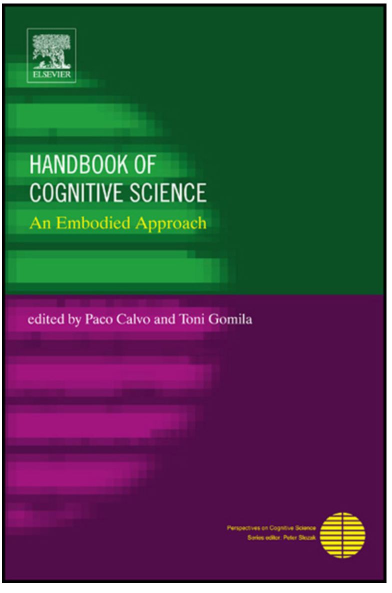

From: Handbook of Cognitive Science: An Embodied Approach, Edited by Paco Calvo and Antoni Gomila
ISBN: 978-0-08-046616-3

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Mathematics, the Ultimate Challenge to Embodiment: Truth and the Grounding of Axiomatic Systems

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The human body is an animal body. A body that has evolved over millions of years coping with real-world properties such as temperature, gravity, humidity, color, space, texture and so on. With this same body humans have been able to create concepts—and think with them—in a way that transcends immediate bodily experience. Today, millions of modern humans effortlessly operate in everyday life with abstract notions like "democracy," "black humor," "inflation," and the "flow of time." In technical domains, like mathematics, humans have created abstract concepts, such as "square root of minus one" and "transfinite numbers"—rich and precise entities that lack any concrete instantiation in the real world. These entities are the product of the human imagination, and exist in the realm of mental abstractions and social practices. How do humans achieve this with the body of a primate? In what sense are the abstract ideas humans create *embodied*? And then there is a question of what *is* mathematics in the first place? What is the nature of this body of knowledge that appears to be timeless, eternal, absolute, and effective to the point that many scholars firmly believe it is part of the very fabric of the universe, transcending human existence?

I must make it clear, right up front, that to address these questions within embodied cognitive science, we must go beyond a concrete understanding of embodiment—"material embodiment" as I called it nearly a decade ago (Núñez, 1999)—a view that centers primarily on *physical corporality*. Moreover, we need to go beyond usual views in embodied cognition that tend to focus on individual perception, action, motor control, emotional states, and neural correlates of given phenomena. As important as this work is, I suggest that relative to the above questions, beyond individuals' performances, behavioral observations, and biological measurements, this time the focus should be on non-material supra-individual cognitive products and their genesis—*conceptual systems*. And by concepts I do not mean pre-defined notions as evoked or thought by someone, but the concepts *themselves*, with their semantic properties and inferential organization. The idea is to explain what *is* mathematics, what makes it possible, and what brings it into being, rather than how individual people learn about *it* (as a pre-existing entity), what people feel in their bodies when they think about it, what parts of people's bodies help them think mathematically, and so on. George Lakoff and I called such an endeavor the Cognitive Science *of* mathematics (Lakoff & Núñez, 2000). In this chapter, we will be looking at the embodiment of stable inferential patterns created and sustained by communities of individuals, which exist beyond the individuals themselves. The approach we take here is comparable to the study of, say, speech accents, in that although they are created, manifested, and sustained *by* individuals, they—the speech accents themselves—constitute distinctions we make at a supra-individual level. In the same way that we speak of Welsh or Jamaican accents, or of John as having an Australian accent but not of John-ian or Sally-ian accents (i.e., the Welsh accent is the one we observe among Welsh *people*) here we will talk about abstract concepts—not individual conceptualizations but concepts that constitute collective domains of knowledge. In sum, we will analyze the embodied cognitive mechanisms that make human abstraction and their supra-individual crystallization possible, and we will see how they bring mathematics, its concepts, and inferential organization, into being.

My goal in this chapter is to provide a brief overview of what is the nature of mathematics from the perspective of embodied cognitive science and conceptual systems. I want to show how the inferential organization of mathematics emerges from everyday cognitive mechanisms of human imagination realized via embodied conceptual mappings such as metaphor, metonymy, conceptual blends, and so on. This, of course, is a vast enterprise, so here I will only concentrate on the fundamental concept of *axiom*, which modern formal mathematics takes to be an essential building block for the study of the foundations of mathematics itself. Contrary to the widespread belief among mathematicians and logicians who see axioms as meaningless formal statements, I want to show that axioms are the product of embodied cognitive mechanisms. Through the analysis of hypersets—a specific branch of contemporary set theory—I intend to show how the quintessential abstract conceptual system we call mathematics (1) emerges from embodied cognitive mechanisms for imagination such as conceptual metaphor; (2) that truth and objectivity comes out of the collective use of these mechanisms; (3) that it can have domains that are internally consistent but mutually inconsistent; (4) and that these domains built on corresponding axiom systems that while grounded in embodied meaning provide different "truths" and inferential organization. Finally, I want to show that these properties are not unique to mathematics but that they exist in everyday abstract conceptual systems as well. I will illustrate this point with empirical observations from my investigation contrasting spatial construals of time in the western world with that in the Aymara culture of the Andes' highlands. I will defend the idea that everyday conceptual systems possess elementary embodied forms of "truth," "axioms," and "theorems" (i.e., true statements derived within a logical system) that are "objective" within the communities that operate with them. These properties of ordinary human imagination serve as grounding for developing more complex and refined forms of abstraction, which find the most sublime form in mathematics.

