# Conceptual Metaphor and the Cognitive Foundations of Mathematics: Actual Infinity and Human Imagination*

**Rafael E. Núñez**

Department of Cognitive Science, University of California,

San Diego, USA

nunez@cogsci.ucsd.edu

The Infinite is one of the most intriguing ideas in which the human mind has ever engaged. Full of paradoxes and controversies, it has raised fundamental issues in domains as diverse and profound as theology, physics, and philosophy. The infinite, an elusive and counterintuitive idea, has even played a central role in defining mathematics, a fundamental field of human intellectual inquiry characterized by precision, certainty, objectivity, and effectiveness in modeling our real finite world. Particularly rich is the notion of *actual infinity*, that is, infinity seen as a "completed," "realized" entity. This powerful notion has become so pervasive and fruitful in mathematics that if we decide to abolish it, most of mathematics as we know it would simply disappear, from infinitesimal calculus, to projective geometry, to set theory, to mention only a few.

From the point of view of cognitive science, conceptual analysis, and cognitive semantics, the study of mathematics, and of infinity in particular, raises several intriguing questions: How do we grasp the infinite if, after all, our bodies are finite, and so are our experiences and everything we encounter with our bodies? Where does then the infinite come from? What cognitive mechanisms make it possible? How does an elusive and paradoxical idea such as the infinite structure an objective and precise field such as mathematics? Why do the various forms of infinities in mathematics have the exact conceptual structure they have? These, of course, are not simple questions. Nor are they new questions. Some of them have been already approached in the fields of philosophy, philosophy of

* This article has been adapted from a more elaborated piece to be published in the *Journal of Pragmatics*.

mathematics, and formal logic. The problem, however, is that these disciplines when dealing with the nature and structure of ideas and concepts often ignore important empirical data. That is, they fail to consider the constraints imposed by findings in the contemporary scientific study of the human mind, the nervous system, and of real human cognitive phenomena. As a result, the study of the nature and the foundation of mathematical entities is often ultimately reduced to discussions over formal proofs or the use of axioms. The contemporary scientific study of the mind tells us that human reasoning and conceptual structures are far from functioning in terms of formal proofs and axioms. What we need in order to answer the above questions is to seriously take into account how the human mind works, and at the very least provide cognitively plausible answers, that eventually could be tested empirically.

In this article, I intend to accomplish three things. First, I want to provide answers to the above questions based on findings in Conceptual Metaphor and Blending Theories, and building on the work I have done in collaboration with George Lakoff in the field of *cognitive science of mathematics*. In the process I'll be using an approach we have called *Mathematical Idea Analysis* (Lakoff and Núñez, 2000). Second, I will analyze in more detail a particular kind of actual infinity, namely, *transfinite cardinals*, as conceptualized by one of the most imaginative and controversial characters in the history of mathematics, the 19$^{th}$ century mathematician Georg Cantor (1845-1918). As we will see later, Cantor created a very precise and sophisticated hierarchy of infinities that opened up entire new fields in mathematics giving shape, among others, to modern set theory. Many celebrated counterintuitive and paradoxical results follow from his work. In this article I will try to explain the cognitive reasons underlying such paradoxes. Finally, I want to analyze what Lakoff and I called the BMI - the Basic Metaphor of Infinity (Lakoff & Núñez, 2000) - in terms of a conceptual blend (Fauconnier & Turner, 1998, 2002). Lakoff and I have hypothesized that the BMI is a human everyday conceptual mechanism that is responsible for the creation of all kinds of mathematical actual infinities, such as points at infinity in projective and inversive geometry, infinite sums, mathematical induction, infinite sets, infinitesimal numbers, least upper bounds, and limits. In this paper then I will take the BMI to be a conceptual blend where BMI stands for the more generic term Basic *Mapping* of Infinity.

## Potential and Actual infinity

Since the time of Aristotle, the infinite has been treated with extreme care. Many Greek thinkers considered the infinite as an entity with no order, chaotic, unstructured. They used the word απειρον, which evoked the idea of illimited and undefined. According to Rucker (1982) *apeiron* not only designated the idea of infinitely big, but also the idea of disorder, infinitely complex. The infinite, therefore, was seen as an entity to be avoided in proper reasoning. For Plato (427-348 BC) and Pythagoras (580-500 BC) there was no room for apeiron. Aristotle (384-322 BC), in his *Physics*, stated quite clearly that infinity should be considered as something that “has potential existence” but never as an actual realized thing. This view dominated most of the debates (in Europe) involving the infinite all the way up to the Renaissance. In mathematics this was no exception and the distinction between *potential* and *actual* infinity has ever since been made, by readily accepting the former and by questioning or simply rejecting the latter.

Potential infinity is the kind of infinity characterized by an ongoing process repeated over and over without end. It occurs in mathematics all the time. For instance, it shows up when we think of the unending sequence of regular polygons with more and more sides. We start with a triangle, then a square, a pentagon, a hexagon, and so on. Each polygon in the sequence has a successor and therefore there is the potential of extending the sequence again and again without end. The process, at any given stage encompasses only a final number of repetitions, but as a whole doesn’t have an end and therefore does not have a final resultant state.

