**MAGNET HARDWARE**

1. \( B_\phi \) field from superconducting magnet
2. Gradient coils
3. Body RF transmit/receive
4. RF receive-only
5. Shim coils

\( B_\phi \rightarrow z \) (longitudinal)
\( B_1 \rightarrow x, y \) (transverse)

**SUPERCONDUCTING MAGNETS IN LIQUID HELIUM**
(no power required after current injected to bring up field using induction)

\[ 1 T = 10,000 \text{ Gauss} \]
\[ \text{Earth: 0.25 - 0.5 G} \]
\[ 25 - 65 \text{ mT} \]
\[ \frac{B_1}{2\pi} = \text{42 MHz/T} \]

**RF TRANSMIT/RECEIVE COILS**

(4) RF receive-only head coils

(3) RF transmit body coil

RF transmitter (30 kW)

Circularly polarized \( B_1 \) field rotating to \( B_\phi \) at Larmor frequency

Shim coils also embedded in here (not shown)

Max gradient:
\[ 80 \text{ mT/m} \]
\[ 200 \text{ T/m/sec} \]

Three 1.5 million watt amplifiers to add ramps to \( B_\phi \) field

\( B_\phi \) non-superconducting water-cooled, external shield

9 Nov 2019
**Spin & Precession**

1. **Nuclei act like a spinning sphere of matter with an embedded equatorial charge** (nuclei with odd atomic weight or odd proton numbers).
2. **Moving charge creates magnetic field**
   - Classical picture
   - Current loop from spinning charge (right-hand rule)
   - N.B.: Classically this would cause EM radiation, spin-down

**Stern-Gerlach Experiment**
- Pass silver atoms through strong magnetic field → split into just 2 beams

**Microscopic Picture**
- No strong magnetic field
  - \( B_0 = 0 \)

**Strong magnetic field, \( B_0 \)**
- Strong \( B_0 \) plus oscillating \( B_1 \)

**Precession**
- Distinguish precession (slow) from spin (fast)
- Treat classically, like spinning top

\[
2\pi \frac{\hbar}{B_0} = \omega_0 = \frac{Y B_0}{I_{\text{gyro}} \text{ (eg. } 63 \text{ MHz)}}
\]

- Bulk equilibrium magnetization (parallel to \( B_0 \))
  - \( M_0^z = \frac{\mathbf{M} \cdot \mathbf{z}}{2} = \frac{Y^2 \hbar^2 B_0 N}{4K T S} \)

N.B.: Compared to top gravity:
- Frictionless spin, doesn't slow
- Signed gravity can change precession direction
- Can stick under floor
- Neighbor bumping causes decay (T2)

**Macroscopic Picture**
- All vectors some length, random directions
  - Slight excess of "up" (1 ppm)
  - Already precessing x, y components still random
  - Precessing vectors are "bunched" at any one moment around circle
  - Bulk magnetization precesses

\[
Y = \frac{1}{I - \omega \sin \phi}
\]

Left-hand rule: "Y-fingers = precess, right-hand rule: thumb = \( B_0 \)"
**Bloch Equation**

- **Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation + relaxation)**
  
  \[
  \frac{d\vec{M}}{dt} = \vec{M} \times \vec{v} - \frac{\vec{M} \times \vec{v}}{T_2} - \left( \frac{M_2 - M_2^o}{T_1} \right) \vec{k}
  \]

  - **Precession:** \( \vec{B} = B_0 \)
  - **Excitation:** \( \vec{B} = B_1 \)

  - **Equilibrium:** \( \vec{M} \) (value in only the \( B_0 \) field)

- **In the Larmor-rotating coordinate system, a tilt who a phase shift for a standard \( B_1 \) excitation is rotation around \( x \)-axis**

- **In general case rotating (distinguish \( B_1 \) & \( M \))**

- **Longitudinal and Transverse relaxations**
  
  \[
  \frac{dM_z(t)}{dt} = - \frac{M_z(t) - M_z^o}{T_1}
  \]

  \[
  \frac{dM_{x'y'}(t)}{dt} = - \frac{M_{x'y'}(t)}{T_2}
  \]

- **Solution to equations above:** 
  - **Time-dependent free precession**
  - **Re-growing from 0 left after pulse-decaying**

\[
M_z'(t) = M_z^o (1 - e^{-t/T_1}) + M_{z}^o (Q_1) e^{-t/T_1}
\]

\[
M_{x'y'}(t) = M_{x'y'}(0) e^{-t/T_2}
\]

