartes, Hume, and Kant. The idea of cognitive conceptions has been largely misunderstood and misinterpreted. This is because cognitive conceptions, such as sensation, perception, and knowledge, have been defined as a set of rules and procedures that are independent of the physical symbols governing the processing of information.

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information. In all these cases, gestures were completely ignored and felt out of the picture that defined what constituted genuine subject matters for the study of the mind. At best gestures were considered a kind of epiphenomenon, secondary to other more important and betterdefined phenomena.

But in the last decade or so, the field of gesture studies has moved forward dramatically, thanks to the work of pioneers such as Kendon (1982, 2004), McNeil (1992, 2000), Goldin-Meadow (2003) and many others. Research in a variety of areas, from child development, to neuropsychology, to linguistics, and to anthropology, has shown the intimate link between oral and gestural production. Finding after finding has confirmed that gestures are produced in synchrony with speech, that they develop in close relation with speech, and that brain injuries affecting speech production also affect gesture production. The following is an abbreviated list of sources of evidence supporting (1) the view that speech and gesture are in reality two facets of the same cognitive linguistic reality, and (2) the embodied approach for understanding language, conceptual systems, and high-level cognition:


Largely unconscious production: Gestures are less monitored than speech, and they are to a great extent unconscious. Speakers are often unaware that they are gesturing at all (McNeil, 1992).

Speech-Gesture synchronicity: Gestures are co-processed with speech, in co-timing patterns which are specific to a given language (McNeil, 1992).

Gestural production with no visible interlocutor: Gestures can be produced without the presence of interlocutors, e.g., people gesture while talking on the telephone, and in monologues: congenitally blind subjects gesture as well (Iversen & Goldin-Meadow, 1998).

Speech-Gesture co-processing: Stutterers stutter in gesture too, and impeding hand gestures interrupt speech production (Mayberry & Jaques, 2009).

Speech-Gesture development: Gestural and speech development are closely linked (Iversen & Thelen, 1999; Bates & Dick, 2002; Goldin-Meadow, 2003).

Speech-Gesture complementarity: Gesture can provide complementarity (as well as overlapping) content to speech context. Speakers synthesize and subsequently cannot distinguish information taken from the two channels (Kendon, 2000).

Gestures and abstract metaphorical thinking: Linguistic metaphorical mappings are parallelly systematically in gesture (McNeil, 1992; Cienki, 1998; Sweetser, 1998; Núñez & Sweetser, 2006).

In all these studies, a careful analysis of important parameters of gestures such as handshapes, hand and arm positions, palm orientation, type of movements, trajectories, starting, and speed, as well as a careful examination of timing, indexing properties, levels of iconicity, and the coupling with environmental features, give deep insight into human thought. Among many properties, gestures usually have three well-defined phases, called preparation, stroke, and retraction (McNeil, 1992). The

stroke is in general the fastest part of the gesture's motion, and it tends to be highly synchronized with speech accentuation and sound movement. The preparation phase is the motion that precedes the stroke (usually slower), and the retraction is the motion after the stroke has been produced (usually slower as well), when the hand goes back to a resting position or to whatever activity it was engaged in.

With these tools from gesture studies and cognition, we can now analyze metathetical expressions like the ones we saw before, but this time focusing on the gesture production of the speaker (in this case a mathematician teaching a university-level mathematics course). The following gestures have been recorded during upper division mathematics classes at a major university in California. Keep in mind that these gestures are abstract (metaphorical) in nature, in the sense that the entities that are indexed with the various handshapes—like points and numbers—are purely imaginary entities.

Figure 6-6 shows a professor of mathematics teaching about convergence of sequences of real numbers. In this particular situation, he is talking about a case in
which the values of an infinite sequence do not get closer and closer to a single value as \( n \) increases, but "oscillate" between two fixed values. His right hand, with the palm towards his left, has a handshape called habit O in American Sign Language for "in gesture studies, where the index finger and the thumb are touching and are slight- ly bent, while the other three fingers are fully bent. In this gesture the touching tip of the index and the thumb is metaphorically indexing a metronomical value standing for the value in the sequence as \( n \) increases (it is almost as if the subject is carefully holding a very tiny object with these two fingers). Holding that fixed handshape, he moves his right hand horizontally back and forth while he says "oscillating."

Hands and arms are essential body parts involved in gesturing. But often it is also the entire body that participates in enacting the inferential structure of an idea.

In the following example (Figure 6-7) a professor of mathematics is teaching a course involving notions of calculus. In this scene he is talking about some particular theorems regarding monotone sequences.

