Numbers and Arithmetic: Neither Hardwired Nor Out There

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Abstract

What is the nature of number systems and arithmetic that we use in science for quantification, analysis, and modeling? I argue that number concepts and arithmetic are neither hardwired in the brain, nor do they exist out there in the universe. Innate subitizing and early cognitive preconditions for number—which we share with many other species—cannot provide the foundations for the precision, richness, and range of number concepts and simple arithmetic, let alone that of more complex mathematical concepts. Numbers and arithmetic, and mathematics in general, have unique features—precision, objectivity, rigor, generalizability, stability, symbolizability, and applicability to the real world—that must be accounted for. They are sophisticated concepts that developed culturally only in recent human history. I suggest that numbers and arithmetic are realized through precise combinations of non-mathematical everyday cognitive mechanisms that make human imagination and abstraction possible. One such mechanism, conceptual metaphor, is a neurally instantiated inference-preserving cross-domain mapping that allows the conceptualization of abstract entities in terms of grounded bodily experience. I analyze how the inferential organization of the properties and "laws" of arithmetic emerge metaphorically from everyday meaningful actions. Numbers and arithmetic are thus—outside of natural selection—the product of the biologically constrained interaction of individuals with the appropriate cultural and historical phenotypic variation supported by language, writing systems, and education.

Keywords

abstraction, arithmetic, conceptual metaphor, conceptual systems, embodiment, imagination, inferential organization, mathematics, numerical cognition, number concepts

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Numbers and arithmetic, and mathematics in general, have played a crucial role in the advancement of science. Most practicing scientists, however, rather than being active developers of the mathematics they practice, are users of the mathematics they need. Questions regarding the foundations or nature of numbers and arithmetic don't usually arise in the prototypical practice of science as such. In biology, for instance, most measurements are, in practice, carried out using natural and rational numbers. Most data analyses are performed using real numbers (sometimes complex numbers), and little attention is paid to issues involving the foundations of such numbers and the related arithmetic processes. Numbers and arithmetic, as well as many pieces of the extraordinary edifice of mathematics, from Cartesian geometry to differential equations to inferential statistics, are taken as given and rarely the biologist would question, say, the nature of the axiomatic structure of the number systems she is using. But, what is the nature of these number systems? And more generally, what is the nature of quantification in science? Where do numbers and arithmetic come from? My goal in this article is to address such questions from the perspective of cognitive science—the scientific investigation of the mind, which gathers interdisciplinary efforts from neuroscience to linguistics and from anthropology to cognitive psychology to computer modeling. Contrary to a prevailing view that numbers and number lines find their roots over millions of years of evolution (Shepard 2001; Dehaene et al. 2003), and even that the roots of arithmetic reside in single neurons (Dehaene 2002), I contend that number systems and arithmetic are sophisticated human concepts that were developed culturally in recent human history, and therefore, outside of natural selection proper. I will argue that these concepts have been brought forth by specific combinations of everyday cognitive mechanisms that make human imagination and abstraction possible. I will defend the idea that the astonishingly small genetically shaped set of mechanisms that sustain numerosity estimations and the *cognitive preconditions for numbers*[1]—which many take as the cornerstone of mathematics (and that we share with other species)—cannot be directly extended to provide the precision, richness, and sophistication of the natural numbers and simple arithmetic, let alone that of more complex number systems and other mathematical concepts. For that to happen, precise (not approximate) inferential mechanisms and notation systems are needed. And here is where qualitatively different high-order mechanisms for human imagination—not intrinsically related to numerosity—play a fundamental role in generating precise and abstract concepts such as number and arithmetic. From this perspective, mathematics, and number systems and arithmetic in particular—even in their simplest forms—are not hardwired, but rather they emerge as culturally shaped sophisticated forms of sense-making. They are the product of the interaction of certain communities of individuals with the appropriate culturally and historically shaped phenotype supported by language, writing systems, artifacts, education, and specific forms of environmental dynamics.

Posing the Very Question of the Nature of Numbers Scientifically

Addressing the question of the nature of numbers and arithmetic and their relation to mathematics is not simple. The usual picture of science sees mathematics as the "Queen of sciences," and therefore, mathematics as such, cannot be studied by other scientific disciplines. According to this view, psychologists and neuroscientists may investigate people performing mathematically, sociologists and ethnographers may study the practice of mathematics, and developmental scientists and educators may explore how kids learn mathematics, but no discipline is in a position to investigate mathematics *as such*. After all, Galileo, one of the founding fathers of modern science, is often quoted as having said, "The laws of the universe are written in mathematics. It is our role to learn how to read them." The practice of mathematics in contemporary science seems to unfold in line with this view.

In the preface of our book *Where Mathematics Comes From*, George Lakoff and I had described the widespread folk and academic conception of the nature of mathematics as being essentially independent of human beings (Lakoff and Núñez 2000). As an extended form of Platonism, this view sees mathematics as being predominantly about timeless eternal objective truths, providing structure and order to the universe. Simple arithmetic facts are seen as timeless truths that are part of the universe itself. As mathematician Alain Connes puts it, they are part of the *realité archaïque*—a timeless archaic reality (Connes et al. 2000). Lakoff and I called this view the *romance of mathematics*, a kind of mythology in which mathematics has a truly objective existence, providing structure to this universe and any possible universe, independent of and transcending the existence of human beings or any beings at all. However, despite its immediate intuitiveness and despite being supported by many outstanding physicists and mathematicians, the romance of mathematics is *scientifically* untenable. It is a mythology, and as such, arguing for or against it is a matter of faith, not a matter of scientific discussion. An alternative to this platonic view of mathematics has been a move to an extreme reductionistic approach where mathematics (or any human set of concepts or behavior) is seen as fully explainable in terms of brain mechanisms and the neuroscience of individual cognitive processing. Nowhere is this opposition more clearly stated than in the book *Conversations on Mind, Matter, and Mathematics*, which features a dialogue between mathematician Alain Connes—the hardcore Platonist cited above, and neuroscientist Jean-Pierre Changeux (Changeux and Connes 1998). In such dialog the reader is presented with a sharply divided dichotomy: Either mathematics exists out there independently of human beings, or it is human, and, as such, fully reducible to species-specific neural pathways, cortical activity, and brain dynamics. I will argue that neither of these approaches is on the right track. As we will see, nativist genome- and localization-driven approaches in mainstream neuroscience cannot explain the nature of number systems and arithmetic. The very stating of the question of the nature of number systems and arithmetic requires a subtler and less reductionistic approach.

Numerosity Discrimination and Cognitive Preconditions for Numerical Abilities

We can now proceed with the following scientific question: Are numbers hardwired in the human brain? Or to be more specific, are small natural numbers hardwired in the brain? In order to properly address the question, important distinctions ought to be made between terms, such as numerosity, approximate estimation, counting, magnitude, small everyday numbers, numerals (notations for numbers), and number concepts. In this section we'll discuss the first three.

In the late 1940s, experimental psychologists established that humans have a capacity for making quick, error-free, and precise judgments of the numerosity of small collections of items (Kaufmann et al. 1949). This capacity, which is different from counting or estimating, was called *subitizing*, from the Latin word for "sudden." Humans can subitize—that is, accurately and quickly discriminate the numerosity of—up to about four items (see Figure 1). Today there is a fair amount of robust evidence suggesting that the ability to subitize is inborn, that it exists in other vertebrates as well, and that it is not limited to visual arrays, but is manifested also when sequences of knocks or beeps, or flashes of light are presented (Davis and Pérusse 1988). A now classic survey of the range of subitizing experiments can be found in Mandler and Shebo (1982).