- **Initial condition**

---

**Notes:**
- These can be ignored during short excitation, so magnetic vector stays same length as it spirals down (vs. relaxation, where it shrinks, then grows)
- Ref: Moe [2021]
**Vector Add, Multiply**

- **Adding vectors** is easy
  \[ \mathbf{c} = \mathbf{a} + \mathbf{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Generalizes to any \( D \)
  \[ \| \mathbf{c} \| = \sqrt{(a_x+b_x)^2 + (a_y+b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**
(= Inner Product)
(= "Scaled projection onto")
\[
\mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z
\]
- Generalizes to any \( D \)
  \[ \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \cos \theta \]
  \( \Leftrightarrow \) zero if \( \mathbf{a}, \mathbf{b} \) orthogonal

**Cross Product**
(= Outer Product)
(= "Geometric algebra")
\[
\mathbf{c} = \mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = [a_y b_z - a_z b_y, a_x b_z - a_z b_x, a_z b_y - a_y b_x]
\]
- Unique orthogonal specific to 3D
  \[ \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \sin \theta \]
  \( \Leftarrow \) max if orthogonal

**Complex Multiply**
(See also quaternions, geometric algebra generalization)
\[
\mathbf{c} = \mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = [a_x b_x - a_y b_y, a_x b_y + a_y b_x]
\]
- Angles add
- Magnitudes multiply
  \[ \| \mathbf{c} \| = \| \mathbf{a} \| \| \mathbf{b} \| \]
  \( \Leftarrow \) like real nums

N.B. equals: \( \mathbf{a} \cdot \mathbf{a} = \| \mathbf{a} \|^2 \)
\[
= \sqrt{a_x^2 + a_y^2 + a_z^2}
\]
EFFECTS OF \( \vec{M}, \vec{B}, \) and \( \Theta \) ON PRECESSION FREQ.

Bloch

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}
\]

Cross prod. properties review:

\[
\left\| \frac{d\vec{M}}{dt} \right\| = \|\vec{M}\| \|\vec{B}\| \sin \Theta
\]

Starting condition

\( \Rightarrow \) now see effects of changing \( \vec{M}, \vec{B}, \Theta \)

Change \( \vec{M} \) length

\( \Rightarrow \frac{d\vec{M}}{dt} \) proportionally larger, so canals effect of larger \( \vec{M} \)

\( \Rightarrow \) same precession freq. as starting cond.

Change \( \Theta \) between \( \vec{M} \) and \( \vec{B} \)

\( \Rightarrow \frac{d\vec{M}}{dt} \) goes up (then down) as \( \sin \Theta \)

\( \Rightarrow \) but circumference also goes up as \( \sin \Theta \), cancelling again

\( \Rightarrow \) same precession freq.

Change \( \vec{B} \) length

\( \Rightarrow \frac{d\vec{M}}{dt} \) goes up, proportional to \( \vec{B} \)

\( \Rightarrow \) but circumference is same at starting cond.

\( \Rightarrow \) increased precession freq. \( (\omega = \gamma \vec{B}) \)
**Simple Matrix Operations**

**Basic Idea**
- A matrix \( \begin{bmatrix} \text{rotates} \\ \text{scales} \end{bmatrix} \) acts on a vector \( \vec{b} = \mathbf{M} \vec{a} \).

**3D Example**

\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z \\
\end{bmatrix} = \begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{21} & M_{22} & M_{23} \\
    M_{31} & M_{32} & M_{33} \\
\end{bmatrix} \begin{bmatrix}
    a_x \\
    a_y \\
    a_z \\
\end{bmatrix}
\]

**Add Translate (after rotate/scale)**
- Commonly used "hack" for aligning 3d vectors.
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} \\ \text{then} \\ \text{translates} \end{bmatrix} \) (4th col = 1).
- N.B.: Have to keep track of order!!
  - Rotate/scale then \( \neq \) trans, then rotate/scale changes rot component: untranslate, rot, retranslate.