MATHEMATICS, A REAL CHALLENGE TO EMBODIMENT

Mathematics is a unique body of knowledge. The very entities that constitute what it is are idealized mental abstractions, which cannot be perceived directly through the senses. The empty set, for instance—the simplest entity in set theory—cannot be actually perceived. We cannot physically observe collections with no members. Or take the simplest entity in Euclidean geometry, the point. As defined by Euclid, a point is a dimensionless entity, which has only location but no extension! The empty set and the Euclidean point, with their precision and clear identity, are idealized abstract entities that do not exist in the real physical world, and therefore they are not available for empirical investigation. Yet, they are fundamental building blocks for the construction of set theory and Euclidean geometry, respectively. But nowhere can the imaginary nature of mathematics be seen more clearly than in concepts involving infinity. Because of the finite nature of our bodies and brains, no direct experience can exist with the infinite itself! Yet, infinity in mathematics is essential. It lies at the very core of many fundamental domains such as projective geometry, infinitesimal calculus, point-set topology, mathematical induction, and set theory, to mention a few. Taking infinity away from mathematics would mean the collapse of this extraordinary edifice, as we know it.

Moreover, mathematics has a unique collection of features. It is (extremely) precise, objective, rigorous, generalizable, and, of course, applicable to the real world. It is also extraordinarily stable, in that a theorem once proved, stays proved forever! Any attempt to address the nature of mathematics must explain these features. What is, then, the nature of mathematics? What makes it possible? What is the cornerstone of such a fabulous objective and precise logical edifice? Such questions have been treated extensively in the realm of the philosophy of mathematics, which are becoming, in the 20th century, specific subject matters for rather technical fields of formal logic and metamathematics. Ever since, the foundations of mathematics have taken to be intramathematical (i.e., inside mathematics proper), as if the tools of formal logic alone are to provide the ultimate answers about the nature of mathematics. But can the foundations of mathematics be, themselves, mathematical entities? Or do they lie outside of mathematics? And if they do, where do these entities come from? And what forms do they have? As we will see, within the formalist approaches, the quest for *axioms* and the study of their deductive power have become a fundamental issue in the investigation of the nature of mathematics.

In an attempt to answer these questions, another (very influential) approach has come from good old Platonism, which relying on the existence of transcendental worlds of ideas beyond human existence, sees mathematical truths and entities as existing independently of human beings. This view, however does not have any support based on scientific findings and does not provide any link to current empirical work on human ideas and conceptual systems (although, paradoxically, as a matter of faith it is supported by many mathematicians, physicists, and philosophers). For scholars who endorse a socio-cultural view (often along the lines of postmodernism), the question of the nature of mathematics is relatively straightforward: Mathematics, much like art, poetry, architecture, music, and fashion, is a "social construction" (Lerman, 1989). Although I endorse the relevance of socio-cultural dimensions in mathematics (e.g., see Lakoff & Núñez, 2000, pp. 355–362), I defend the idea that mathematics is *not just* the result of socio-cultural practices. It is not clear, in a purely socio-cultural constructivist view, what makes mathematics so special. What distinguishes mathematics from other forms of social constructions, say, fashion or poetry? Any precise enough explanatory proposal of the nature of mathematics should give an account of the peculiar collection of features that make mathematics so unique: precision, objectivity, rigor, generalizability, stability, and applicability to the real world. This is what makes the scientific study of the nature of mathematics so challenging: mathematical entities (organized ideas and stable concepts) are abstract and imaginary, yet they are realized through the biological and social peculiarities of the human animal. For those studying the human mind scientifically, the question of the nature of mathematics is indeed a real challenge, especially for those who endorse an embodied oriented approach to cognition. The crucial question is: *How can an embodied view of the mind give an account of an abstract, idealized, objective, precise, sophisticated, and powerful domain of ideas if direct bodily experience with the subject matter is not possible?* In *Where Mathematics Comes From*, Lakoff and I propose some preliminary answers to this question (Lakoff & Núñez, 2000). Several basic elements of this proposal are analyzed in the next section.

EVERYDAY EMBODIED MECHANISMS FOR HUMAN IMAGINATION

Building on findings in mathematical cognition and the neuroscience of numerical cognition, and using mainly methods from cognitive linguistics, a branch of cognitive science, Lakoff and I asked, what cognitive mechanisms are used in structuring mathematical ideas? And more specifically, what cognitive mechanisms can characterize the inferential organization observed in mathematical ideas themselves? We suggested that most of the idealized abstract technical entities in mathematics are created via everyday human cognitive mechanisms that extend the structure of bodily experience while preserving inferential organization. Such "natural" mechanisms are, among others, conceptual metaphors (Lakoff & Johnson, 1980, 1999; Johnson, 1987; Sweetser, 1990; Lakoff, 1993; Lakoff & Núñez, 1997; Núñez & Lakoff, 1998, 2005), conceptual metonymy (Lakoff & Johnson, 1980), and conceptual blends (Fauconnier & Turner, 1998, 2002; Núñez, 2005). Using a technique we called *Mathematical Idea Analysis* we studied in detail many mathematical concepts in several areas in mathematics, from set theory to infinitesimal calculus, to transfinite arithmetic, and showed how, via these everyday embodied mechanisms, the inferential patterns drawn from bodily experience in the real world get extended in very specific and precise ways to give rise to a new emergent inferential organization in purely imaginary domains.

Consider the following two everyday linguistic expressions: "The election is *ahead* of us" and "the long Winter is now *behind* us." Literally, these expressions do not make any sense. "An election" is not something that can physically be "ahead" of us in any measurable or observable way, and the "Winter" is not something that can be physically "behind" us. Hundreds of thousands of these expressions, whose meaning is not literal but *metaphorical*, can be observed in human everyday language: "he is a *cold* person," "she has *strong* opinions," "the market is quite *depressed*." Metaphor, in this sense, is not just a figure of speech, or an exceptional communicational tool in the hands of poets and artists. It is an ordinary mechanism of thought, which, usually operating unconsciously and effortlessly, permeates nearly every aspect of human everyday (and technical) language, making imagination possible.