There is now a clear consensus that subitizing is not merely a pattern-recognition process. The neural mechanism underlying subitizing, however, is still in dispute. Gallistel and Gelman have claimed that subitizing is just very fast counting—serial processing, with visual attention placed on each item (Gelman and Gallistel 1978). Dehaene has instead hypothesized that subitizing is all-at-once; that is, it is accomplished via "parallel preattentive processing," which does not involve attending to each item one at a time (Dehaene 1997). Dehaene's evidence for his position comes from patients with brain damage that prevents them from attending to things in their environment serially and therefore from counting them. They can nonetheless subitize accurately and quickly up to three items (Dehaene and Cohen 1994).

Other than subitizing, the field of numerical cognition has established during the last couple of decades that, at a very early age, human babies exhibit a variety of cognitive capacities that are preconditions for numerical competence. For instance, at three or four days, a baby can discriminate between collections of two and three items (Antell and Keating 1983), and under certain conditions, infants can even distinguish three items from four (Strauss and Curtis 1981; van Loosbroek and Smitsman 1990). By four and a half months, babies exhibit behaviors that can be interpreted as having some basic understanding of "simple arithmetic," as in "one plus one is two" and "two minus one is one" (Wynn 1992). Like in subitizing, these abilities are not restricted to visual arrays. Babies can also discriminate small numerosities of sounds. At three or four days, a baby can discriminate between sounds of two or three syllables (Bijeljac-Babic et al. 1991). Also, as early as six months infants are able to discriminate between large collections of objects on the basis of numerosity, provided that they differ by a large ratio (8 vs. 16 but not 8 vs. 12; Xu and Spelke 2000).

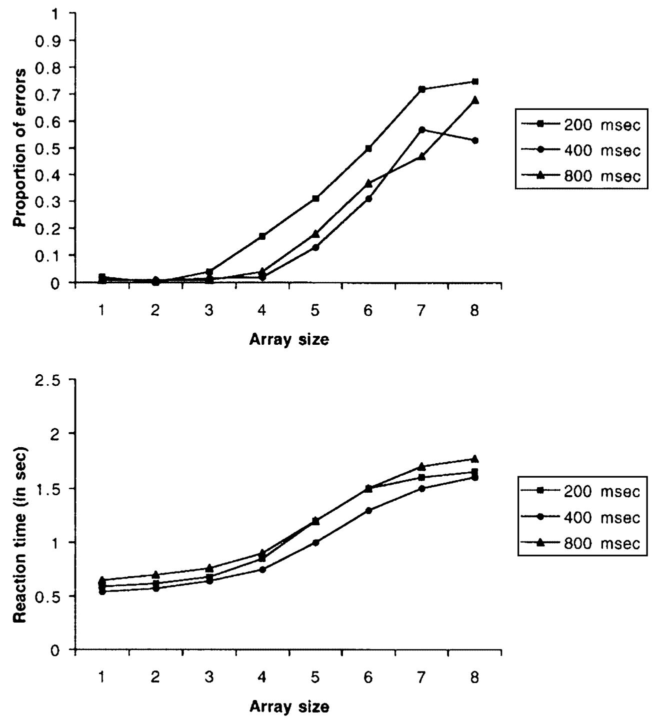

Figure 1.
Numerosity judgments under three experimental conditions. The levels of accuracy (top graphic) and reaction time (bottom graphic) stay stable and low for arrays of sizes of up to four items. The numbers increase considerably for larger arrays (adapted from Mandler and Shebo 1982).

The evidence that babies have these abilities is robust, but many questions remain open. What exactly are the mechanisms—neurophysiological, psychological, and others—underlying these abilities? How stimulus-dependent are these findings? When an infant's expectations are violated in such experiments, exactly what expectations are being violated? How do these abilities relate to other developmental processes? And so on. The experimental results to date give a limited picture. For example, there is no evidence that infants have a notion of the fundamental property of *order* before the age of eleven months (Brannon 2002). If they indeed lack the concept of order before this age, this would suggest that young infants can do what they do without realizing that, say, three is greater than two or two is greater than one (Dehaene 1997). In other words, it is conceivable that very young babies make the distinctions they make, but without a rudimentary concept of order. If so, when, exactly, does the fundamental property of order emerge from the rudiments of baby's abilities and how? Moreover, it is not known how much of the emergence of order is made possible by early cultural and linguistic phenomena. As it has been known for a couple of decades now, already at around eight months infants start to be involved in quite sophisticated social interactions. At eight months, infants exhibit word comprehension, produce deictic gestures, and display gestural routines (Bates et al. 1979; Bates and Snyder 1987), and by 11 months—when infants manifest some understanding of order (Brannon 2002)—they exhibit word production (naming) and recognitory gestures (Bates and Snyder 1987; Bates et al. 1989; Caselli 1990; Shore et al. 1990). These facts of early human cultural, linguistic, and brain development suggest that even a fundamental numerical property like order is not hardwired in the brain, and that, in order to be brought forth, it may require cultural and linguistic stimulation available at a very early age. Despite the evidence discussed, experimenters do not necessarily agree on how to answer these questions and how to interpret many of the findings (for a brief summary, see Bideaud 1996; Bates and Dick 2002).

Early Cognitive Preconditions for Numerical Abilities Don't Provide the Foundations of Number Systems and Arithmetic

Many interesting questions remain open in the study of early cognitive preconditions for numerical abilities. What are the process and mechanisms underlying the development of these abilities? What is the neuroscience of subitizing and numerosity estimation? And, so on. But, with respect to our question of the nature of numbers and arithmetic, how informative can the answers to these questions really be if at this point we haven't even properly addressed the issue of how and when the fundamental numerical property of order enters into the picture of early cognitive abilities? Whatever the case, there is a widespread belief that by studying the basis of the cardinality of "counting" numbers we learn about mathematics *itself* (Butterworth 1999; Dehaene 2002). And with respect to numbers, some experimentalists in child psychology and number neuroscience think that the concept of natural number is bootstrapped from early quantitative skills such as estimating magnitudes and enumerating (e.g., Carey 2004). But are these positions tenable?

A major problem in most accounts of the concept of number is that scholars often introduce crucial elements of the *explanans* in the very *explanandum*.[2] That is, they take number systems as pre-given and introduce them as a part of the explanatory proposal itself (Núñez 2008a). Gallistel et al. (2006: 247), for instance, speak of "mental magnitudes" referring to a "real number system in the brain," where the very *real numbers* are taken for granted, and put them "in the brain." But, we must not forget that the system of real numbers is an extremely sophisticated concept (e.g., it is *infinitely* precise!), shaped historically over centuries and supported by technical notions like completed order field and the least upper bound axiom. How could such a number system be simply "in the brain"? For the purposes of a biological brain dealing with approximate magnitudes in the real world, the dense *ordered field* of rational numbers, for instance—with infinitely many rationals between any two rationals—would suffice. But, again, rational numbers cannot be taken for granted, either, and they must be explained as well: where do they come from?