**3 Special Cases (3d)**: Rotate around each major axis without changing length (Scale = 1.0)

- Rotate around X-axis:
  \( \mathbf{R}_x(\alpha) = \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos \alpha & \sin \alpha \\
    0 & -\sin \alpha & \cos \alpha \\
  \end{bmatrix} \)
  - e.g., 90° flip

- Rotate around Y-axis:
  \( \mathbf{R}_y(\alpha) = \begin{bmatrix}
    \cos \alpha & 0 & -\sin \alpha \\
    0 & 1 & 0 \\
    \sin \alpha & 0 & \cos \alpha \\
  \end{bmatrix} \)
  - e.g., 180° flip to avoid add 180° phase after 90° flip on x'

- Rotate around Z-axis:
  \( \mathbf{R}_z(\alpha) = \begin{bmatrix}
    \cos \alpha & \sin \alpha & 0 \\
    -\sin \alpha & \cos \alpha & 0 \\
    0 & 0 & 1 \\
  \end{bmatrix} \)
  - e.g., precession with 3D along z'

**General Case**
- Rotate around general 2'-axis:
  \( \mathbf{R}_{a'}(\gamma) = \mathbf{R}_z(\theta) \mathbf{R}_y(\phi) \mathbf{R}_z(\alpha) \mathbf{R}_y(\phi) \mathbf{R}_z(\theta) \)
  - Quaternions are more efficient
Solutions to Simple Differential Eq.

**Diff. Eq.** \[ \frac{dM}{dt} = \frac{M}{T_2} \]

**Solution:** \[ M(t) = \frac{M_0}{T_2} \cdot e^{-t/T_2} \]

**Goal:**
1. Find \( M(t) \) where derivative satisfies diff. eq.
2. Also find \( M(t) \) (one of many) that passes thru init condition.

\[ \Rightarrow - \text{ since our diff eq is: } \text{ derivative} = \text{ const } \cdot \text{ same funct.} \]

\[ \Rightarrow - \text{ try exponential, since derivative } (e^x) = e^x \]

<table>
<thead>
<tr>
<th>diff. eq.</th>
<th>[ M(t) = \frac{-1}{T_2} \cdot \frac{M(t)}{T_2} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>denu. of var</td>
<td>OK - we have recovered only diff eq.</td>
</tr>
<tr>
<td>const</td>
<td>N.B., this function is the &quot;unknown&quot; like the ( x ) in ( x + 1 = 3 )</td>
</tr>
<tr>
<td>same var</td>
<td></td>
</tr>
</tbody>
</table>

**One Soln:** \[ M(t) = e^{-t/T_2} \]

**Take deriv. to check:** \[ M'(t) = \frac{-1}{T_2} \cdot e^{-t/T_2} \]

\[ \Rightarrow - \text{ same as } M(t) \]

**Another Soln:** \[ M(t) = \text{const} \cdot e^{-t/T_2} \]

**Take deriv. to check:** \[ M'(t) = \frac{-1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} \]

\[ \Rightarrow - \text{ so: any constant } \text{OK!} \]

\[ \text{e.g. const } = 2 \]
\[ \text{const } = -0.5 \]

**Initial Condition:**

\[ \text{information added to soln (not from diff eq!)} \]

\[ M'(t) = \frac{M_0}{T_2} \cdot e^{-t/T_2} \]

**Magnetization immedi. after RF (B1) ends:** \[ M_x y(0) \]
**Bloch Eq. - Matrix Version**

Differential Eq.:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}_\phi$$

Solution:

$$\mathbf{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = \mathbf{R}_z(\omega t) \mathbf{M}(0)$$

Include Relaxation

Differential Eq.:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}_\phi - \frac{M_x \hat{e}_z + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1}$$

Solution:

$$\mathbf{M}(t) = \begin{bmatrix} e^{-\frac{t}{T_2}} & 0 & 0 \\ 0 & e^{-\frac{t}{T_2}} & 0 \\ 0 & 0 & e^{-\frac{t}{T_1}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1-e^{-\frac{t}{T_1}}) \end{bmatrix}$$
Excitation in the Rotating Frame

- Original Bloch eq. in laboratory frame:
  \[ \frac{d \vec{M}}{dt} = \vec{M} \times \vec{B} \]

- Add on-resonance B1 to \( \vec{B} \)
  \[ \vec{B} = B_1(t) (\cos \omega_t \hat{i} - \sin \omega_t \hat{j}) + B_0 \hat{k} \]

  - Lab frame < on-resonance
  - *Basic excite
  - Matrix version

- Substitution to convert to the rotating frame
  \[ \frac{d \vec{M}}{dt} = \begin{bmatrix}
  \frac{dM_x}{dt} \\
  \frac{dM_y}{dt} \\
  \frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
  0 & -\omega_t & w(t) \\
  \omega_t & 0 & -w(t) \\
  -w(t) & w(t) & 0
\end{bmatrix} \begin{bmatrix}
  M_x \\
  M_y \\
  M_z
\end{bmatrix} \]