Cognitive linguistics (and more specifically, cognitive semantics) has studied this phenomenon in detail and has shown that the meaning of these hundreds of thousands metaphorical linguistic expressions can be modeled by a relatively small number of conceptual metaphors. These conceptual metaphors, which are inference-preserving cross-domain mappings, are cognitive mechanisms that allow us to project the inferential structure from a grounded *source domain*, for instance, thermic experience, into another one, the *target domain*, usually more abstract, say, affection. As a result, specific temperature-related notions like "cold" and "warm" get mapped onto "lack of affection" and "presence of affection," respectively, and open up an entire world of inferences where the relatively abstract domain of "affection" is conceived and understood in terms of a more concrete one, namely, thermic experience. A crucial component of what is modeled is inferential organization, the network of inferences that is generated via the mappings.

We can illustrate how the mappings work with the above temporal examples where events are conceived as being in front of us or behind us. Note that although the expressions use completely different *words* (i.e., one refers to a location *ahead* of us, whereas the other to a location *behind* us), they are both linguistic manifestations of a single general conceptual metaphor, namely, TIME EVENTS ARE THINGS IN SAGGITAL UNIDIMENSIONAL SPACE.[1] As in any conceptual metaphor, the inferential structure of target domain concepts (time, in this case) is created via a precise mapping drawn from the source domain (in this case saggital unidimensional space: the linear space in front and behind an observer). The general mapping of this metaphor is shown in the following table[2]:

Source Domain Saggital unidimensional space relative to ego		Target Domain Time
Objects in front of ego	→	Future times
Objects behind ego	→	Past times
Object co-located with ego	→	Present time
The further away in front of ego an object is	→	The "further away" an event is in the future
The further away behind ego an object is	→	The "further away" an event is in the past

The inferential structure of this mapping accounts for a number of linguistic expressions, such as "The summer is still *far away*," "The end of the world is *near*" and "Election day is *here*." Many important entailments—or truths—follow from the mapping. For instance, transitive properties applying to spatial relations between the observer and the objects in the source domain are preserved in the target domain of time: if, relative to the front of the observer, object *A* is further away than object *B*, and object *B* is further away than object *C*, then object *C* is closer than object *A*. Via the mapping, this implies that time *C* is in a "nearer" future than time *A*. The same relations hold for objects behind the observer and times in the past. Also, via the mapping, time is seen as having extension, which can be measured; and time can be extended, like a segment of a path, and conceived as a linear bounded region, and so on.

Of course spatial construals of time and conceptual mapping, in general, present many more subtleties and complexities. We will come back to some of them later (in section "Everyday Abstraction: the Embodiment of Spatial Construals of Time and Their 'Axioms'"). For the moment, let us stop here and

[1] Following a convention in cognitive linguistics, small capitals here serve to denote the name of the conceptual mapping as such. Particular instances of these mappings, called metaphorical expressions (e.g., "she has a great future in front of her") are not written with capitals.

[2] There are two main forms of this general conceptual metaphor defined according to the nature of the moving agent—the relative motion of ego with respect to the objects, or the objects with respect to ego (as in *Easter is approaching* vs. *We're approaching Easter*). Their analysis goes beyond the scope of this chapter. For details see Lakoff (1993), Núñez (1999), and Núñez & Sweetser (2006).

see how this theoretical framework applies to more sophisticated imaginary ideas such as the ones constituting mathematical concepts and their axiomatic systems.

MATHEMATICAL ABSTRACTION: THE EMBODIMENT OF AXIOMS, SETS, AND HYPERSETS

Axioms are the modern, and more technical manifestation of the old Euclidean idea of *postulate*. The great mathematician Euclid (ca. 325–265BC) is best known for having systematized the knowledge of geometry and developed what is known as Euclidean geometry today. Although Greek thinkers had been developing geometry at least since the time of pre-Socratic philosophers, three centuries before Euclid, they generated a body of knowledge that was far from constituting a unified discipline. It was Euclid who put together the results and advancements in geometry in a systematic manner. He organized this body of knowledge in such a way that he could derive, through logical deduction, all the known facts of geometry from a few simple and fundamental facts. He took these essential facts to be trivial, intuitive, and too self-evident to be deduced from other facts (e.g., "A straight line may be extended to any finite length"). He called these facts *postulates*. Euclid claimed that only five postulates were required to characterize, using a ruler and compass only, the essence of the entire domain of plane geometry as a subject matter, and believed that from these essential facts all other geometric truths could be derived by deduction alone (i.e., theorems). From this came the idea that every subject matter in mathematics could be characterized in terms of a few essential facts—a short list of postulates, taken as truths, from which all other truths about the subject matter could be deduced. The rest is history. From over two millennia, from Euclid until Kurt Gödel in the 20th century, it has been assumed that an entire mathematical subject matter should follow from a small number of logically independent postulates or axioms. Following this influential view, axioms became the deductive source of all the properties of a given mathematical system (the theorems). When written symbolically in formal logic, a collection of axioms symbolically represents, in compact form, the essence of an entire mathematical system. Euclid's deductive method is still, today, the backbone of mathematics.

For more than a century now, axiomatizing mathematical subject matters has become a crucial enterprise in mathematics, serving as an engine for developing new mathematics: the axiomatization of different forms of geometry, number systems, different types of set theory, of statistics, and so on. Generations of mathematicians have developed entire careers seeking to find the smallest number of logically independent axioms for specific subject matters. The quest for the most appropriate and logically fruitful set of axioms for given subject matters became the ultimate goal for many who were investigating the foundations of mathematics. But, whereas Euclid understood his postulates for geometry to be meaningful to human beings, modern axiomatic mathematics has

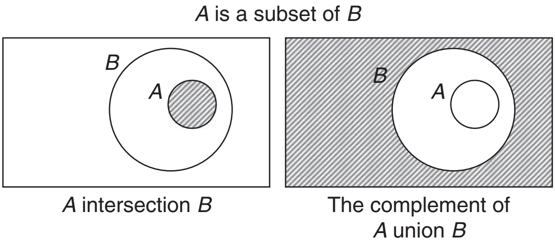

FIGURE 17.1 Venn diagrams representing the case "*A* is a subset of *B*." The diagram on the left depicts "*A* intersection *B*," and the one on the right "the complement of *A* union *B*." Venn diagrams implicitly assume a metaphorical conception of sets as container schemas, deriving their logics from the logic of conceptual container schemas. Members of set *A* are thus conceptualized as being *inside* the container or bounded region *A*, whereas non-members are *outside* of it (Lakoff & Núñez, 2000).

taken axioms as mind-free sequences of symbols. Most of modern mathematics today sees axioms as defined to be free of human conceptual systems and human understanding, characterizing the essence of each mathematical subject matter. But are axioms genuinely mind-independent? Are they simply meaningless strings of symbols? Recent developments in set theory provide rich insight into these questions.