As he is talking about an unbounded monotone sequence, he is referring to the important property of "going in one direction" (i.e., taking increasingly large values). As he says this he is predicting iterative unfolding circles with his right hand and at the same time he is walking forward, accelerating at each step (Figure 6-7a through 6-7e). His right hand, with the palm toward his chest, displays a shape called index V (Thumbs relatively extended and touching the upper part of an extended index bent in right angle, like the other fingers), which he keeps in a relatively fixed position while doing the iteration circular movement. A few milliseconds later he completes the sentence by saying "it takes off to infinity" at the very moment when his right arm is fully extended and his hand shape has shifted to an extended shape called spread with a fully (almost over) extension, and the tips of the fingers pointing forward, slightly at eye-level.

Sometimes, when the sequence exhibits a peculiar property, handshapes adopt specific forms that match the meaning of those properties. In Figure 6-8 we see the same professor talking this time about a situation where the sequence is constant. His dominant hand (the right one, with which he has been writing) curls back, his elbow is bent in 90 degrees and his wrist is maximally bent with the palm oriented down. His fingers are also bent pointing downward (Figure 6-8a). Then while keeping that handshape he extends his elbow (and wrist) producing a small outward (and slightly downward) motion with his right hand. In the meantime his left hand, with palm toward right, raises slowly, forming a five handshape (Figure 6-8b). As he says the word "constant", he abruptly stops the forward motion with his right hand marking a location situated a couple of inches in front of his open left hand (Figure 6-8c). While keeping his left hand totally fixed holding the same "five" handshape, he iterates a couple of times the sitting forward movement with his right hand always stopping sharply at the same location, just a few inches from the open palm of his left hand. These abruptly stopped movements performed with the curled handshape while referring to a constant sequence sharply contrast with the smooth open ended fully extended arm, hand and fingers, of the previous example produced when referring to an unbounded monotone sequence (Figure 6-7).

It is important to mention that in these three cases the blackboard is full of mathematical expressions containing formalisms like the ones we saw in the previous sections (e.g., formalisms, with universal and existential quantifiers, which have no indication or reference to motion). The gestures (and the linguistic expressions used), however, tell us a very different conceptual story. In these examples, these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about. In the first example, the oscillating gesture matches, and it is produced synchronically with, the linguistic expressions used. In the second example, the unfolding iterative circular gesture matches the inferential organization of the iteration involved in the monotone sequence, and the entire body moves forwards as the sequence unfolds. Since the sequence is unbounded, it "takes off to infinity". A idea which is precisely characterized in a synchronous way with the full frontal extension of the arm and the hand. That motion contrasts with the one in the third example, where a closed shaped hand moves slightly forward but hits repeatedly the same location never being able to go further.

We can conclude from these examples that:

First, gestures provide converging evidence for the psychological and embodied reality of the linguistic expressions analyzed with classic techniques in cognitive linguistics, such as metaphor and fictive motion analysis. In these cases gesture analysis shows that the metaphorical expressions we saw earlier are not cases of dead metaphors. The above gestures show that the dynamism involved in these ideas has full psychological and cognitive reality, which is enacted in real time while speaking and thinking in an inter-sensory context.

Second, these gestures show that the fundamental dynamic contents involving finite sequences, limits, continuity, and so on, are in fact ontologically of the inferential structure of these ideas. Formal language in mathematics, however, is not as rich as everyday language and cannot capture the full complexity of the inferential structure of mathematical ideas. It is the job of the cognitive science of mathematics to characterize the full richness of mathematical ideas (it is not the job of mathematics itself). These findings should inform mathematics education in order to conceive a more human cognitive-friendly way of teaching mathematics.

CONCLUSION

For many, mathematics is a timeless set of truths about the universe, transcending our human existence. For others mathematics is what is characterized by formal definitions and axiomatic systems. And for others it is "just" a social construction, very much like music or literature. These different views of the nature of mathematics produce corresponding views of mathematics education that interfere with a healthy process of teaching and learning mathematics. The first two, which are predominant in our culture, ignore the fact that mathematics is a genuine creation of the abstraction and imagination of the human animal. Under these accounts natural human meaning and understanding are not part of the picture, and therefore...
FOUNDATIONS FOR THE FUTURE IN MATHEMATICS EDUCATION

Edited by

Richard A. Lesh
Indiana University

Eric Hamilton
U.S. Air Force Academy—Colorado

James J. Kaput