- After substitution any off-resonance appears as residual \( B_0 \) (see off-res notes page)
  \[ \vec{M} = R_2(\omega_t) \cdot \vec{M}_{\text{rot}} \]
  \[ \vec{B} = R_2(\omega_t) \cdot \vec{B}_{\text{rot}} \]
  "Subtract off" rotating frame
  \[ \frac{d \vec{M}_{\text{rot}}}{dt} = \vec{M}_{\text{rot}} \times \vec{B}_{\text{eff}} \]

  - Rotating frame < on-resonance
  - *Basic excite
  - Removes \( \omega_t, \cos/\sin \)

  - Rotating frame < off-resonance
  - General < incl gradients
  - Gradient: \( \omega(t) = \gamma g_z \hat{z} \)
  - Off-res: appears as residual \( B_0 \), lifting \( B_1 \) vect. out of \( x-y \) plane

  - Rotating frame < on-resonance
  - *Small tip approx.
  - Small tip

  \[ \frac{d \vec{M}_{\text{rot}}}{dt} = \begin{bmatrix}
  0 & w(t) & 0 \\
  -w(t) & 0 & 0 \\
  0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
  M_x' \\
  M_y' \\
  M_z'
\end{bmatrix} \]

This means \( \vec{M} \) vect. update will contain component that rotates \( \vec{M} \) around \( z \)-axis (in rotating coords \( \equiv \) phase)
**BLOCH EQ. SUMMARY**

\[ \frac{d\hat{M}}{dt} = \hat{M} \times \hat{\mathbf{B}} - \left( \frac{\hat{M}_x \hat{i} + \hat{M}_y \hat{j}}{T_2} \right) - \left( \frac{(M_z - M_0^z) \hat{k}}{T_1} \right) \]  

(lab-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \( \frac{d\hat{M}}{dt} \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2, T_1 \) decay
  - \( \hat{B}_1 \) is also rapidly wiggling
- Conceptual simplification in 4 stages:

1) **lab frame**
- Just precession

2) **rotating frame**
- \( \hat{\mathbf{M}} \) stopped
- That is, \( \hat{B}_0 = 0 \)

3) **add \( \hat{B}_1 \)**
- \( \hat{B}_1 \) also stopped!
- But \( \hat{\mathbf{M}} \times \hat{\mathbf{B}} \) still works!!
- "Precess" around \( \hat{B}_1 \) axis

4) **off-resonance**
- Slow precess, now around tilted \( \hat{B}_{eff} \)
  - Tilted plane
  - Apparent \( B_z \) comp. from residual precess. around \( z \) from off-resonance
RF FIELD POLARIZATION

- Polarization (change of direction) of magnetic field (vs. electric field)

\[ \mathbf{B}_1(t) = B_1 \cdot \cos\omega t \hat{\mathbf{z}} \]

- Linearly polarized field

N.B.: \( \mathbf{B}_1 \) adds to much larger \( \mathbf{B}_0 \)

- Circularly polarized field (quadrature)

\[ \mathbf{B}_1^{\text{circ}} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) = B_1 \cdot e^{-i\omega t} \]

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF spin

Typical 90° flip (around x-axis)

(around x-axis)

(same coords as above, z at top)

Typical 180° flip (around opposite y-axis)

180° flip reg

~6X power of 90°
**SIGNAL EQUATION**

\[
\Phi(t) = \int \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r}
\]

**magnetic flux** thru coil scalar
(integral mag. field perpendicular to area)

\[
\mathbf{V}(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r}
\]

Faraday law of Induction

\[\text{evaluate using free precession e.g. (solution to Bloch) ignoring relaxation, rewrite w/ complex notation in time dependence from lab frame Bloch}\]

- ignore change in \(z\)-comp. \(\mathbf{M}\) because so slow \(\rightarrow\) i.e., we only see \(M_{xy}\), not \(M_z\)
- substitute \(\mathbf{M}(t)\) with lab frame \(M_{xy}(t) = M_{xy}(0) e^{-i\omega t} e^{-i\omega t}\)
- simplify:
  1) ignore decay (assume this \(t=0\))
  2) assume phase-sensitive detection \(\text{[data complex]} \rightarrow\) rotating frame!