Let us start with a simple question. What is a set? Intuitively, many people (including mathematicians) would say that a set is some kind of collection or aggregate.[3] Many authors speak of sets as "containing" their members and most students think of sets this way. Even the choice of the word "member" suggests such a reading, as do the Venn diagrams used to introduce the subject (Figure 17.1).

Implicit in this form of understanding of sets is the conceptual metaphor SETS ARE CONTAINER SCHEMAS, whose mapping and inferential organization is shown in the following table:

Source Domain Container schemas		Target Domain Sets
Display of container schemas and entities	→	The membership structure of a set
Interior of the container	→	The membership relation
The containers themselves	→	Sets
Entities inside a container	→	Members of a set

[3]The father of modern set theory, the German mathematician Georg Cantor (1845–1918) referred to it as *Menge*: "any collection into a whole (*Zusammenfassung zu einem Ganzen*) *M* of definite and separate objects *m* of our intuition or our thought" (Cantor, 1915/1955, p. 85).

If we operate with this conceptual metaphor, the understanding of Venn diagrams follows immediately. On the modern formalist view of the axiomatic method, however, a "set" is not a container but rather any mathematical structure that "satisfies" the axioms of set theory as written in symbols. The traditional axioms for set theory (the Zermelo–Fraenkel axioms) are often taught as being about sets conceptualized as containers. But if we look carefully through those axioms, we will find nothing in them that characterizes a container. The terms "set" and "member of" are both taken as undefined primitives. In formal mathematics, it means that they can be anything that fits the axioms. Here are the classic Zermelo–Fraenkel axioms, including the axiom of choice, what are commonly called the ZFC axioms.

- *The axiom of extension*: Two sets are equal if and only if they have the same members. In other words, a set is uniquely determined by its members.
- *The axiom of specification*: Given a set A and a one-place predicate, $P(x)$ that is either true or false of each member of A, there exists a subset of A whose members are exactly those members of A for which $P(x)$ is true.
- *The axiom of pairing*: For any two sets, there exists a set that they are both members of.
- *The axiom of union*: For every collection of sets, there is a set whose members are exactly the members of the sets of that collection.
- *The axiom of powers*: For each set A, there is a set $P(A)$ whose members are exactly the subsets of set A.
- *The axiom of infinity*: There exists a set A such that (1) the empty set is a member of A and (ii) if x is a member of A, then the successor of x is a member of A.
- *The axiom of choice*: Given a disjointed set S whose members are non-empty sets, there exists a set C that has as its members one and only one element from each member of S.

There is nothing in these axioms that explicitly requires sets to be containers. What these axioms do, collectively, is to *create* entities called "sets," first from elements and then from previously created sets. The axioms do not say explicitly how sets are to be conceptualized.

The point is that, within formal mathematics, where all mathematical concepts are mapped onto set-theoretical structures, the "sets" used in these structures are not technically conceptualized as container schemas. They do not have container schema structure with an interior, boundary, and exterior. Indeed, within formal mathematics, human ideas are not supposed to exist at all, and hence sets are not supposed to be conceptualized as anything in particular. They are undefined entities whose only constraints are that they must "fit" the axioms. For formal logicians and model theorists, sets are those entities that fit the axioms and are used in the modeling of other branches of mathematics. Of course, most of us do conceptualize sets in terms of container schemas (as in the case of Venn diagrams) and that is perfectly consistent with the axioms just described.

But when we conceptualize sets as container schemas, a constraint follows automatically: Sets cannot be members of themselves, as containers cannot be inside themselves. Strictly speaking, this constraint does not follow from the axioms but from our metaphorical understanding of sets in terms of containers. The axioms do not rule out sets that contain themselves. However, an extra axiom was proposed by the mathematician John von Neumann (1903–1957) that does rule out this possibility.

- *The axiom of foundation*: There are no infinite descending sequences of sets under the membership relation. That is, $\ldots S_{i+1} \in S_i \in \ldots \in S$ is ruled out.

Since allowing sets to be members of themselves would result in such a sequence, this axiom has the indirect effect of ruling out self-membership.

But despite the fact that this axiom is somewhat fixing the "self-containing problem" (by ruling out self-membership), certain model-theorists have found that for special cases they would like to preserve the possibility of allowing "self-membership." For example, consider an expression like

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}$$

If we observe carefully, we can see that the denominator of the main fraction has, in fact, the value defined for x itself. In other words, the above expression is equivalent to

$$x = 1 + \frac{1}{x}$$

Such recursive expressions are common in mathematics and computer science. The possibilities for modeling such expressions using "sets" are ruled out if the only kind of "sets" used in the modeling must be ones that cannot have themselves as members. And these mathematicians have pointed out that despite that "containment" in itself is not part of the Zermelo–Fraenkel axioms, still our implicit ordinary grounding metaphor that SETS ARE CONTAINER SCHEMAS gets in the way of modeling kinds of phenomena (especially recursive phenomena) like the one above. They realized that a new non-container metaphor (not based on what they called the "box metaphor") was needed for thinking about sets, and explicitly constructed one (see Barwise & Moss, 1991).