Similarly, Dehaene (2003: 147) argues for the existence of a "logarithmic mental number line," which somehow "can explain why nature selected an 'internal slide rule' as its most efficient way of doing mental arithmetic." Here again, we see that "logarithms"—and even the "number line," which are part of the explanans (i.e., the numbers whose origin is to be explained), are taken as pre-given and brought teleologically into the explanandum (i.e., the explanation that include those very entities, such as number lines, and logarithms). Not only the slide rule but also the logarithms themselves are human inventions and should not be taken for granted when addressing the question of the nature and the origin of mathematics and number systems.

In a study of numerosity judgments and in support of the idea of an innate mental number line, Dehaene et al. (2008a: 1217) compared Western and Mundurukú people from the Amazon, known for having a language with a reduced lexicon for precise numbers, and argued that "the mapping of numbers onto space is a universal intuition and that this initial intuition of number is logarithmic." According to the authors the results presented therein support the idea that "mathematical objects may find their ultimate origin in basic intuitions of space, time, and number that have been internalized through millions of years of evolution in a structured environment and that emerge early in ontogeny, independently of education." Here again, "mathematical objects" and "numbers," which are part of the explanans, are, in a teleological move, taken as pre-given to the point that they are seen as "internalized through millions of years of evolution." As I pointed out elsewhere (Núñez, 2008b, submitted), the systematic number-space mapping, or more precisely, the concept of the "number line" itself

Figure 2.
A clay tablet from the Old Babylonian period known as YBC 7289. This tablet is one of the few containing drawings. It has a depiction of a square with both diagonals, and several numbers written on it. Experts interpret the content as an exercise of square root approximation (Fowler and Robson 1998). No reference to a number line is made. No representations of a number line are known to exist in Babylonian mathematics. (© Bill Casselman. Yale Babylonian collection; reprinted with permission from http://www.math.ubc.ca/∼cass/Euclid/ybc/ybc.html)

is a sophisticated concept concocted in Europe toward the 17th century, which does not seem to be as "hard-wired" as Dehaene and collaborators suggest. If that were the case, we should expect ubiquitous manifestations of number lines—linear or logarithmic—all over the world since the dawn of civilizations. Moreover, we should find them in the early arithmetic developed by civilizations known for their mathematical sophistication such as those found in Mesopotamia, Egypt, China, and Mesoamerica. However, no such evidence exists. Indeed, 4000-year-old clay tablets show that Babylonians, for instance, developed a sophisticated knowledge of arithmetical bases, fractions, and operations without the slightest reference to number lines as such. Out of the roughly half a million published cuneiform tablets, from which no more than 5,000 tablets contain mathematical knowledge, only about 50 tablets have diagrams on them (Robson 2008) but *none* provides evidence of number lines. Recent research on abundant archaeological material (Fowler and Robson 1998; Robson 2008) shows that until the 3rd century BC *number* was conceptualized essentially as an adjectival property of a collection or a measured object (Figure 2).

Dehaene et al.'s report (2008a) seems to be less about mathematical concepts and more about the affordance of a line segment for reporting approximative numerosity judgments that are subject to the ubiquitous psychophysical Weber–Fechner Law that the magnitude of a perceived intensity of a stimulus is a logarithmic function of objective stimulus intensity. But, crucially, Dehaene et al.'s piece (2008a) presents a much deeper problem. Unanalyzed data reported only in the corresponding "supporting online material" (Dehaene et al. 2008b: 11), show that the subgroup of Mundurukú participants that matters the most for testing the claim of innateness and universality of the number line—the Mundurukú uneducated adults—simply failed to establish the expected number-to-line mapping. These participants, for instance, on average failed to map the lowest number "one" with the left endpoint of the presented line segment, and when the stimuli were presented in tones they even failed to map them on the line preserving the fundamental property of order for the basic numerosities "one," "two," and "three" (for details, see Núñez submitted). Paradoxically, these crucial data, presented only in passing, in fact strongly support the idea that the number line *is not* innately built-in in the human mind. These unanalyzed results seem to be in line with the findings in several studies that have challenged the idea that there is such thing as a fundamental "number line" in the mind (Bächtold et al. 1998; Fischer 2006; Ristic et al. 2006; Santens and Gevers 2008).

Finally, there are at least two main problems that need to be addressed when trying to explain the nature of numbers and arithmetic (and of mathematics, in general) with a primarily bottom-up approach that builds exclusively on early cognitive preconditions for numerical abilities. First, as Rips et al. (2008) had pointed out, these basic skills cannot be directly extended to provide the richness, precision, and sophistication of concepts as fundamental as "natural number." These authors propose that children construct the concept of natural number and arithmetic relying on top-down processes and by constructing "mathematical schemas." Although their notion of "mathematical schema" is quite generic and disembodied, it addresses some of the concerns I analyze in this article, namely, the need to investigate in detail conceptual systems that are concocted and sanctioned externally to the child's mind (Núñez 2008a). The second problem is that unlike numbers, which are precise and operate over large ranges, numerosity judgments and individual neuron tuning are only approximate and operate on extremely small ranges. Explaining the origin of numbers and arithmetic requires an explanation that gives an account of their precision and the highly developed range extension, as well as the specificity and precision of their combinatorial power. Mere training at improving numerosity judgments, whether it is at the level of the individual or the neuron, doesn't provide the answer to the question on the nature of number systems and arithmetic. A less reductionistic, and far more comprehensive approach is required. One that addresses, beyond psychophysical and basic individual cognitive processing, what makes mathematics, and numbers and arithmetic in particular, such a unique conceptual system.

What Is Special About Numbers, Arithmetic, and Mathematics?

Numbers and arithmetic are part of mathematics. And mathematics is unique. It is an extraordinary conceptual system characterized by the fact that the very entities that constitute it are imaginary, idealized, and mental abstractions. These entities cannot be perceived directly through the senses. A Euclidean point, for instance—the simplest entity in Euclidean geometry, cannot actually be perceived. A point, as defined by Euclid is an entity that has only location but no extension. However, humans, or rather, some specially trained humans, can create via *imagination* a Euclidean point in a clear, precise, and nonambiguous manner. And they can build with it more complicated purely imaginary entities, such as segments, planes, polygons, and spheres. A Euclidean point, like mathematical infinities, proofs by reductio ad absurdum, and empty sets, is an *idealized* abstract entity realized via human cognitive mechanisms.The entire edifice of mathematics, involves, in one-way or another, human imagination and idealization.

But if arithmetic, and mathematics in general, is the product of human imagination, how can we explain its nature with those unique features such as precision, objectivity, rigor, generalizability, stability, and, of course, applicability to the real world? And how can we do this when the subject matter is truly abstract and apparently detached from anything concrete, such as complex, infinitesimal, and transfinite numbers? And how can a bodily grounded view of the mind give an account of an abstract, idealized, precise, sophisticated, and powerful domain of ideas if direct bodily experience with the subject matter is not possible?