But more than potential infinity, what is really interesting and mathematically fruitful is the idea of *actual* infinity, which characterizes an infinite process as a *realized* thing. In this case, even though the process is *in*-finite, that is, it does not have an end, it is conceived as being "completed" and as having a *final resultant state*. Following on the example of the sequence of regular polygons, we can focus our attention on certain aspects of the sequence and observe that because of the very specific way in which the sequence is built certain interesting things happen. After each iteration the number of sides grows by one, and the sides become increasingly smaller. As we go on and on with the process we can see that the shape of the polygon becomes closer and closer to the shape of a circle. Thinking in terms of actual infinity imposes an end *at infinity* where the entire infinite sequence *does have* a final resultant state, namely a circle *conceived as* a regular polygon with an infinite number of sides. This circle has all the prototypical properties circles have (i.e., area, perimeter, a center equidistant to all points on the circle, $\pi$ being the ratio between the perimeter and the diameter, etc.) but it *is* a polygon.

It is the fact that there is a final resultant state that makes actual infinity so rich and fruitful in mathematics. But it is also this same feature that has made the idea of actual infinity extremely controversial because it has often lead to contradictions, one of the worst evils in mathematics. A classic example is the "equation" $k/0 = \infty$, where $k$ is a constant. This "equation" is based on the idea that (when finite values are concerned) as the denominator gets progressively smaller the value of the fraction increases indefinitely. So *at* infinity the denominator *is* 0 and the value of the fraction *is* $\infty$. The problem is that by accepting this result we are forced to accept that $(0 \cdot \infty) = k$, that is, the multiplication of zero times infinity could be equal to any number. This, of course, doesn't make any sense. Because of contradictions like this one many brilliant mathematicians, such as Galileo (1564-1642), Carl Friedrich Gauss (1777-1855), Augustin Louis Cauchy (1789-1857), Karl Weierstrass, Henri Poincaré (1854-1912), among others had energetically rejected actual infinity. Up to the $19^{th}$ century there was a well-established consensus among mathematicians that at best actual infinity could provide some intuitive ideas when dealing with limits, for instance, but that no consistent and interesting mathematics could possibly come out of an infinity actually realized. Georg Cantor, following some preliminary work by Bernard Bolzano (1781-

1848) and Richard Dedekind (1831-1916) radically challenged this view, seeing in actual infinity a genuine mathematical entity. His controversial, unconventional, and highly disputed work generated amazing new mathematics.

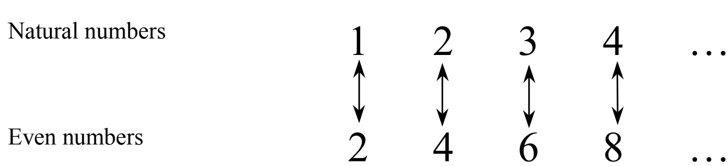
## A brief history of transfinite cardinals

The 19th century was a very productive period in the history of mathematics, one that saw fundamental developments such as non-euclidean geometries, and the so-called arithmetization of analysis. The latter, a program lead by Karl Weierstrass, Richard Dedekind, and others, intended to ban geometrical and dynamic intuition (thought to be the source of paradoxes) by reducing the whole field of calculus developed in the 17th century by Newton and Leibniz, into the realm of numbers. Counting and focusing on discrete entities, like numbers, became essential. It is in this *zeitgeist* that Georg Cantor was brought into his development of transfinite numbers, dispelling well-established views that abolish the use of actual infinities in mathematics. Today, Cantor is best known for the creation of a mathematical system where numbers of infinite magnitude define very precise hierarchies of infinities with a precise arithmetic, giving mathematical meaning to the idea of some infinities being greater than others. His work was highly controversial, produced many counter-intuitive results and for most of his professional life Cantor had to struggle against heavy criticism (for details, see Dauben 1990).

A basic problem for Cantor was to determine the number of elements in a set. This is, of course, a trivial problem when one deals with finite sets, but when one deals with sets with infinitely many elements, such as the set of counting numbers 1, 2, 3, … , (i.e., the set of so-called natural numbers) things are quite different. How do you count them if they are infinitely many? Cantor focused on the fact that when comparing the relative size of finite sets, not only can we count their elements, we can also set up pairs by matching the elements of the two sets. When two finite sets have the same number of elements, a one-to-one correspondence between them can be established. And conversely, when a one-to-one correspondence between two finite sets can be established we can conclude that they have the same number of elements.

Cantor elaborated on the idea of one-to-one correspondence and extended it to infinite sets. He turned to the following question: Are there more natural numbers than even numbers? A similar question had already been asked in the first half of the 17$^{th}$ century by Galileo. He observed that it was possible to match, one-by-one, the natural numbers with their respective squares but because the squares are contained in the collection of natural numbers they form a smaller collection than the natural numbers. Facing this paradoxical situation Galileo concluded that attributes such as “bigger than,” “smaller than,” or “equal to” shouldn’t be used to compare collections when one or both had infinitely many elements. But in the 19$^{th}$ century Cantor could get around the “paradox” by building on the previous very creative though not well-recognized work by Bernard Bolzano and by Richard Dedekind. These two mathematicians were the first to recognize the possibility of matching the elements of an infinite set with one of its subsets as an *essential* property of infinite sets and not as a weird pathology. Dedekind in fact provided for the first time in history a definition involving the infinite in positive terms (i.e., not in negative terms such as *in*-finite or *non*-finite). He said (in modern terminology) that a set *S* is infinite if and only if there exists a proper subset *S'* of *S* such that the elements of *S'* can be put into one-to-one correspondence with those of *S*. Only infinite sets have this property.