The idea is to use graphs, not containers, for characterizing sets. The kinds of graphs used are accessible pointed graphs or APGs. "Pointed" indicates an asymmetric relation between nodes in the graph, indicated visually by an arrow pointing from one node to another —or from one node back to that node itself (Figure 17.2). "Accessible" indicates that there is a single node, that is linked

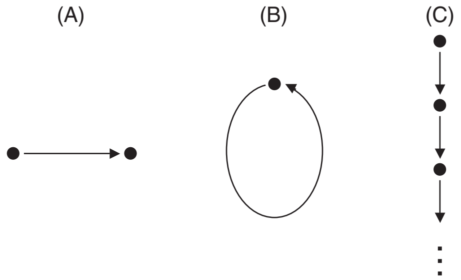

FIGURE 17.2 Hypersets: Sets conceptualized as graphs, with the empty set as the graph with no arrows leading from it. The set containing the empty set is a graph whose root has one arrow leading to the empty set (A). Illustration (B) depicts a graph of a set that is a "member" of itself, under the SETS ARE GRAPHS metaphor. Illustration (C) depicts an infinitely long chain of nodes in an infinite graph, which is equivalent to (B).

to all other nodes in the graph and can therefore be "accessed" from any other node.

From the axiomatic perspective, what has been done is to replace the Axiom of Foundation with another axiom that implies its negation, the "Anti-Foundation Axiom." But from the perspective of Mathematical Idea Analysis the creators of this new conception of "sets" have implicitly used a radically different conceptual metaphor, with graphs—not container schemas—as a source domain. The following table shows the mapping of such a powerful metaphor:

Source Domain Accessible pointed graphs		Target Domain Sets
An APG	→	The membership structure of a set
An arrow	→	The membership relation
Nodes that are tails of arrows	→	Sets
Decorations on nodes that are heads of arrows	→	Members
APG's with no loops	→	Classical sets with the Axiom of Foundation
APG's with or without loops	→	Hypersets with the Anti-Foundation Axiom

The effect of this conceptual metaphor is to eliminate the notion of containment from the concept of a "set." The graphs have no notion of containment built into them at all, and containment is not modeled by the graphs. Graphs that have no loops satisfy the ZFC axioms and the Axiom of Foundation. They, thus, work just like sets conceptualized as container schemas. But graphs that *do* have loops model sets that can "have themselves as members." They do not work like sets that are conceptualized as containers, and they do not satisfy the Axiom of Foundation. A "hyperset" is an APG that may or may not contain loops. Hypersets, thus, do not fit the Axiom of Foundation but rather another axiom with the opposite intent:

- *The anti-foundation axiom*: Every APG pictures a unique set.

With this example we can see the power of the embodied mechanism of conceptual metaphor in mathematics, playing a crucial role even at the foundational level of axiomatic systems. Sets, conceptualized in everyday terms as containers, do not have the right properties to model everything needed. So some mathematicians have metaphorically reconceptualized "sets" to exclude containment by using other more appropriate conceptual metaphors—certain kinds of graphs. The only confusing thing is that this special case of graph theory is still called "set theory" for historical reasons. Because of this misleading terminology, it is sometimes said that the theory of hypersets is "a set theory in which sets can contain themselves." From a cognitive point of view, this is completely misleading because it is not a theory of "sets" as we ordinarily understand them in terms of containment. The reason that these graph theoretical objects are called "sets" is a functional one: they play the role in modeling axioms that classical sets with the Axiom of Foundation used to play.

The moral is that mathematics has (at least) two *internally consistent* but *mutually inconsistent* metaphorical conceptions of sets: one in terms of container schemas and another in terms of graphs. And in both cases, corresponding axioms have been especially concocted to organize the inferential structure (theorems) of both kinds of "set" theory, namely, the Axiom of Foundation and the Anti-Foundation Axiom, respectively. Is one of these conceptions right and the other wrong? Are truths in one system "higher" than the truths in the other one? What axiom system is providing the ultimate truth about sets? A Platonist might want to think that there must be only one literal correct notion of a "set" transcending the human mind. But from the perspective of Mathematical Idea Analysis, these two distinct notions of "set" define different and mutually inconsistent subject matters, conceptualized via radically different human conceptual metaphors. Interestingly, in mathematics, cases like this one are more of a rule than an exception!

Let us now go back to the discussion about our ordinary forms for conceiving Time, and see how some of the embodied mechanisms that bring mathematics into being are the same that make everyday imagination possible.

EVERYDAY ABSTRACTION: THE EMBODIMENT OF SPATIAL CONSTRUALS OF TIME AND THEIR "AXIOMS"

Time, which for centuries has intrigued philosophers, physicists, and theologians, is a fundamental component of human experience. It is intimately related with everything we do, yet it is abstract, in the sense that we do not experience it directly as an isolated thing we can point to. Besides, our brains do not seem to have specific areas dedicated to process pure temporal experience in the way it does with, say, visual or auditory stimulation. Still, humans from all cultures must cope, implicitly or explicitly—with time-related entities, whether it is for cooking, dancing, hunting, traveling, or raising children. So, how do humans make up time concepts? As we saw earlier, the short answer is by treating "time" metaphorically as being *spatial* in nature, and one widespread form allows us to conceive the future as being in front of us, and the past behind us. This (mostly unconscious) way of thinking seems extremely obvious and natural, to the point that we barely notice that this is a major form of comprehension of temporal experience shared by many cultures around the globe. Even though nobody explicitly taught us this way of thinking about time, we master it effortlessly. It is simply part of who we are. This form of conceiving future and past, however, despite being spread across countless unrelated cultures around the world, is not universal! In collaboration with linguist Eve Sweetser from the University of California at Berkeley, we were able to reach this conclusion after studying in detail the conceptions of time in the Aymara people of the South American Andes (Núñez & Sweetser, 2006). This constituted the first well-documented case violating the postulated universality of the metaphorical orientation future-in-front-of ego and past-behind-ego.