In the book mentioned earlier, *Where Mathematics Comes From*, George Lakoff and I have proposed some preliminary answers to such questions (Lakoff and Núñez 2000). Building on the findings in mathematical cognition and the neuroscience of numerical cognition, and using (for the moment) mainly the methods from cognitive linguistics, a branch of cognitive science, we asked, which cognitive mechanisms are used in structuring mathematical ideas? And more specifically, what cognitive mechanisms can characterize the *inferential organization* observed in number systems and arithmetic, and in mathematical ideas themselves? We suggested that most of the idealized abstract technical entities in mathematics are made possible by everyday human cognitive mechanisms that extend the structure of bodily experience while preserving crucial inferential organization. Such ordinary mechanisms are, among others, conceptual metaphors (Lakoff and Johnson 1980; Sweetser 1990; Lakoff 1993; Lakoff and Núñez 1997; Núñez and Lakoff 2005), conceptual blends (Fauconnier and Turner 1998, 2002; Núñez 2005, in press), conceptual metonymy (Lakoff and Johnson 1980), fictive motion, and dynamic schemas (Talmy 1988, 2003; Núñez and Lakoff 1998; Núñez 2006). Using a technique that we called *mathematical idea analysis*, we studied in detail many mathematical concepts in several areas of mathematics, from number systems, to set theory, to infinitesimal calculus, to transfinite arithmetic, and showed how, via everyday human embodied mechanisms, such as conceptual metaphor and conceptual blending, the inferential patterns drawn from bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains.

Conceptual Mappings and Inferential Organization in Everyday Abstraction

Let us now leave numbers and mathematics aside for a moment, and look at how humans generate everyday abstraction and conceptual systems as they manifest themselves naturally and effortlessly in ordinary language and discourse. Consider the following two everyday linguistic expressions: "The spring is *ahead* of us" and "the presidential election is now *behind* us." Literally, these expressions don't make any sense. "The spring" is not something that can physically be "ahead" of us in any measurable or observable way, and an "election" is not something that can be physically "behind" us. Hundreds of thousands of these expressions, whose meaning is not literal but *metaphorical*, can be observed in human everyday language. They are the product of human imagination, which convey precise meanings, and when people use them in everyday conversations, they can make precise inferences about them. Cognitive semantics has studied this phenomenon in detail and has shown that the inferential organization of these hundreds of thousands metaphorical linguistic expressions can be modeled by a relatively small number of *conceptual metaphors* (Lakoff and Johnson 1980; Lakoff 1993). These conceptual metaphors, which are inference-preserving cross-domain mappings, are cognitive mechanisms that allow us to project the inferential structure from a source domain, which usually is grounded in some form of basic bodily experience, into another one, the target domain, usually more abstract. A crucial component of what is modeled is *inferential organization*. A substantial body of research has studied these (and other) mechanisms for imagination in many conceptual domains and through various theoretical and empirical methods, from cross-cultural and cross-linguistic studies to experiments in psycholinguistics and cognitive neuroscience, to computer modeling (for an overview, see Gibbs 2008).

In the above examples, although the expressions use completely different words (i.e., the former refers to a location *ahead of us*, and the latter to a location *behind us*), they are linguistic manifestations of a single general conceptual metaphor, namely, TIME EVENTS ARE THINGS IN SAGGITTAL UNIDIMENSIONAL SPACE.[3] As in any conceptual metaphor, the

Table 1. The TIME EVENTS ARE THINGS IN SAGGITTAL UNIDIMENSIONAL SPACE Metaphor.

Source domain Saggittal unidimensional space relative to ego		**Target domain Time**
Objects in front of ego	→	Future times
Objects behind ego	→	Past times
Objects co-located with ego	→	Present times
The further away in front of ego an object is	→	The "further away" an event is in the future
The further away behind ego an object is	→	The "further away" an event is in the past

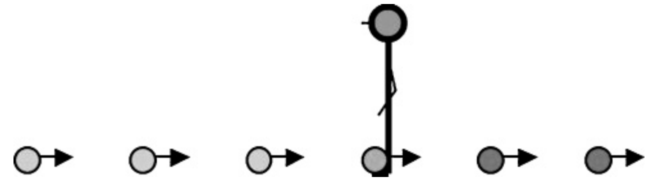

Figure 3.
A graphic representation of the TIME PASSING IS MOTION OF AN OBJECT metaphor.

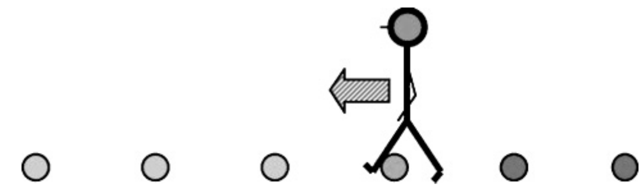

Figure 4.
A graphic representation of the TIME PASSING IS MOTION OVER A LANDSCAPE metaphor.

inferential structure of target domain concepts (time, in this case) is via a precise mapping drawn from the source domain (unidimensional space, in this case). The general mapping of this conceptual metaphor is given in Table 1.

The inferential structure of this mapping accounts for a number of linguistic expressions, such as "the summer is still *far away*," "the end of the world is *near*" and "election day is *here*." Many important entailments—or truths—follow from this mapping. For instance, transitive properties applying to spatial relations between the observer and the objects in the source domain are preserved in the target domain of time: If, relative to the front of the observer, object A is further away than object B, and object B is further away than object C, then object C is closer than object A. Via the mapping, this implies that time C is in a "nearer" future than time A. The same relations hold for objects behind the observer and times in the past. Also, via the mapping, time is seen as having extension, which can be measured; and time can be extended (like a segment of a path), can be conceived as a linear bounded region, and so on.

In what concerns time expressions, cognitive linguists have identified two main forms of this general conceptual metaphor defined according to the nature of the moving agent—the relative motion of ego with respect to the objects, or the objects with respect to ego. This gives rise to the subforms, TIME PASSING IS MOTION OF AN OBJECT (which models the inferential organization of expressions such as *Christmas is coming*) and TIME PASSING IS MOTION OVER A LANDSCAPE (which models the inferential organization of expressions such as *we are approaching the end of the month*) (Lakoff 1993).[4] The former mapping has a fixed canonical observer where times are seen as entities moving with respect to the observer (Figure 3), while the latter has times as fixed objects where the observer moves with respect to the events in time (Figure 4).

These two forms share some fundamental features: Both map (preserving transitivity) spatial locations in front of ego with temporal events in the future, co-locations with ego with events in the present, and locations behind ego (also preserving transitivity) with events in the past. Spatial construals of time are, of course, much more complex, but a detailed analysis of them goes beyond the scope of this piece (for details, see Lakoff 1993; Lakoff and Johnson 1999; Núñez 1999; for cross-linguistic and gestural studies, see Núñez and Sweetser 2006; for experimental psychological studies based on priming paradigms, see Boroditski 2000; Gentner 2001; Núñez et al. 2006).

For the purposes of this article, there are four main morals that we should keep in mind regarding conceptual metaphors (and conceptual mappings in general):

(1) The subject matter of conceptual metaphor analysis is inferential organization

At the level of the cognitive idea analysis we are discussing, the primary focus is not on how *single individuals* learn how to use these conceptual metaphors, or on what difficulties they encounter while they learn them, or on how they may lose the ability to use them after a brain injury, but the focus is on the characterization (i.e., model) across hundreds of linguistic expressions and co-speech gesture productions of the structure of the inferences that can be drawn from such conceptual metaphors. For example, from "the spring is ahead of us," we can infer that the summer is not just ahead of us but *further away in front of us*. Similarly, from "the presidential election is behind us," we can infer that the various effects immediately following the election are not only behind us but also *much closer to us* than the election itself.