Figure 1.
A mapping establishing the one-to-one correspondence between the sets of natural and even numbers.

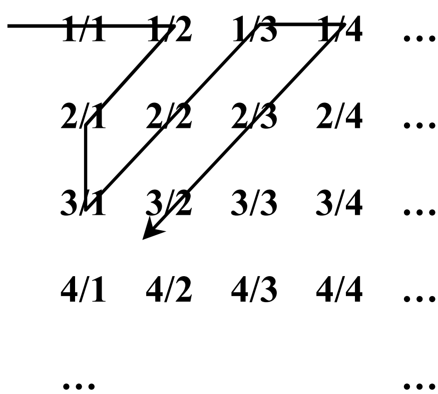



With this background, Cantor had the way paved for answering his question regarding the size of natural and even number. He then declared that “whenever two sets - finite or infinite - can be matched by a one-to-one correspondence, they have the same number of elements” (Maor, 1991, p. 57). Because such a correspondence between natural and even numbers, can be established (Figure 1), he concluded there are just as many even numbers as there are natural numbers. In this framework, the fact that all even numbers are contained in the natural numbers (i.e., the former constitute a proper subset of the latter) doesn’t mean that the set of natural numbers is bigger. Following Dedekind’s definition above, that fact simply shows a property of infinite sets.

And what about other kinds of infinite sets, with more challenging properties? Could such sets be put in one-to-one correspondence with the natural numbers? For instance, natural numbers and even numbers can be ordered according to magnitude such that every member has a definite successor. So what about say, rational numbers, which don’t have this property? Rational numbers are *dense*, that is, between any two rational numbers, even if they are extremely close, we can *always* find another rational number. Rationals don’t have successors. The set of rational numbers seems to have infinitely many more elements than the naturals because not only can we find infinitely many rationals bigger or smaller than a given number (i.e., towards the right or the left of the number line, respectively), we can also find infinitely many rationals in any portion of the number line defined by two rationals. Is then the set of rationals bigger than the naturals?

In order to try to establish a one-to-one correspondence between the rationals and the naturals one needs, first of all, to display both sets in some organized way. In the case of even and natural numbers that organization was provided by order of magnitude. But because rationals are dense they can’t be ordered by magnitude. Cantor, however, found a way of displaying *all* rationals, one by one, in a clever *infinite array*. Figure 2 shows such array, which displays all possible fractions. Fractions with numerator one are displayed in the first row, fractions with numerator two are in the second row, and so on. And similarly, fractions with denominator one are in the first column, fractions with denominator two in the second column, and so on. In 1873 Cantor was able to show, with this array, and against his own intuition (!), that it was possible to establish a one-to-one correspondence between the rationals and the naturals. All you need to do is to assign a natural number to each fraction encountered along the path indicated in Figure 2. The path covers all possible fractions going diagonally up and down ad infinitum[1].

Figure 2.
Cantor's infinite array of rational numbers conceived for the proof of their denumerability. Each fraction covered by the arrow can be mapped to a natural number thus establishing a one-to-one correspondence between the naturals and the rationals.




When such a correspondence is established between two infinite sets, Cantor said that they have the same *power* (Mächtigkeit). Later, thanks to Gottlob Frege (1848-1925) this term became known as *cardinal number*. Cantor called the power of the set of natural numbers, $\aleph_0$, the smallest transfinite number (denoted with the first letter of the Hebrew alphabet, aleph). Today, infinite sets that can be put in a one-to-one correspondence with the natural numbers are said to be *denumerable* or *countable*, having cardinality $\aleph_0$.
Cantor's next question was, are all infinite sets countable? Towards the end of 1873 he found out that the answer was no. He was able to provide a proof that the real numbers can't be put into one-to-one correspondence with the natural numbers: the set of real numbers is not denumerable.

[1] A rational number can be expressed by different fractions. For the purpose of the one-to-one correspondence only the simplest fraction denoting a rational is considered. For example, 2/4, 3/6, 4/8, etc. are equivalent to 1/2, and therefore they are skipped.

In order to proceed with his proof Cantor reasoned by reductio ad absurdum and by considering only the real numbers between zero and one. He started by assuming that a correspondence between the natural numbers and the real numbers between zero and one was possible. Since every real number has a unique non-terminating decimal representation he wrote down the correspondence as follows[2]:

| | | |
|---|---|---|
| 1 | ➔ | $0.a_{11}a_{12}a_{13}\ldots$ |
| 2 | ➔ | $0.a_{21}a_{22}a_{33}\ldots$ |
| 3 | ➔ | $0.a_{31}a_{32}a_{33}\ldots$ |
| … | ➔ | …… |

The list, according to the original assumption includes *all* real numbers between 0 and 1. He then showed that he could construct a real number that wasn't included in the list, a number of the form $0.b_1b_2b_3\ldots$, where the first digit $b_1$ of this number would be different from $a_{11}$ (the first digit of the first number in the list), the second digit $b_2$ of the new number would be different from $a_{22}$ (the second digit of the second number in the list), and so on. As a result, the new number $0.b_1b_2b_3\ldots$, which is bigger than zero but smaller than 1, would necessarily differ from any of the numbers in the list in at least one digit. The digit $b_k$ (the $k$-th digit of the new number) will always differ from the digit $a_{jk}$ given by the diagonal (the $k$-th digit of the $j$-th number of the list when $j = k$). This leads to a contradiction since the original list was supposed to include *all* real numbers between 0 and 1, and therefore the one-to-one correspondence between the naturals and the reals in the interval (0, 1) can't be established. Since the naturals are a subset of the reals this means that the reals form a non-denumerable set which has a power higher than the naturals: A transfinite number bigger than $\aleph_0$. Cantor called it *c*, for the power of the continuum.