Aymara, an Amerindian language spoken by nearly 2 million people in the Andean highlands of western Bolivia, southeastern Peru, and northern Chile, present a fascinating contrast to the well-known spatial-temporal mappings described earlier, and a clear challenge to the cross-cultural universals of metaphoric cognition studied so far. In Aymara, the basic word for "front" (*nayra*, "eye/front/sight") is also a basic expression meaning "past," whereas the basic word for "back" (*qhipa*, "back/behind") is a basic expression meaning "future". For example, *nayra mara*, whose literal translation is "eye/front year" means "last year," and *qhipa pacha*—"back time"—means future time. Many more temporal expressions in Aymara follow this pattern. But here is where, as cognitive scientists, we had to remain very cautious in reaching fast conclusions regarding possible exotic conceptions of time. To proceed, we needed to address two important research questions:

1. What exactly are the mappings involved in these metaphorical expressions?
2. Is there evidence of their psychological reality? That is, do Aymara people really *think* metaphorically in this manner, or are they simply using *dead* fossilized expressions with no inherent metaphorical meaning?

The first question pushed us to make further theoretical distinctions. In cases like "The election is *ahead* of us" and "the long Winter is now *behind* us" the terms "ahead," "behind", and so on, are defined relative to *ego*. In other words, ego is the *reference point* and therefore the conceptual metaphor described earlier—Time Events Are Things in Saggital Unidimensional Space—is said to be an instance of an Ego-reference-point (Ego-RP) metaphorical mapping. It is crucial not to confuse this mapping with another type called Time-reference-point (Time-RP), that underlies metaphorical expressions such as "the day *before* yesterday" or "revive your *post* summer skin," where morphemes like *fore* (front) and *post* (posterior) denote earlier than and later than relations, respectively.[4] The Time-RP mapping is the following:

Source Domain (unmarked) Unidimensional space		Target Domain Time
Objects	→	Times
Sequence of objects	→	Chronological order of times
Object *A* is in front of object *B*	→	Time *A* occurs earlier than time B
Object *A* is in behind object *B*	→	Time *A* occurs later than time B
Object *A* is co-located with object *B*	→	Time *A* occurs simultaneously with time *B*

This mapping is in many respects, simpler than the Ego-RP one. As it does not have an ego, it does not have a "now" in the target domain of time, and, therefore, it does not have built in the intrinsically deictic categories, past, present, and future. The Time-RP mapping has only earlier than and later than relationships. But when a particular moment is picked as "now," then "earlier than now" (past) and "later than now" (future) can be obtained. According to this mapping, however, "earlier than now" (past) gets its meaning from a "front" relationship, and "later than now" events (future) from a "behind" relationship. This may create confusion as in the case of the Ego-RP mapping the opposite seems to be happening: "front" (of us) means "future" and "behind" (us) means "past." The confusion, however, is immediately clarified by asking the following simple question: in front of *what*? or, behind *what*? Technically, this means identifying the underlying reference point. In "the day before yesterday," the reference point is "yesterday," "in front" of which is located the day the expression refers to. In "revive your post summer skin" the reference point is "summer," with the phrase targeting the times that follow the sunny season. To understand the Aymara case, we must keep this fundamental distinction between Ego-RP and Time-RP mappings clearly in our minds.

[4]For empirical evidence of the psychological reality of this metaphor, see Núñez & Sweetser (2006) (gestural), and Núñez et al. (2006) (priming experiments).

The crucial question we needed to address was: What are the reference points involved in the uses of *nayra* (front) and *qhipa* (back) in Aymara? That is, what is "in front of" or "behind," when these terms are used for temporal meaning? If the reference points are temporal entities such as "winter," "sunrise," "lunch time," or "rainy season," as opposed to "us" or "me," then there is absolutely nothing intriguing or exotic in the above Aymara temporal expressions. In such cases Aymara uses of "front" and "back" would be equivalent to English Time-RP cases like "the day before yesterday" or "post summer." In fact, this is what occurred with some Polynesian and African languages that had been claimed to be "special" with respect to space–time metaphors, but whose data, after proper analysis, turned out to be standard Time-RP cases (Moore, 2000). If in Aymara, however, the reference point is indeed ego, that is, "front of *us*" means past and "behind *us*" means future, then this finding would be critical since it would provide a counterexample to the largely universal Ego-RP mapping.

The second question we needed to investigate was how people—Aymara or otherwise—actually think about time. For this, we had to go beyond the mere analysis of words and their etymological roots. We needed to investigate empirically the psychological reality of these space–time metaphors, and ask: Do people actually *think* this way? Or perhaps the expressions simply used "dead" lexical items from a distant past that lost its original metaphorical meaning? And how can we tell?

Along with my colleagues, I addressed both these questions. Regarding the first one—the question of reference points—we quickly hit some dead ends. It turns out that in Aymara, due to grammatical reasons too involved to explain here, it is not possible to simply find markers like "us" in temporal uses of "front" (*nayra*) or "back" (*qhipa*). In short, using purely linguistic methods, we could not tell whether the Aymara expressions given earlier were Ego-RP or Time-RP. This is precisely the nightmare scenario for a scientist: not being able to provide an answer to the research question with the methods at hand. We therefore looked for other methods. The best candidate—which turned out to be essential in answering the second question as well—ended up being a long forgotten dimension of human language: Embodied spontaneous gestures.

Why gestures? When speaking, humans from all cultures spontaneously produce gestures. These are effortless but complex sequences of motor activity—especially hand movements—that are co-produced with speech. The study of human gestures, after being ignored in academic circles for a long time, has made a substantial progress over the last couple of decades. Research in a variety of areas, from child development (Bates & Dick, 2002) to neuropsychology (Kelly et al., 2002; McNeill, 2005), to linguistics (McNeill, 1992; Cienki, 1998), and to anthropology (Haviland, 1993), has shown the intimate link between oral and gestural production. Moreover, it is known that linguistic metaphorical mappings are paralleled systematically in gesture (Cienki & Müller, 2008; Núñez, 2006). We, therefore, could reliably ask what kinds of gestures Aymara speakers produce when uttering temporal expressions using "front" (*nayra*) or "back (*qhipa*).