(2) There is no pre-given ultimate truth in human conceptual systems

When imaginary entities are concerned (and this is crucial for the study of numbers and arithmetic), *truth* is always relative to the inferential organization of the mappings involved in the underlying conceptual metaphors. For instance, "last summer" can be conceptualized as being *behind us* as long as we operate with the general conceptual metaphor Time Events Are Things In Saggittal Unidimensional Space mentioned above, which determines a specific bodily orientation with respect to metaphorically conceived events in time, namely, the future as being "in front" of us, and the past as being "behind" us. But I have showed in collaboration with Eve Sweetser (Núñez and Sweetser 2006) that this mapping is not universal. Through ethnographic field work, as well as cross-linguistic gestural and lexical analysis of the Aymara language of the Andes' highlands, we provided the first well-documented case that violated the postulated hardwired universality of the metaphorical orientation future-in-front-of-ego and past-behind-ego. In Aymara, for instance, "last summer" is lexicalized and conceptualized as being *in front of ego*, not behind ego, and "next year" not as being in front of ego, but *behind* ego. Moreover, Aymara speakers not only utter these words when referring to time but also produce co-timed corresponding gestures, strongly suggesting that these metaphorical spatial construals of time are not merely about words but also about deeper conceptual phenomena that show in largely unconscious real-world motor action co-produced with speech (Figures 5 and 6).

The moral is that there is no *ultimate transcendental truth* regarding these imaginative structures. Human abstraction allows for different internally consistent forms of sense making, which can be mutually inconsistent, and still have a large overlap in terms of the extentionality of the entailments. For example, for English speakers, as well as for Aymara speakers, it is true that the time of our great-grandparents occurred earlier in the past than the time of our parents (i.e., both cases, with their own internal consistency, extentionally provide the same entailment for the "earlier than" relation). But whereas, for us it is true that the time of our great-grandparents is "further away behind us," this is not true for Aymara people, for whom that time is "further away in front of the them" (i.e., both cases are mutually inconsistent). Therefore, there is no ultimate single truth about where, really, is the ultimate metaphorical location of the future (or the past), or how is future ultimately "represented" in the human brain. Truth will depend on the details of the mappings of the underlying conceptual metaphor. As we'll see, this will turn out to be of paramount importance when mathematical concepts are concerned. Their ultimate truth is not hidden in the structure of the universe, but it will be relative to the underlying human conceptual mappings (e.g., conceptual metaphors) used to create them (for a discussion of how this relates to the nature of mathematics and the embodiment of axiomatic systems, see Núñez 2008c).

(3) Human abstraction is embodied in nature

It is crucial to keep in mind that the abstract conceptual systems we develop are possible *because* we are biological beings with specific morphological and anatomical features. In this sense, human abstraction is *embodied* in nature. It is because we are living creatures with a salient and unambiguous front and a back that we can build on the bodily experiences that these properties provide, and bring forth stable and robust concepts such as "the future in front of us." This wouldn't be possible if we had the body of a jellyfish or an amoeba (even leaving memory and planning capabilities aside). As an analogy, in order to have the possibility to ever come up with a notion of natural number, we (or any living organism) would have to have some kind of morpho-physiological organization sustaining some kind of perception (or propioception) capable of discriminating figure from ground, and capable of enacting individuation and discrete entities. In other words, the embodiment of a living organism whose perceptual systems only allow for perception of gradients, like a sponge, for instance, cannot generate the conditions for the notion of number (or even for the simplest form of numerosity to emerge).

(4) Human abstraction is intrinsically culturally shaped

From the previous points it follows that abstract concepts—even everyday ones that do not rely on writing and technical practices—are intrinsically cultural and not hardwired in the human brain. Because of this reason, when addressing the nature of abstract concepts, such as number and arithmetic, one must not ignore top-down and supra-individual mechanisms that are shaping the phenotype of the cognizing individual (Núñez 1995, 1997). And it should be clear that invoking the essential cultural and supra-individual component of abstract systems doesn't mean that abstract conceptual systems are "simply" socially constructed, as a matter of mere convention. Biological properties and specificities of human bodily grounded experience impose very strong constraints on what concepts can be created. While social conventions usually have a huge number of degrees of freedom, human abstract concepts don't. For example, the color pattern of the Euro bills was socially constructed via convention (and so were the design patterns they have). But virtually any color ordering would have done the job. Primary metaphorical construals of time, on the contrary, are, as far as we know, *only* based on a spatial source domain. And this is not an arbitrary or speculative statement but an *empirical* observation: There is no language or culture on earth, as far as we know, where time is conceived in terms of thermic or chromatic source domains, for instance. Moreover, not any spatial domain does the job. Such spatial construals of time are, as far as we know, always based on unidimensional space.[5] Human abstraction is thus not merely

(a)

(b)

Figure 5.
The speaker, at right, is referring to the Aymara expression *aka marat(a) mararu,* literally "from this year to next year." (a) When saying *aka marat(a),* "from this year," he points with his right index finger downwards and then, (b) while saying *mararu*, "to next year," he points backwards over his left shoulder. (© 2008 Rafael Núñez and Carlos Cornejo)

"socially constructed." It is constructed through strong nonarbitrary biological and cognitive constraints that play an essential role in constituting what human abstraction is. Again, this turns out to be very important when mathematical concepts are concerned.

With these morals in mind, we are now in a position to go back to our discussion of the nature of mathematical entities and analyze what mechanisms of everyday imagination bring the numbers systems and arithmetic to being.

Natural Numbers, Arithmetic, and Conceptual Mappings

Numbers systems and arithmetic are *qualitatively* more complex than subitizing, single-neuron numerosity approximate tuning, and early cognitive preconditions for numerical abilities of monkeys and newborn babies. To understand what is the nature of numbers and arithmetic, we need to understand questions such as, why does arithmetic have the properties it has? where do the laws of arithmetic come from? what cognitive

Figure 6.
The speaker, at left, is talking about the Aymara phrase *nayra timpu,* literally "front time," meaning "old times." When he translates that expression into Spanish, as he says *tiempo antiguo* he points straight in front of him with his right index finger. (© 2008 Rafael Núñez and Carlos Cornejo)

mechanisms are needed to go from the insignificant early abilities we are born with to full-blown arithmetic? In order to go beyond subitizing and estimation we need further cognitive capacities. Subitizing is certain and precise within its range. But we have additional capacities that allow us to extend this certainty and precision. To do this, for instance, we must count. And in order to count (say, on our fingers), several capacities are required:

- Grouping capacity: To distinguish what we are counting, we have to be able to group discrete elements visually, mentally, or by touch.
- Ordering capacity: Fingers come in a natural order on our hands. But the objects to be counted typically do not come in any natural order in the world. They have to be ordered—that is, placed in a sequence, as if they corresponded to our fingers or are spread out along a path.
- Pairing capacity: We need a cognitive mechanism that enables us to sequentially pair individual fingers with individual objects, following the sequence of objects in order.
- Memory capacity: We need to keep track of which fingers have been used in counting and which objects have been counted.
- Exhaustion–detection capacity: We need to be able to tell when there are "no more" objects left to be counted.
- Cardinal-number assignment capacity: We need to be able to operate in a way such that the last number in the count is an ordinal number, a number in a sequence. And we need to be able to assign that ordinal number as the size—the cardinal number—of the group counted. That cardinal number, as such—the size of the group, has no notion of sequence in it.
- Independent-order capacity: We need to realize that the cardinal number assigned to the counted group is independent of the order in which the elements have been counted. This capacity allows us to see that the result is always the same.