But Cantor's work went beyond these two transfinite numbers, $\aleph_0$ and *c*. He showed that in fact there is an entire infinite and very precise hierarchy of transfinite numbers. In order to do so Cantor elaborated on the idea of power set (i.e., the set whose elements are all the subsets of the original set, including the empty set and the original set itself),

[2] The proof requires that all real numbers in the list to be written as non-terminating decimals. For example, a fraction such as 0.3 should be written as 0.2999…

observing that for finite sets when the original set has $n$ elements, its power set has exactly $2^n$ elements. Cantor extended this idea to infinite sets showing that the power (cardinality) of the power set of natural numbers was exactly $2^{\aleph_0}$. This new set in turn formed a power set whose cardinality was $2^{2^{\aleph_0}}$, ans so on. This remarkable result defined a whole infinite hierarchy of transfinite cardinals holding a precise greater than relationship:

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \dots$$

Cantor was able to prove a further extraordinary result: The number of elements in the set of real numbers is the same as the number of elements in the power set of the natural numbers. In other words he proved the equation $c = 2^{\aleph_0}$ to be true, meaning that the number of points of the continuum provided by the real line had exactly $2^{\aleph_0}$ points. But Cantor didn't stop there. He was also able to show an extremely counter-intuitive result : Dimensionality of a space is not related with the numbers of points it contains. Any tiny segment of the real line has the same number of points as the entire line, and the same as in the entire plane, the entire three-dimensional space, and in fact in any « hyper-space » with a denumerable number of dimensions. Cantor added many more counter-intuitive and controversial results to his long list of achievements. He developed a very rich work on transfinite ordinals (see Dauben , 1983, and Sondheimer & Rogerson, 1981), and defined a precise arithmetic for tranfinite cardinals where unorthodox equations such as the following hold:

$\aleph_0 + 1 = 1 + \aleph_0 = \aleph_0$;

$\aleph_0 + k = k + \aleph_0 = \aleph_0$, for any natural number $k$;

$\aleph_0 + \aleph_0 = \aleph_0$;

$\aleph_0 \times k = k \times \aleph_0 = \aleph_0$, for any natural number $k$;

$\aleph_0 \times \aleph_0 = \aleph_0$;

$(\aleph_0)^k = \aleph_0$, for any natural number $k$.

These equations represented an extraordinary improvement in approcahing and studying the infinite when compared to the old and vague idea represented by the symbol ∞. With Cantor infinite numbers acquired a precise meaning, and constituted the corner stone of the development of extremely creative and ingenious new mathematics.

## Actual Infinity: Aspect, conceptual blending, and the BMI

In order to understand the cognitive nature of actual infinity and the conceptual structure underlying transfinite cardinals, we need to refer to two main dimensions of human cognitive phenomena: One is *aspect,* as it has been studied in cognitive semantics, and the other one is the BMI (originally described in Lakoff and Núñez, 2000, as the

Basic Metaphor of Infinity) and here treated as the Basic Mapping of Infinity, a form of conceptual blend.

*Aspect*

In cognitive semantics, aspectual systems characterize the structure of event concepts. The study of aspect allows us to understand, for instance, the cognitive structure of iterative actions (e.g., "breathing", "tapping") and continuous actions (e.g., "moving") as they are manifested through language in everyday situations. Aspect can tell us about the structure of actions that have inherent beginning and ending points (e.g., "jumping"), actions that have starting points only (e.g., "leaving"), and actions that have ending points only (e.g., "arriving"). When actions have ending points, they also have resultant states. For example, "arriving" (whose aspectual structure has an ending point) in *I arrive at my parents' house*, implies that once the action is finished, I am located *at* my parents' house. When actions don't have ending points they don't have resultant states. Many dimensions of the structure of events can be studied through apsect.

For the purpose of this article, the most important distinction regarding aspect is the one between *perfective aspect* and *imperfective aspect*. The former has inherent completion while the latter does not have inherent completion. For example, the prototypical structure of "jumping" has inherent completion, namely, when the subject performing the action touches the ground or some other surface. We say then that "jumping" has perfective aspect. "Flying," on the contrary, does not have inherent completion. The prototypical action of "flying" in itself does not define any specific end, and does not involve touching the ground. When the subject performing the action, however, touches the ground, the very act of touching puts an end to the action of flying but does not belong to "flying" itself. We say that "flying" has imperfective aspect.

Processes with imperfective aspect can be conceptualized both as continuative (continuous) or iterative processes. The latter have intermediate endpoints and intermediate results. Sometimes continues processes can be conceptualized in iterative terms, and expressed in language in such a way. For example, we can express the idea of sleeping continuously by saying "he slept and slept and slept." This doesn't mean that he slept three times, but that he slept uninterruptedly. This human cognitive capacity of conceiving something continuous in iterative terms turns out to be very important when infinity is concerned. Continuous processes without end (e.g., endless continuous monotone motion) can be conceptualized as if they were infinite iterative processes with intermediate endpoints and intermediate results (for details see Lakoff & Núñez, 2000).