\[\text{even from one coil!}\]

**Laboratory frame Bloch solutions:**

\[M_L \rightarrow \text{same} \]

\[M_T = M_{xy}(0) e^{e^{-i\omega t}}\]

**demodulate**

\[e^{-i\omega t}\]

\[\text{sum across object}\]

**Spatially dependent resonant freq. in rotating frame} \rightarrow i.e. after subtraction of \(w_0\) \(\Rightarrow \text{RF signal related to changes across object}\)

\[\text{Standard Signal Expression}\]

\[\mathbf{S}(t) = \int \mathbf{M}(\mathbf{r}, 0) e^{-i \frac{\omega_0}{2} t} \, d\mathbf{r}\]

**i.e., at a single time point, RF signal is vector sum across object of local transverse magnetization vectors**

**phase angle in rotating frame**

\[\frac{\omega t}{\text{rad/sec}} = \frac{\text{rad/sec}}{\text{rad/sec}} (\phi = \omega \Delta t)\]

[getting difference converts lab \(\rightarrow\) rotating frame]
**Phase-Sensitive Detection**

- Method for moving very high frequency Larmor oscillations down to tractable frequency range

\[ V(t) \rightarrow \text{multiply} \rightarrow \text{Low-Pass Filter} \rightarrow S(t) \]

Ref: Signal (1.23 MHz)

Demodulated signal \( \propto \) RF coil signal \( \cdot \) reference (transmitter)

\[ \propto \sin[(\omega_0 + \delta\omega) t] \cdot \sin[\omega_0 t] \]

\[ \propto \frac{1}{2} \left[ \cos(\delta\omega t) - \cos(2 \omega_0 + \delta\omega) t \right] \]

This signal is digitized.

- One freq - freq domain

Signal

Reference

Demodulated

After filter

- Chirp - time domain

Chirp

Center

Demodulated

Low freq signal

Two signals are made from a single receiving RF coil

- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{1}{2} \) phase, then PSD)

- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{1}{2} \) better)
**FID - FREE INDUCTION DECAY, $T_2^*$**

- Signal (FID) resulting from RF pulse w/ angle $\alpha$
  
  $$\mathbf{S}(t) = \sin \alpha \int_{-\infty}^{\infty} \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \, dw$$

- $\rho(w)$ is the spectral density function.
- $T_2$ - unrecoverable (= rapid)
- $T_2^*$ - add recoverable (= rapid + static)
- $\omega = Y\Delta\phi (\text{fixed})$
- $\Delta\phi = Y\Delta B\phi$

- An example spectral density ("Lorentzian inhomogeneity")
  
  $$\rho(w) = \frac{M_0^2}{\Delta w^2 + (\omega - \omega_0)^2}$$

- $\Delta w = Y\Delta B\phi$
- $\omega_0 = Y\Delta B\phi$

- $M_0^2$ width is proportional to $\Delta B\phi$

- $S(t) = \pi \cdot M_0^2 \cdot Y\Delta B\phi \cdot \sin \alpha \cdot e^{-tY\Delta B\phi} \cdot e^{-t/T_2} \cdot e^{-i\omega_0 t}$

- Combine $T_2 + \text{static terms}$

- $\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$

- $T_2$ - overall decay rate including inhomogeneous $B\phi$
- $T_2'$ - e.g. extra decay from BOLD

- $\rho(w) = \frac{1}{C+x^2}$ (N.B. Lorentzian)

- $\rho(w) = \frac{1}{C+x^2}$ (N.B. not Lorentzian!)
**ECHOES — spin echo**

1. Just after 90° x' pulse, \( f_{10} + f_{hi} \) have same phase.
2. Relaxation + phase dispersion of \( f_{10} + f_{hi} \) (both from \( B > B_0 \)).
3. Just after 180° y' pulse (y' pulse like x' pulse but RF has +90° phase).
4. Echo caused by re-phasing of \( f_{10} + f_{hi} \) (w/ decay due to \( T_2 \)).

- Remember: brief RF just tips vectors while retaining length.
- Relaxation includes tips and shrinks (\( M_T \)) and grows (\( M_c \), echo).
- 180° x' pulse works too, but echo will have +\( \pi \) phase (left side in figs above).
- Echo generated even if second pulse not 180° (see next).

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**Time Graph:**

- FID decay (and echo growth/decay) described by \( T_2^* \), from inhomogeneity.
- Reduction in height of echo compared to initial described by \( T_2 \), echo fixes the 'star'.