When these capacities are used within the subitizing range (between 1 and 4), we get stable results because cardinal-number assignment is done by subitizing, say, the fingers used for counting. To count beyond four—the range of the subitizing capacity—we need not have only the cognitive mechanisms listed above but additional capacities that allow us to put together perceived or imagined groups to form larger groups, and a capacity to associate physical symbols (or words) with conceptual numbers.

But to go beyond subitizing and counting, which only provide some of the cognitive preconditions for numerical abilities, we need to characterize arithmetic operations and their properties. At this point, we need mechanisms for human imagination like the ones analyzed in the previous section, namely, conceptual mappings.

- Metaphorizing capacity: We need to be able to conceptualize cardinal numbers and arithmetic operations in terms of basic experiences of various kinds—experiences with groups of objects, with the part–whole structure of objects, with distances, with movement and locations, and so on.
- Conceptual-blending capacity: We need to be able to form correspondences across conceptual domains that bring emerging inferential structure (e.g., combining subitizing with

counting; numbers with lines to make "numbers-lines") and put together different conceptual metaphors to form complex metaphors. Conceptual metaphor and conceptual blending are among the most basic everyday cognitive mechanisms that take us beyond minimal early abilities and simple counting to the elementary arithmetic of natural numbers.

Since conceptual metaphors preserve inferential organization, such metaphors allow us to ground our understanding of arithmetic in our prior understanding of extremely commonplace physical activities. Our understanding of elementary arithmetic is based on a correlation between (1) the most basic literal aspects of arithmetic, such as subitizing and counting, and (2) everyday activities, such as collecting objects into groups or piles, taking objects apart and putting them together, taking steps, and so on. Such correlations allow us to form metaphors by which we greatly extend our subitizing and counting capacities. Thus, when we conceptualize numbers as collections, we project the logic of collections onto numbers. In this way, experiences like grouping that correlate with simple numbers give further logical structure to an expanded notion of number.

The metaphorizing capacity is central to the extension of arithmetic beyond mere subitizing, early cognitive preconditions for numerical abilities, and counting. George Lakoff and I have suggested (Lakoff and Núñez 2000) that the inferential organization of basic arithmetic with natural numbers comes from four conceptual metaphors that ground our numerical understanding on basic bodily experience, which Lakoff and I called the 4Gs. We named these mappings ARITHMETIC IS OBJECT COLLECTION, ARITHMETIC IS OBJECT CONSTRUCTION, THE MEASURING STICK METAPHOR, and ARITHMETIC IS MOTION ALONG A PATH. The detailed analysis of these mappings can be found elsewhere (Lakoff and Núñez 2000: chs. 3 and 4). But in order to give you a flavor of the robustness and inferential richness of such conceptual mappings, we can flesh out important components of at least one of these conceptual metaphors, i.e., ARITHMETIC IS OBJECT COLLECTION.

When an infant is given a group of blocks of numerosity three, she will naturally and unconsciously subitize them as having three items. If one item is taken away, she will subitize the resulting group as having numerosity two. Such everyday experiences of subitizing, "addition," and "subtraction" with small collections of objects involve correlations between addition and adding objects to a collection, and between subtraction and taking objects away from a collection, respectively. Such regular correlations, Lakoff and I hypothesized, result in neural connections between sensory-motor physical operations like taking away objects from a collection and arithmetic operations like the subtraction of one number from another. Such neural connections, we believe, sustain a conceptual metaphor at the neural level—in this case, the metaphor that ARITHMETIC IS OBJECT COLLECTION. This metaphor, we hypothesize, is—when accompanied and scaffolded by appropriate language, emotional support, and behavior—learned at an early age, prior to any formal arithmetic training. Indeed, arithmetic training assumes this unconscious conceptual (not linguistic) metaphor: In teaching arithmetic, we all take it for granted that the adding and subtracting of numbers can be understood in terms of adding and taking away objects from collections.

Table 2. The ARITHMETIC IS OBJECT COLLECTION Metaphor.

Source domain Object collection		Target domain Arithmetic
Collections of objects of the same size	→	Numbers
The size of the collection	→	The size of the number
Bigger	→	Greater
Smaller	→	Less
The smallest collection	→	The unit (one)
Putting collections together	→	Addition
Taking a smaller collection from a larger collection	→	Subtraction

The ARITHMETIC IS OBJECT COLLECTION metaphor is a precise conceptual mapping from the domain of physical objects to the domain of numbers. The metaphorical mapping consists of (1) the source domain of object collection (based on our commonest experiences with grouping objects); (2) the target domain of arithmetic (structured nonmetaphorically by subitizing and counting); and (3) a mapping across the domains (based on our experience subitizing and counting objects in groups). The basic mapping of this conceptual metaphor is given in Table 2.

We can see empirical evidence of this conceptual metaphor in the actual practice of everyday language. The word "add" has the physical meaning of placing a substance or a number of objects into a container (or group of objects), as in "add sugar to my coffee," "add some logs to the fire," and "add onions and carrots to the soup." Similarly, "take . . . from," "take . . . out of," and "take . . . away" have the physical meaning of removing a substance, an object, or a number of objects from some container or collection. Linguistic examples include "take some books out of the box," "take some water from this pot," and "take away some of these logs." These are not random or superficial occurrences. This reflects deep patterns of how real-world actions such as "to add" and "to take away" are actively recruited to conceptualize imaginary arithmetic facts via the metaphor. By virtue of the ARITHMETIC IS OBJECT COLLECTION metaphor, these expressions are used for the corresponding arithmetic operations of addition and subtraction. If you add four apples to five apples, how many do you have? If you take two apples from five apples, how many apples are left? It follows from the metaphor that adding yields something bigger (more) and subtracting yields something smaller (less). Accordingly, lexical items like “big” and “small,” which indicate literal size for objects and collections of objects, are metaphorically extended so they apply to numbers, as in “which is bigger, 5 or 7?” and “two is smaller than four.” This conceptual metaphor is so deeply ingrained in our unconscious minds that we have to think twice to realize that numbers are not physical objects and so do not literally have a size.

The ARITHMETIC IS OBJECT COLLECTION metaphor has many entailments, which can be stated precisely if we take the basic truths about collections of physical objects, and map them, via the metaphorical mapping, onto statements about numbers. The result is a set of “truths” about the natural numbers under the operations of addition and subtraction. For example, suppose we have two collections, A and B, of physical objects, with A bigger than B. Now suppose we add the same collection C to each. Then A plus C will be a bigger collection of physical objects than B plus C. This is a fact about the collections of physical objects of the same size. Using the mapping NUMBERS ARE COLLECTIONS OF OBJECTS, this physical truth that we experience in grouping objects can now be conceptualized as an arithmetical truth about numbers: If a number A is greater than number B, then A plus number C is greater than B plus C. All of the following truths about numbers arise in this way, via the metaphor ARITHMETIC IS OBJECT COLLECTION.

What we normally call “laws” of arithmetic are in fact metaphorical entailments of the conceptual mapping we are operating with. For instance, in each of the following cases, the conceptual metaphor ARITHMETIC IS OBJECT COLLECTION maps a property of the source domain of object collections to a unique corresponding property of the target domain of numbers. This metaphor extends properties of subitizing 1 through 4, such as precision, to an indefinitely large collection of natural numbers. We can see how properties of object collections are mapped by this metaphor onto properties of natural numbers in general.