With these elements we can now try to understand how human cognitive mechanisms bring *potential infinity* into being. From the point of view of aspect, potential infinity involves processes that may or may not have a starting point, but that explicitly *deny the possibility of having an end point*. They have no completion, and no final resultant state. We arrive then to an important conclusion:

- Processes involved in potential infinity have imperfective aspect.

*BMI, the Basic Mapping of Infinity*

Let's now analyze actual infinity, which is what we really care about in mathematics. It is here where the BMI becomes crucial. The BMI is a general conceptual mapping which is described in great detail elsewhere (Lakoff & Núñez, 2000). It occurs inside and outside of mathematics, but it is in the precise and rigorous field of mathematics that it can be best appreciated. Lakoff and Núñez have hypothesized that the BMI is a single human everyday conceptual mechanism that is responsible for the creation of all kinds of mathematical actual infinities, from points at infinity in projective geometry, to infinite sums, to infinite sets, and to infinitesimal numbers and limits. When seen as a conceptual blend[3] the BMI has two input spaces. One is a space involving Completed Iterative Processes (with perfective aspect). In mathematics, these processes correspond to those defined in the finite realm. The other input space involves Endless Iterative Processes (with imperfective aspect), and therefore it characterizes processes involved in potential infinity. In the blended space what we have is the emergent inferential structure required to characterize processes involved in actual infinity. Figure 3 shows the correspondences between the input spaces and the projections towards the blended space.

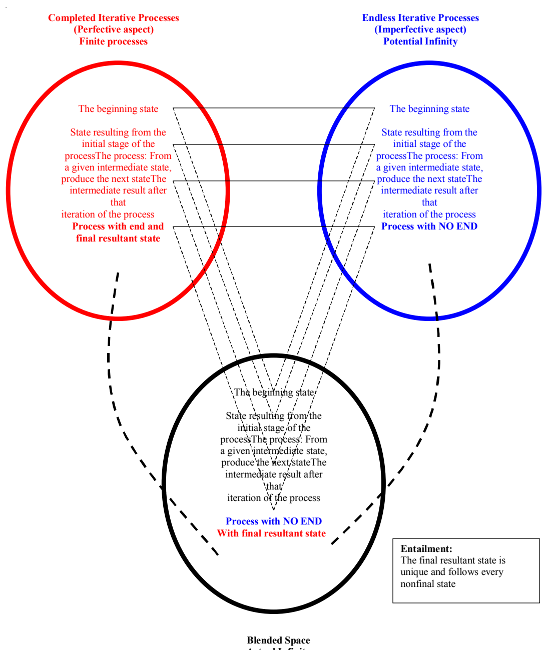



Figure 3.
The BMI, the Basic Mapping of Infinity, as a conceptual blend.

[3] Details of conceptual blending can be seen in other contributions in this volume, and in the original work of Fauconnier and Turner (1998, 2002).

The correspondence between the two input spaces involves all the elements with the exception of the very last one, the single element that distinguishes a finite process from a potentially infinite process. All the corresponding elements are projected into the blended space. But what makes this blend really rich is the fact that both distinctive elements of the input spaces are also projected creating thus the very specific and peculiar inferential structure observed in actual infinity: that an *endless process does have an end and a final resultant state*.

In order to illustrate how the BMI works, let's take the example mentioned earlier of the sequence of regular polygons. The first input space (located on the left in Figure 3) has a finite process with perfective aspect. The process then involves a sequence of regular polygons starting with a triangle, then a square, a pentagon, and so on, all the way to a polygon with a finite number of sides, say 127 sides. The second input space (located on the right in Figure 3), involves the endless sequence of regular polygons discussed earlier (which has imperfective aspect). In the blend, all the corresponding elements are projected, which gives us the sequence of regular polygons with a triangle, a square, a pentagon, and so on. But this time the "endless" component of the sequence (which is projected from input space 2) blends with the "final resultant state" component (from input 1) to give us "an endless sequence of regular polygons whose final resultant state at infinity is a circle." This is a very peculiar kind of circle because it is actually a polygon, a polygon with an infinite number of sides.

## The cognitive mechanisms underlying transfinite cardinals

With these cognitive tools in mind let us now turn to Cantor's work and analyze, one by one, the cognitive mechanisms underlying transfinite cardinals.

<u>1. Cantor's Metaphor: Same Number As IS Pairability</u>

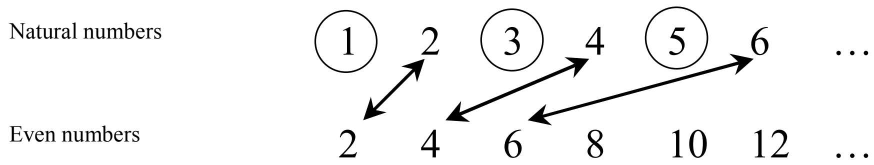
In order to characterize his notion of power (Mächtigkeit) for infinite sets, Cantor makes use of a conceptual metaphor: Same Number As IS Pairability (for details, see Lakoff & Núñez, 2000). This metaphor allows him to create the conceptual apparatus for giving a precise metaphorical meaning to the comparison of number of elements (i.e., power, cardinality) in infinite sets. Let's see how this works.