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**Diagram Notes:**

- Just suggest fit (really about 1,000,000 cyc here).
- Hidden/dephased transverse mag. here.
- Echo
- N.B.: would be complete disk above.
ECHOES — spin echo

\[ \alpha_1 - \tau - \alpha_2 - \tau \] (both pulses along y' for simplicity)

**General transforms (operations)**

\[ M_x' \rightarrow M_x', \cos \alpha - M_z', \sin \alpha \]
\[ M_y' \rightarrow M_y' \]
\[ M_z' \rightarrow M_x', \sin \alpha - M_z', \cos \alpha \]
\( \approx \) (etc for \( \alpha_1, \alpha_2 \))

**Effect of \( \alpha_1 \) pulse**

\[
\begin{align*}
M_x'(w, \omega_1) &= -M_z'(w) \sin \alpha_1 \\
M_y'(w, \omega_1) &= 0 \\
M_z'(w, \omega_1) &= M_z'(w) \cos \alpha_1
\end{align*}
\]

**After \( \tau \) delay**

\[
\begin{align*}
M_x'(w, \tau) &= -M_z'(w) \sin \alpha_1, \cos \omega \tau, e^{-\tau/2} \\
M_y'(w, \tau) &= M_z'(w) \sin \alpha_1, \sin \omega \tau, e^{-\tau/2} \\
M_z'(w, \tau) &= M_z'(w) \left[ 1 - (1 - \cos \alpha_1) e^{-\tau/2} \right]
\end{align*}
\]

**Effect of \( \alpha_2 \) pulse** (no effect on \( M_y' \); rewrite \( M_x' \) and \( M_z' \) eqns)

\[
\begin{align*}
M_x^{y_1}(w, \tau_1) &= M_z'(w) \sin \alpha_1, \left( \sin^2 \alpha_2 e^{-i \omega \tau} - \cos^2 \alpha_2 e^{-i \omega \tau} \right) e^{-\tau/2} \\
&\quad - M_z'(w) \left[ 1 - (1 - \cos \alpha_1) e^{-\tau/2} \right] \sin \alpha_2
\end{align*}
\]

**Time dependent**

- Free precession around \( \hat{z} \) (rewrite \( M_x^{y_1}(w, \tau_1) \))

**For a large num of freq's:**

\[
\begin{align*}
M_{x^{y_1}}(w, \tau) &= M_{x^{y_1}}(w, \tau_1) e^{-(\tau - \tau_1)/2} e^{-i \omega \tau_1} \\
&\quad = M_z'(w) \sin \alpha_1, \sin^2 \alpha_2 e^{-\tau/2} e^{-i \omega \tau_1} \\
&\quad - M_z'(w) \sin \alpha_1, \cos^2 \alpha_2 e^{-\tau/2} e^{-i \omega \tau_1} \\
&\quad - M_z'(w) \left[ 1 - (1 - \cos \alpha_1) e^{-\tau/2} \right] \sin \alpha_2 e^{-i \omega \tau_1}
\end{align*}
\]

\([\text{terms } 1,2,3 \text{ are dephasing} \rightarrow \text{FID of echo}]

\([\text{term } 1 \text{ is dephasing} \rightarrow \text{nephase at } t = 2 \tau] \]

**Echo signal**

\[
S(t) = \sin \alpha_1, \sin^2 \alpha_2 e^{-t/2} e^{-i \omega (t - TE)} \int_{-\infty}^{\infty} \rho(w) e^{-t/2(w) e^{-i \omega (t - TE)}} \, dw
\]

\[
A_E = \sin \alpha_1, \sin^2 \alpha_2 e^{-t/2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2(w) e^{-i \omega (t - TE)}} \, dw
\]

\[90^\circ - \tau - 90^\circ\]

\[
S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2} e^{-i \omega (t - TE)} \, dw
\]

\[90^\circ - \tau - 180^\circ\]

\[
S_2(t) = \frac{\text{no } 1/2 \text{ factor}}{}
\]

\[90^\circ - \tau - 180^\circ\]

\[
S_3(t) = \text{multiply by } i \rightarrow \text{add } 1/2 \text{ phase}
\]

Etc for \( A_E \) ... like
Echo TRAIns - spin-echo trains

- it's (too) easy to make echos...

\[
E_n = \frac{3^{(n-1)} - 1}{2}
\]

- three RFs \(\rightarrow\) four echos (here)
- six RFs \(\rightarrow\) 12 echos (!)