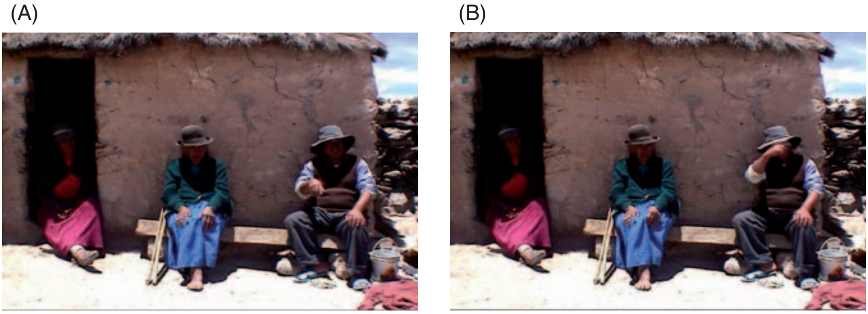

FIGURE 17.3 The speaker, at right, is referring to the Aymara expression *aka marat(a) mararu*, literally "from this year to next year." (A) When saying *aka marat(a)*, "from this year," he points with his right index finger downward and then (B) while saying mararu, "to next year," he points backwards over his left shoulder. (©2008 Rafael Núñez. Published by Elsevier Ltd. All rights reserved) (See color plate)

Where are they pointing when doing so? What is the built-in reference point of such pointings?

In order to find out, in collaboration with Chilean colleagues Manuel Mamani and Vicente Neumann from the University of Tarapacá, and Carlos Cornejo from the Pontifical Catholic University of Chile, I conducted videotaped ethnographic interviews with Aymara people from the north-easternmost tip of Chile, up in the Andes, along the border with Bolivia. As we were interested in *spontaneous* gestures (with high ecological validity), the interviews were informal and were designed to cover discussions involving reference to time. Participants were asked to talk about, make comments, compare, and explain a series of events that had happened or that were expected to happen in the context of their communities. They were also asked to talk about traditional "sayings," anecdotes, and expressions in Aymara involving time and to give examples of them. To our amazement, what we found was that Aymara speakers gestured in Ego-RP patterns! Alongside the Ego-RP spatial language used to represent time as in front (*nayra*) and in back (*qhipa*) of ego, they gesturally represented time as deictically centered space: the speaker's front surface was essentially "now," as in English speakers' gestures (Figure 17.3a). The space behind the speaker was the Future (Figure 17.3b), whereas the space in front of the speaker was the Past (Figure 17.4).

Moreover, locations in front and closer to the speaker were more recent past times, while locations in front and farther from the speaker corresponded to less recent times. For instance, speakers contrasted "last year" with "this year" by pointing first at a more distant point and then at a nearer one. When talking about wider ranges of time, rather than particular points in time, we saw speakers sweeping the dominant hand forward to the full extent of the arm as they talked about distant past generations and times. In sum, our data showed, on the one hand, that the reference point in the above temporal expressions in Aymara

FIGURE 17.4 The speaker, at left, is talking about the Aymara phrase *nayra timpu*, literally "front time," meaning "old times." When he translates that expression into Spanish, as he says *tiempo antiguo* he points straight in front of him with his right index finger. (©2008 Rafael Núñez. Published by Elsevier Ltd. All rights reserved) (See color plate)

is indeed ego centered (our first question) and on the other hand, thanks to the analysis of gestures, that for Aymara speakers the Ego-RP metaphorical spatial conception of time has genuine psychological reality (our second question). With these empirical data at hand we were thus able to characterize the actual mapping of the Aymara form of the conceptual metaphor as shown in the following table:

Source Domain Saggital unidimensional space relative to ego		Target Domain Time
Objects in front of ego	→	Past times
Objects behind ego	→	Future times
Object co-located with ego	→	Present
The further away in front of ego an object is	→	The "further away" an event is in the Past
The further away behind ego an object is	→	The "further away" an event is in the Future

This analysis of Aymara language and gesture provides the first empirically demonstrated case of a counterexample to the largely spread space–time metaphors where "future" is conceived as being "in front" of ego and "past" behind ego. Aymara has the opposite pattern (and it may not be the only such culture). Beyond its anecdotal flavor, this finding is crucial as it shows that human abstraction is not pre-wired in the brain. It tells us that there is no single way for achieving abstraction, not even for a fundamental domain such as time. Human biology is certainly fundamental in providing the basis for human imagination. But, building on universal species-specific body morphology and neural organization, different aspects of bodily experience may be recruited for the systematic construction of more abstract concepts, which allow for plasticity and cultural variation. Regarding temporal metaphorical uses of front–back relationships, we tend to profile frontal motion. Based on this, our basic postulate (or "axiom") builds on prototypical frontal motion. If we walk (forward) at any given time we will reach a location that is in front of us, leaving behind us the original location. *That* location is reached in the future relative to the moment we started the action, with the initial position where we were initially (past) located behind us. Aymara people, however, although do walk in the same way as the rest of the world does, operate with a radically different postulate (or "axiom"). They profile a fundamentally different aspect of front–back features: what is seen (and therefore known), lies in front of the observer and behind them lies what is outside the visual inspection. These features parallel essential temporal properties, namely, past events are known, whereas future events are not. In Aymara, visual perception appears to play the leading role in bringing temporal concepts to being, and several data sources support this explanation, from evidential grammatical markers to special social practices and values.

The moral is that humans have at least two forms for conceiving time along a bodily front–back axis, which are—like in set theory—internally consistent but mutually inconsistent. These forms are defined by mutually exclusive ways of orienting the body in saggital unidimensional space, providing a radically different collection of truths. By profiling different aspects of bodily grounded experience we get one case with a built-in postulate ("axiom") that puts the observer "facing" the future and the other case with the very opposite postulate with the observer "facing" the past. Once the orientation of the observer is defined, a series of theorem-like entailments follow. Which one is the correct one? Where really is the past? In front of us? Behind us? Like in mathematics, no ultimate transcendental answer can be provided. Both forms have their own postulates (or axioms), and truth rests on the underlying embodied mappings that made these very abstractions possible.