The everyday notions of "same numbers as" and "more than" are, of course, based on the experience we have with finite - not infinite - collections. The following are everyday (non-formal) characterizations of these notions:

- **Same Number As**: Collection A has the same number of elements as collection B if, for every member of A, you can take away a corresponding member of B and not have any members of B left over.
- **More Than**: Collection B has more elements than collection A if, for every member of A, you can take away a member of B and still have members left in B.

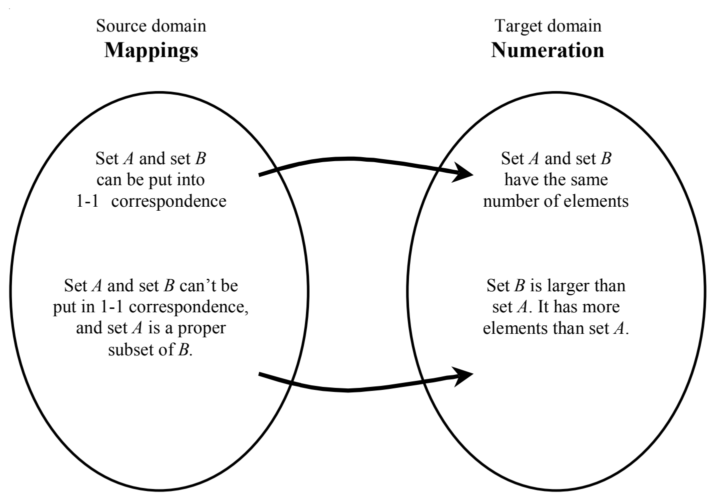
There is nothing uncontroversial about these everyday notions. In fact, decades ago, the Swiss psychologist Jean Piaget described in detail how these fundamental notions get organized quite early in children's cognitive development without explicit goal-oriented education (Piaget, 1952, Núñez, 1993). So, when we approach the question "Are there more natural numbers than even numbers?" equipped with the ordinary notions of "same number as" and "more than", the answer is pretty straightforward. We can match the elements of both sets as shown in Figure 4 and arrive at the conclusion that there are indeed more natural numbers, because there are the odd numbers left over.

Figure 4.
A mapping between the natural and even numbers based on the ordinary notion of "same as" and "more than." The mapping shows that one can pair elements of the two collections and have the odd numbers left over (shown with a circle). The entailment of this natural mode of reasoning is that there are more natural numbers than even numbers.



But, it is true that the two sets are also pairable in the sense that we can put them in a one-to-one correspondence as shown earlier in Figure 3. Pairability and "same number as," however, are two very different ideas. They do have the same extension for finite collections (i.e., they cover the same cases giving the same results), but they are cognitively different and their inferential structure differs in important ways. In his investigations into the properties of infinite sets, Cantor used the concept of *pairability* in place of our everyday concept of *Same Number As*. In doing so, he established a conceptual metaphor, in which one concept (same number as) is conceptualized in terms of the other (pairability). Figure 5 shows the mapping of this simple but crucial conceptual metaphor.

Figure 5.
Georg Cantor's fundamental conceptual metaphor SAME NUMBER AS IS PAIRABILITY. This simple but ingenious metaphor is at the core of transfinite numbers and modern set theory.



We often see in mathematics books, textbooks, and articles statements like “Cantor proved that there are *just as many* positive even integers as natural numbers.” According to a cognitive account of our ordinary notion of “As Many As” Cantor proved no such thing. What Cantor did was simply to prove that the sets were *pairable*. It is only via Cantor’s metaphor that it makes sense to say that he “proved” that there are, metaphorically, “just as many” even numbers as natural numbers. Unfortunately, mathematics texts in general ignore the metaphorical nature of Cantor’s new meaning given to the idea of pairability, ascribing to it a kind of transcendental truth, and failing to see its truth as derived from a conceptual metaphor. As a consequence, they often conclude that there is something fundamentally wrong with human intuition when dealing with infinity. Consider for instance the following citation: “Would it be possible, for example, to match on a one to one basis the set of all counting numbers with the set of all even numbers? At first thought this seems impossible, since there seem to be twice as many counting numbers as there are even numbers. And yet, if we arrange all the even numbers in a row according to their magnitude, then this very act already shows that such a matching is possible … So our intuition was wrong!” (Maor, 1991, p. 56).

The same applies to views within mathematics regarding the role of everyday langauge. Consider the following citation concerning the problem of comparing similar infinite sets: “The confusions and apparent paradoxes in this subject arise from the transfer of everyday language, acquired from experience with finite collections, to infinite sets where we must train ourselves to work strictly with the mathematical rules of the game even though they lead to surprising results.” (Sondheimer & Rogerson, 1981, p. 149).

Our cognitive analysis shows that there is nothing wrong with our “intuition.” And there is nothing wrong with “everyday language” either. Extensive work in cognitive linguistics shows that conceptual metaphor and conceptual blending are not mere linguistic phenomena, but they are about thought and cognition. In the practice of mathematics what is often called “intuition” or naïve ideas expressed by “everyday language” are in fact very well organized conceptual structures based on bodily-grounded systems of ideas with very precise inferential structures. But unfortunately in mathematics, what counts are the “strict” and rigorous “mathematical rules” (which from a cognitive perspective need to be explained as well!). “Intuition” and “everyday language” are seen as vague and imprecise (for further discussion of this and its implications for formal programs in mathematics see Núñez & Lakoff, 1998, and Núñez & Lakoff, submitted).