- Magnitude: “Object collections have a magnitude” (source domain) maps to “numbers have a magnitude” (target domain).
- Stability of results for addition: “Whenever you add a fixed object collection to a second fixed object collection you get the same result” (source domain), maps to “whenever you add a fixed number to a second fixed number you get the same result” (target domain).
- Stability of results for subtraction: “Whenever you take away a fixed object collection from a second fixed object collection you get the same result” (source domain), maps to “whenever you subtract a fixed number from a second fixed number you get the same result” (target domain).
- Uniform ontology: “Object collections play three roles in addition: what you add to something, what you add something to, and the result of adding. Despite their differing roles they all have the same nature with respect to the operation of the addition of object collections” (source domain) maps to “numbers play three roles in addition: what you add to something; what you add something to, and the result of adding. Despite their differing roles, they all have the same nature with respect to the operation of the addition of numbers” (target domain).
- Closure for addition: “The process of adding an object collection to another object collection yields a third object collection” (source domain) maps to “the process of adding a number to a number yields a third number” (target domain).
- Inverse operations: “For collections, whenever you take away what you added or add what you took away, you get the original collection” (source domain) maps to “for numbers, whenever you subtract what you added, or add what you subtracted, you get the original number” (target domain).
- Unlimited iteration for addition: “You can add object collections indefinitely” (source domain) maps to “you can add numbers indefinitely” (target domain).
- Limited iteration for subtraction: “You can take away object collections from other object collections until nothing is left” (source domain) maps to “you can subtract numbers from other numbers until nothing is left” (target domain).
- Sequential operations: “You can do combinations of adding and taking away object collections” (source domain) maps to “you can do combinations of adding and subtracting numbers” (target domain).

The inferential organization of the ARITHMETIC IS OBJECT COLLECTION metaphorical mapping provides a wide range of essential arithmetic properties. Without going into the details, we can mention several *equational properties* that applying to object collections get metaphorically extended to numbers.

- Equality of result: One can obtain the same resulting object collection/number via different operations.
- Preservation of equality: Adding/subtracting equals to/from equals yields equals.
- Commutativity: Adding A to B gives the same result as adding B to A.
- Associativity: Adding B to C and then adding A to the result is equivalent to adding A to B and adding C to that result.

And several *relationship properties*

- Linear consistency: If a collection/number A is greater than B, then B is less than A.
- Linearity: If A and B are two collection/numbers, then either A is greater than B, or B is greater than A, or A and B are of the same magnitude.

- Symmetry: If collection/number A is the same size as B, then B is the same size as A.
- Transitivity: If collection/number A is greater than B and B is greater than C, then A is greater than C.

The ARITHMETIC IS OBJECT COLLECTION metaphor has many more entailments. Following the same procedure we have used to identify the pre-images of the metaphorical mappings for addition and subtraction of natural numbers, we can analyze the set of truths involving pooling and repeated addition of object collections, as well as those involving splitting and repeated subtraction. We can then state precisely how, via the metaphorical mappings, these truths in the realm of object collection give rise to further arithmetical laws for multiplication and division of natural numbers, including commutativity and associativity for multiplication, distributivity for multiplication over addition, multiplicative identity, and inverse of multiplication (for details, see Lakoff and Núñez 2000: ch. 3).

Extending Number Systems Beyond Direct Bodily Experience via Conceptual Mappings

What we have seen so far applies just to "counting" numbers—the natural numbers. Most of the number systems used by scientists today, however, transcend correlational patterns with direct bodily experience: Dense rational numbers (with infinitely many of them between any two rationals); irrational numbers which cannot be represented as the ratio of two natural numbers; infinitely precise real numbers whose notation requires an infinite decimal expansion; complex numbers where the imaginary unit "i" requires that there is a number such that squaring it yields the negative number "-1"; and many more, such as infinitesimal numbers, hyper real numbers, transfinite numbers, and so on. These numbers cannot be created based on bodily grounded experience, alone. In order to be understood, they require further conceptual mappings to make them possible. We can get a flavor of how this works by going back to our simple ARITHMETIC IS OBJECT COLLECTION metaphor. This metaphor does not provide "zero" as a number. When we subtract, say, six from six, the result cannot be directly conceived in terms of a collection of objects. In the source domain of object-collection manipulation, the action of taking a collection of six objects from a collection of six objects yields an absence of any objects at all—not a *collection of objects*. But if we want the result of that operation to *be* a number (i.e., we value the property of closure for object collection), then, in order to accommodate the ARITHMETIC IS OBJECT COLLECTION metaphor, we must conceptualize the absence of a collection as a collection. For this, a new conceptual mapping is necessary. One that is not grounded in everyday experience at all, but that is sustained by an ordinary mechanism for ordinary imagination: conceptual metaphor. Indeed, what is needed is a metaphor that creates something out of nothing: From the absence of a collection, the metaphorical mapping creates a unique collection of a particular kind—a collection with no objects in it. This ZERO COLLECTION metaphor maps "the lack of objects to form a collection" in the source domain onto "the empty collection" in the target domain. Given this additional metaphor as input, our initial ARITHMETIC IS OBJECT COLLECTION metaphor will now map the newly created empty collection onto a number—which we call "zero." Once the metaphor is extended, more properties of numbers follow as entailments of the metaphor: additive identity and inverse of addition, for instance.

This type of metaphorical conceptual extension is very common in mathematics—an entity-creating metaphor. In this case, the conceptual metaphor creates zero as an actual number. Although zero is an extension of the ARITHMETIC IS OBJECT COLLECTION metaphor, it is not a natural extension. It does not arise from a direct correlation between the experience of collecting and the experience of subitizing or of making approximate numerosity judgments. Children do not come up, by themselves, with such mappings. In order to operate with this extended version of the conceptual mapping one must be explicitly taught and exposed to such material. Nunes et al. (1993) showed that poor children in urban Brazil, who spend most of the day in the street and without formal education, develop arithmetical concepts and skills for dealing with everyday street commercial transactions without the notion of zero, as such. That is, they may have the concept of "nothing" or "absence" but not necessarily the numerical and arithmetical concept of zero. Zero is in fact a very sophisticated concept, which was introduced quite late to medieval Europe with tremendous difficulties, both at the conceptual and semiotic levels (Menninger 1969; Rotman 1987). As Wilden (1972: 188) puts it, "Zero is not an absence, not nothing, not the sign of a thing, not a simple exclusion . . . It is not an integer, but a meta-integer, a rule about integers and their relationships." The ZERO COLLECTION metaphor is therefore an artificial metaphor, concocted ad hoc for the purpose of extension by "experts" who create a solution to a need—a specific entity that denotes nothingness, and that has the same ontology (i.e., being a number) as the other "counting" numbers. This creation is a logically constrained cultural process mediated by language, notation systems, and the invention of cultural artifacts. Therefore, what today appears to be the simple and obvious concept of "zero" is not innate or hardwired in the brain, but rather it is a high-order concocted concept.