CONCLUSION

We have analyzed two types of human abstract conceptual systems, one in the realm of contemporary technical mathematics and the other in a fundamental domain of human everyday experience—time. We have seen how, ordinary embodied cognitive mechanisms that sustain human imagination, such as conceptual metaphor, are essential in structuring the meaning and the inferential organization of these abstract conceptual systems. And we have seen how even the most fundamental and abstract building blocks of modern mathematics—axioms—find their grounding on human everyday understanding and embodied sense-making. There are, of course, many differences between technical abstraction and everyday abstraction: usually the former requires writing systems, whereas the latter can evolve with oral tradition alone; the former requires explicit (usually effortful) goal-directed instruction, whereas the latter does not; the former defines conceptual systems that usually are shared by specialized communities, whereas the latter tends to be spread over entire cultures and ethnic groups; and so on. Despite these differences, both forms of human abstraction share essential properties—embodiment, supra-individuality, and truth—that show the same origin: human imagination as realized in the body/mind of the human primate.

Embodiment: In this chapter, we showed how the abstract conceptual systems we develop are possible *because* we are biological beings with specific morphological and anatomical features. In this sense, human abstraction is *embodied* in nature. It is because we are living creatures with a salient and unambiguous front and back, for instance, that we can build on these properties and the related bodily experiences we have to bring forth stable and solid concepts such as "the future in front of us." This would not be possible if we had the body of, say, a jellyfish. Similarly, we can have the experiences and the understanding of containers because we have brain mechanisms as topographic maps of the visual field, center-surround receptive fields, and gating circuitry in which container schemas appear to be realized neurally (Regier, 1996). But whereas other non-human primates share these mechanisms with us, and have fronts and backs as well, it is the modern human primate that has an embodied cognitive apparatus, such as conceptual metaphor, that can systematically extend immediate bodily experiences to create imaginary notions like future-as-front-locations and sets-as-containers. Moreover, biological properties and specificities of human bodily grounded experience impose very strong constraints on what concepts can be created. Because of this, abstract conceptual systems are not "simply" socially constructed, as a matter of convention. Although social conventions usually have a huge number of degrees of freedom, many human abstract concepts do not. For example, the color pattern of the Euro bills was socially constructed via convention (and so were the design patterns they have). But virtually any color ordering would have done the job. In the case of metaphorical construals of time, not any source domain serves the purpose: human construals of time are *spatial*. And this is an *empirical* observation, not an arbitrary or speculative statement, since, as far as we know, there is no language or culture on earth where time is conceived in terms of thermic or chromatic source domains. Human abstraction is thus not merely "socially constructed." It is constructed through strong non-arbitrary biological and cognitive constraints that play an essential role in constituting what human abstraction is, from everyday ideas to highly sophisticated mathematics. Human cognition is *embodied,* shaped by species-specific non-arbitrary constraints.

Supra-individuality: We saw that to study the embodiment of conceptual systems, the level of analysis is situated above the individuals. The primary focus is not on how *single individuals* learn to use, say, conceptual metaphors, or what difficulties they encounter when they learn them, or how they may lose the ability to use them after a brain injury, and so on. The focus is on the characterization, across hundreds of linguistic expressions and other manifestations of meaning (e.g., gestures) of the structure of the inferences that can be drawn from these metaphors, which is available for a community of people operating with such mapping. For example, when English speakers hear "the winter is behind us," they can implicitly and effortlessly infer that the previous fall is not just behind them but *further away behind* them. Similarly, if they read "the election is ahead of us," they implicitly infer that the various effects of the political climate building up to the election are not only ahead of them but also *much closer in front of* them than the election itself. And if we operate with the metaphor SETS ARE CONTAINER SCHEMAS, then we implicitly know that an object cannot be, both, inside and outside a container, and therefore, via the metaphor, that an element cannot be a member and a non-member of a set. The focus of embodied idea analysis (mathematical or other) is thus situated at a supra-individual level, at the level of the mappings and networks of metaphorical inferences. Large communities sharing networks of metaphors and mappings constitute cultures. In what concerns everyday ideas such as time these cultures may naturally coincide with ethnic groups located geographically in specific places (e.g., Aymara people in the Andes) but in mathematics, irrespective of ethnicity or geography, one could speak of the culture of mathematicians practicing set-theory with ZFC axioms and the other one practicing with hypersets and the Anti-Foundation axiom.

Truth: One of the most important morals of this chapter is that when imaginary entities are concerned, truth is always relative to the inferential organization of the mappings involved in the underlying conceptual metaphors. "Last summer" can thus be conceptualized as being *behind us* as long as we operate with the general conceptual metaphor TIME EVENTS ARE THINGS IN SAGGITAL UNIDIMENSIONAL SPACE, which determines a specific bodily orientation with respect to metaphorically conceived events in time—future as being "in front" of us and the past as being "behind" us. As we saw, this way of conceptualizing time, although spread worldwide, is not universal. For an Aymara speaker from the Andes' highlands, it is *not true* that the sentence "The Winter is *behind* us" refers to an event that has already occurred. In fact that sentence means the very opposite, namely, that the Winter has not taken place yet! Aymara people operate with a different conceptual time–space metaphor, which provides a different set of truths. The same occurs with sets and hypersets in contemporary set-theory. They have different collections of truths, characterized by different collection of axioms. The moral is that there is no *ultimate truth* regarding human imaginative structures. In the cases we saw, there is no ultimate truth about where, really, lies the ultimate metaphorical location of the future (or the past) or whether sets can allow self-membership. Truth depends on the details of the mappings of the underlying conceptual metaphor. This turns out to be of paramount importance when mathematical concepts are concerned: their ultimate truth is not hidden in the structure of the universe, but it rests on the underlying embodied conceptual mappings used to create them.

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