Consider this other statement: “[Cantor concluded,] there are just as many even numbers as there are counting numbers, just as many squares as counting numbers, and just as many integers (positive and negative) as counting numbers” (Maor, 19, p. 57). In our ordinary conceptual system, this is not true. Not because our intuition is wrong, or because our everyday language is imprecise and vague, but because it is an inference made within a different conceptual structure with a different inferential structure. According to our ordinary notion of “more than” there are indeed more natural numbers than there are positive even integers or squares. And there are more integers than there are natural numbers. There is a precise cognitively-structured logic underlying this inference.

This of course doesn’t lessen Cantor’s brilliant results. Cantor’s ingenious metaphorical extension of the concept of pairability and his application of it to infinite sets constitutes an extraordinary conceptual achievement in mathematics. What he did in the process was create a new technical mathematical concept - pairability - and with it, new mathematics. This new mathematics couldn’t have been invented only with our everyday ordinary notions of “same number as” and “more than.” But Cantor also intended pairability to be a *literal extension* of our ordinary notion of “same number as” from finite to infinite sets. There Cantor was mistaken. From a cognitive perspective, it is a metaphorical rather than literal extension of our very precise everyday concept.

possibility of establishing a one-to-one correspondence between the natural numbers and a dense set such as the rational numbers. What is rarely mentioned in mathematics texts (to say the least) is that this proof makes implicit use of human cognitive mechanisms such as conceptual metaphor and blending. Consider Cantor's infinite array of fractions shown in Figure 2. There the BMI is used over and over, implicitly and unconsciously, in comprehending the diagram. It is used in each row of the array, for assuring that *all* fractions are included. First, the BMI is used in the first row for assuring that a*ll* fractions with numerator one are included in a *completed* collection, without missing a single one. Then, the BMI is used to assure that *all* fractions with numerator two are *actually* included, and so on. In the same way, the BMI is implicitly used in each column of the array to assure that *all* fractions with denominator one, two, three, and so on, are actually included in this infinite array providing completion to it. Finally, the BMI is used in conceptualizing the endless arrow covering a *completed* path. The arrow covers every single fraction in the array assuring, via the BMI, the possibility of the one-to-one correspondence between *all* rationals and naturals. The BMI together with Cantor's metaphor discussed earlier validate the diagram as a proof that the natural numbers and the rational numbers have the same power - that is, the same cardinality.

3. The BMI in Cantor's diagonal proof of the non-denumerability of real numbers

Cantor's celebrated diagonal proof also makes implicit use of the BMI. First, there is the use of the special case of the BMI for infinite decimals. Each line, is of the form $0.a_{j1}a_{j2}a_{j3}\ldots$, where *j* is a natural number denoting the number of the line. Thanks to the BMI each of these unending lines can be conceived as being completed. It is important to remind that the Cantor's diagonal proof requires that all real numbers in the list to be written as non-terminating decimals. It is the BMI that allows a fraction like 0.5 (with terminating decimals) to be conceived and written as 0.4999…, a non-terminating - yet completed - decimal. Then there is the use of the special case of the BMI for the set of *all* natural numbers. Each row corresponds to a natural number, and *all* of them must be there. This provides the conditions for testing the assumed denumerability of the real numbers between zero and one. Third, the proof by reductio ad absurdum assumes that *all* real numbers between zero and one are included in the list. This provides the essential condition for the success of the proof because it guarantees that there is a contradiction if a number is constructed that is not included in the originally assumed *completed* list. This is indeed the case of the new constructed number $0.b_1b_2b_3\ldots$. Fourth, there is the sequence along the diagonal formed by the digits of the form $a_{jk}$ where $j = k$. It, too, is assumed to include *all* such digits on the diagonal. The fact that all real numbers must be written as non-terminating decimals guarantees that a digit $a_{jk}$ when $j = k$ (on the diagonal) is not a part of an endless sequences of zeroes (i.e., a sequence of zeroes for digits $a_{jk}$ when $j < k$, which would be the case of a fraction such as 0.5000…). This is another implicit use of the BMI. And finally, there is the process of constructing the new number $0.b_1b_2b_3\ldots$ by replacing each digit $a_{jk}$ with $j = k$ (on the diagonal) with another digit. The process is unending, but must cover the *whole* diagonal, and must create the new real number, not included in the original list, written as a non-terminating - yet complete - decimal. Another implicit special case of the BMI.

## Conclusion

In this piece I have briefly introduced one aspect of George Cantor's creative work - transfinite cardinals - and I have analyzed some of his celebrated counterintuitive and paradoxical results. Counter-intuitive ideas and paradoxes are very interesting and fertile subject matters for cognitive studies because they allow us to understand human abstraction through conflicting conceptual structures. From the point of view of cognitive science, especially from cognitive linguistics and Mathematical Idea Analysis, it is possible to clarify what makes Cantor's results counterintuitive. These analyses show also that, contrary to many mathematicians' and philosophers of mathematics' beliefs, the nature of potential and actual infinity can be understood not in terms of transcendental (or platonic) truths, or in terms of formal logic, but in terms of human *ideas,* and *human cognitive mechanisms*. Among the most important mechanisms for understanding the cognitive nature of transfinite cardinals and actual infinities are:

– Aspectual systems; with iterative and continuative processes, perfective and imperfective structures with initial states, resultant states, and so on.