The ARITHMETIC IS OBJECT COLLECTION and the ZERO COLLECTION metaphors ground our most basic extension of arithmetic—from the innate subitizing and early cognitive preconditions for numerical abilities to the natural numbers plus zero. As is well known, this understanding of number still leaves gaps: It does not give a meaningful characterization of 5 minus 6 or 3 divided by 4. To characterize how bodily grounded understanding gets extended to bring forth these arithmetic expressions we need further entity-creating metaphors. We need metaphors for the negative numbers, for rational numbers, and so on (for details, see Lakoff and Núñez 2000: ch. 3).

Epilogue: Making Arithmetic Robust via Isomorphism and Conflation

Up to this point we have explored only one of the 4Gs, the four basic grounding metaphors for arithmetic mentioned earlier: ARITHMETIC IS OBJECT COLLECTION. We have seen how this conceptual mapping provides grounding for precise calculation—not just approximate estimation—and unambiguous inferential organization to the concept of natural numbers. But the consolidation of robust understanding of the natural number concept comes when all 4Gs, the four grounding metaphors for arithmetic—ARITHMETIC IS OBJECT COLLECTION, ARITHMETIC IS OBJECT CONSTRUCTION, THE MEASURING STICK METAPHOR, and ARITHMETIC IS MOTION ALONG A PATH—together co-ground similar inferential organization. We saw how truths about object collections can be mapped onto truths about numbers. But remarkably similar entailments apply to cases when the experiential grounding is based on object making, taking steps along a path, and using sticks, fingers, and arms to determine magnitude. These correlations in everyday experience between early cognitive preconditions for numerical abilities and the source domains of these metaphors give rise to the 4Gs. The metaphors—at least in an automatic, unconscious form—arise naturally from such conflations in experience. The significance of the 4Gs is that they allow human beings, who have apparently an innate capacity to form metaphors, to consolidate arithmetic with precision beyond the small subitazable range, while preserving the basic properties of subitizing and other early cognitive abilities. This may seem so obvious as to hardly be worth mentioning, but it is the basis for the systematic extension of innate subitizing (in the 1–4 range) and early abilities way beyond its inherent limits. Because in the subitizable range this knowledge "fits" the experience we have with object collection and construction (i.e., constructing objects from fixed amounts of parts), motion along a path (taking a fixed amount of steps along a path), and manipulation of physical segments to establish magnitude (e.g., palms, feet, etc.), those four domains of concrete experience are suitable for metaphorical extensions of such early knowledge preserving the desired essential properties. Taking one step after taking two steps gets you to the same place as taking three steps, just as adding one object to a collection of two objects yields a collection of three objects. Thus, the properties of early cognitive preconditions for numerical abilities can be seen as "picking out" these four source domains for the metaphorical extension of basic arithmetic capacities beyond the number 4. Indeed, the reason that all these four domains fit the early cognitive abilities in the subitizing range is that there are structural relationships across the domains. Thus, object construction always involves object collection; we can't build an object without gathering the parts together. The two experiences are conflated and, presumably, thereby neurally linked from an early age. Putting physical segments end-to-end is similar to object construction (think of Legos, for instance). When we use a measuring stick to mark off a distance, we are mentally constructing a line segment out of parts—a "path" from the beginning of the measurement to the end. A path of motion from point to point corresponds to such an imagined line segment. In sum, there are structural correspondences between (a) object collection and object construction; (b) the construction of a linear object and the use of a measuring stick to mark off a line segment of certain length; (c) using a measuring stick to mark off a line segment, or "path," and moving from location to location along a path.

As a result of these structural correspondences, there are isomorphisms across the 4G metaphors—namely, the correlations just described between the source domains of those metaphors. This isomorphism defines a one-to-one correspondence between metaphoric definitions of arithmetic operations—addition and multiplication—in the four conceptual metaphors. For such an isomorphism, the following three conditions must hold:

(1) There is a one-to-one mapping, M, between elements in one source domain and elements in the other source domain—that is, the "images" under the "mapping."
(2) M preserves sums: $M(x + y) = M(x) + M(y)$; that is, the images of sums correspond to the sums of images.
(3) M preserves products: $M(x \cdot y) = M(x) \cdot M(y)$; that is, the images of products correspond to the products of images.

Now, consider, for example, the source domains of object collection and motion, which appear quite dissimilar. There is such an isomorphism between those two source domains. First, there is a one-to-one correspondence, M, between sizes of collections and distances moved. For example, a collection of size three is uniquely mapped to a movement of three units of length, and conversely. A close inspection of these two source domains would tell that the other two conditions are also met. This is something crucial about the conceptual organization of number systems and arithmetic: The source domains of all four basic grounding metaphors for the arithmetic of natural numbers are isomorphic in this way. Crucially, there are no numbers in these source domains; there are only object collections, motions, and so on. But given how they are mapped onto the natural numbers, the relevant inferential structures of all these domains are isomorphic.

Aside from the way they are mapped onto the natural numbers, these four source domains are not isomorphic: Object construction characterizes fractions but not zero or negative numbers, whereas motion along a path characterizes zero and negative numbers. In other words, if we look at the complete domains in isolation, we will not see an isomorphism across the source domains. What creates the isomorphism is the collection of mappings from these source domains of the 4Gs onto natural numbers. And what grounds the mappings onto natural numbers are the experiences we have, across the four domains, with innate and early cognitive abilities—with subitizing and counting in such early experiences as forming collections, putting things together, moving from place to place, and so on. The brain sustaining these cognitive activities is, however, not genetically determined to perform in this way. It requires a specific phenotypical variation of body and brain shaped by innate capacities and maturation patterns along with the crucial mediation of culture, language, and specific interactions with created artifacts. Numbers and arithmetic are thus neither hardwired nor out there in the universe. Rather, they are created, not without effort, by the inventive and imaginative social human mind.

Notes

1. The field of numerical cognition often uses the terms "early numerical abilities" and "proto-arithmetic" to refer to the ensemble of innate (or prelinguistically early) abilities that appear to lead to number concepts, such as subitizing and approximate numerosity discrimination. These terms involve a teleological conception where such abilities are seen as already containing a "numerical" component, that is, as if something inherently "numerical" would have been selected in evolution. I contend that this is misleading, and that these abilities are instead *cognitive preconditions for numbers*, but are not intrinsically "numerical" in themselves. The "numerical" status proper, with precision, generalizability, and compositionality, is thus outside of evolution via natural selection.

2. Sometimes potential confusion is introduced when naming or identifying relevant phenomena. For instance, now, thinking retroactively, I consider that the title of the first chapter of our book *Where Mathematics Comes From* (Lakoff and Núñez 2000)—"The Brain's Innate Arithmetic"—is a misnomer in that it conveys the idea that "arithmetic" as such, with all its complexities, is somewhat innate. That is not what the content of the chapter says, but the title conveys that misleading idea.

3. Following a convention in cognitive linguistics, capitals here serve to denote the name of the conceptual mapping as such. Particular instances of these mappings, called metaphorical expressions (e.g., "she has a great future in front of her"), are not written with capitals.

4. For a different and more recent taxonomy based on linguistic data, as well as on gestural and psychological experimental evidence, see Núñez and Sweetser (2006), Núñez et al. (2006), and Cooperrider and Núñez (2009).

5. Although they can, of course, be more complicated. Such is the case of cyclic or helix-like conceptions. But even in those cases the building blocks—a segment of a circle or a helix—preserve the topological properties of the unidimensional segment.

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