– Conceptual metaphors, such as Cantor's Metaphor Same Number As IS Pairability
– Conceptual blending, such as the multiple implicit uses of the BMI, the Basic Mapping of Infinity, in Cantor's proofs.

These mechanisms are not mathematical in themselves. They are human cognitive mechanisms, realized after millions of years of evolution and constrained by the peculiarities of human bodies and brains.

Transfinite cardinals are the result of a masterful combinantion of conceptual metaphor and conceptual blending done by the extremely creative mind of Georg Cantor, who worked in a very prolific period in the history of mathematics. These ideas and the underlying cognitive mechanisms involved in Cantor's work, are bodily-grounded and *not arbitrary*. That they are not arbitrary is a very important point that often gets confused in the mathematical and sometimes the philosophical communities where human-based mechanisms are often taken to be mere "social conventions." What is ignored is that species-specific bodily-based phenomena provide a biological ground for social conventions to take place. This ground, however, is not arbitrary. It is in fact constrained by biological phenomena such as evolutionary facts, morphology, neuroanatomy, and the complexity of the human nervous system. Abundant literature in conceptual metaphor and blending tells us that source and target domains, input spaces, mappings, and projections are realized and constrained by bodily-grounded experience such as thermic experience, visual perception, spatial experience, and so on. In the case of transfinite numbers these constraints are provided by container-schemas for understanding (finite) collections and their hierarchies, visual and kinesthesic experience involved in size comparison and the matching of elements, correlates between motor control and aspect, and so on (for details see Lakoff & Núñez, 2000). The strong biological constraints operating on these mechanisms provide very specific inferential structures which are very different from non (or weakly) constrained "social conventions" like the color of dollar bills or the font used in stop signs. These non-arbitrary cognitive mechanisms, which are essential for the understanding of conceptual structures, can be studied empirically and stated precisely, and cognitive science techniques such as Mathematical Idea Analysis can serve this purpose.

In this article, I mainly referred to transfinite cardinals, as an example of a very rich and interesting case of actual infinity. But this is only one case. Lakoff and Núñez (2000) have shown that there are many other instantiations of actual infinity in mathematics realized via the BMI, such as points at infinity in projective and inversive geometry, infinite sets, limits, transfinite ordinals, infinitesimals, least upper bounds, and so on. These mathematical infinities belong to completely different fields. From a purely mathematical point of view, their existence is guaranteed by very specific tailor-made axioms in various fields. In set theory, for instance, one can make use of infinite sets simply because there is a specific axiom, the *axiom of infinity*, that grants their existence. The existence of other mathematical actual infinities in other fields is guaranteed by similar axioms. The relevance of the BMI then is two-fold. On the one hand it explains with a single mechanism cases of actual infinity ocurring in different non-related mathematical fields. Whereas in mathematics actual infinities are characterized by different sets of axioms in different fields, cognitively, they can be characterized by a single cognitive mechanism: the BMI. On the other hand the BMI provides a cognitively plausible explanation of the nature of actual infinity that is constrained by what is known in the scientific study of human cognition, human conceptual structures, human language, and the peculiarities of the human body and brain. Mathematical axioms don't have to comply with any constraints of this kind, because they only operate within mathematics itself. Therefore, axioms can't provide explanations of the nature of transfinite cardinals, actual infinities, or, for that matter, of mathematical concepts in general. The BMI, along with other cognitive mechanisms, allows us to appreciate the beauty of transfinite cardinals, and to see that the portrait of infinity has a human face.

## References

Dauben, J.W. (1983). Georg Cantor and the Origins of Transfinite Set Theory. *Scientific American*, June: 122–154.

Dauben, J.W. (1990). *Georg Cantor: His Mathematics and Philosophy of the Infinite*. Princeton University Press, Princeton, NJ.

Fauconnier, G. & M. Turner (1998). Conceptual integration networks. *Cognitive Science,* 22 (2): 133–187.

Fauconnier, G. & M. Turner (2002). *The Way We Think. Conceptual Blending and the Mind's Hidden Complexities*. Basic Books, New York.

Lakoff, G. & R. Núñez. (2000). *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. Basic Books, New York.

Maor, E. (1991). *To Infinity and Beyond: A Cultural History of the Infinite*. Princeton University Press, Princeton, NJ.

Núñez, R. (1993). *En deçà du transfini: Aspects psychocognitifs sous-jacents au concept d'infini en mathématiques*. University of Fribourg Press, Fribourg.

Núñez, R. & G. Lakoff (1998). What did Weierstrass really define? The cognitive structure of natural and ε-δ continuity. *Mathematical Cognition*, 4 (2):85-101.

Núñez, R. & G. Lakoff (submitted). An Invitation to the Cognitive Science of Mathematics. *The Mathematical Intelligencer*.

Piaget, J. (1952). *The Child's Conception of Number*. W.W. Norton, New York.

Rucker, R. (1982). *Infinity and the Mind.* Birkhäuser, Boston.

Sondheimer, E. & A. Rogerson (1981). *Numbers and Infinity. A Historical Account of Mathematical Concepts*. Cambridge University Press, Cambridge.