A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2\(\tau\) spacing

\[
\alpha, x' M_L \rightarrow M_T
\]

\[
\alpha_x: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)}
\]

\[
\alpha_{\frac{3}{2}}: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays}
\]

(\text{after being held in limbo between } 180° \text{ FID}_2 \text{ and } \text{FID}_3;\text{ acts like 2-pulse echo})

- typically, 90° and 180° applied in different axes (\(x', y', z', \ldots\))
- which reduces phase errors due to imperfect 180° pulses
- (since slightly-off rotation around \(y'\) affects phase less)
**EXTENDED PHASE GRAPHS**

- Using full Bloch eq. solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ$, $180^\circ$)
- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_z$
- Problem #2: Third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

$\Rightarrow$ Rule for effect of $\alpha$
RF pulse on transverse mag

$\Rightarrow$ Rule for effect of $\alpha$
RF pulse on longitudinal mag

$\Rightarrow$ Echo when phase path crosses zero
3 - Pulse Echo Amplitudes

- Assume $M_z^0 = 1$

**RF transmit**

- $\alpha_1$
- $\alpha_2$
- $\alpha_3$

**RF receive**

- $T_1$
- $T_2$
- $T_3$

Echo: $SE_{1,2}$

- Time: $t = 2T_1$
- Amplitude: $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2T_1/T_2}$

Special cases:
- $\alpha_1 = 90^0, \alpha_2 = 180^0$: $e^{-2T_1/T_2}$
- $2\alpha_1 = \alpha_2$: $\sin^3 \alpha_1 \cdot e^{-2T_1/T_2}$

$Z^0$

- Time: $t = 2T_2$
- Amplitude: $-\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2T_2/T_2}$

$STE$

- Time: $t = 2T_1 + T_2$
- Amplitude: $\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-T_3/T_2} e^{-2T_1/T_2}$

- N.B.: $T_1$

$SE_{2,3}$

- Time: $t = T_1 + 2T_2$
- Amplitude: $\left[ 1 - (1 - \cos \alpha_1) e^{-T_1/T_2} \right] \sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + 2T_2)/T_2}$

$SE_{1,3}$

- Time: $t = 2(T_1 + T_2)$
- Amplitude: $\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + T_2)/T_2}$

- $T_2$-dependence in $STE$ (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse.
**Hyper Echoes**

(N.B. coord sys vs Bloch eq. notes)

- Hennig & Scheffler (2001)
- Sphere surface defines 2D space for \( \mathbf{M} \) moved by:
  1) vector rotation of \( \mathbf{M} \) around tilted axis in transverse \( xy \) plane by RF with flip, \( \alpha \), and phase, \( \phi = P(\alpha, \phi) \)
  2) rotation around \( z \) by phase evolution due to freq offset, \( \omega \) (B0 offset) and time, \( t = P(\omega, t) \)

- three symmetries:
  - solid lines: phase evolve at RF flip or RF
  - dashed lines: just 180° again

By combining long sequences observing these symmetries, can generate a strong echo even w/ many inserted \( \alpha \)-pulses in between

**Practical Use**

- multi-echo example
- can also use to prepare, then separate read-out

- practical prob: 180° pulses deposit a lot of RF (6x 90°)
  - prob at high fields
- by arranging to get big echo in middle of \( k \)-space can get by with much less RF power
**Gradient Echoes** - $T_2^*$, GE chains

- Initial negative gradient dephases spins
- After $t = T_0$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities so echo amplitude is $A_E = e^{-t/T_2^*}$
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decays

\[ \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \Rightarrow A_E = e^{-t/T_2^{**}} \]

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled
  \[ \Rightarrow \text{hence, echoes are } T_2^*-\text{weighted, not } T_2-\text{weighted} \Rightarrow \text{more susceptible to inhomogeneities} \]

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware \[ \Rightarrow 64 \text{ echoes} \]
**IMAGE CONTRAST**

T1 Saturation-recovery (no echo, just FID)

- Contrast \((PD, T1, T2, T2^*)\) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

\[
M_z^0 = M_0^z \text{ magnetization}
\]

\[
M_z^0 (1 - e^{-TR/T1}) + \text{[ignores]}
\]

\[
\text{"steady state" after here}
\]

- Simple saturation/recovery w/ no echo
- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immediately after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, \(M_z\) just before second pulse:

\[
M_z^{(2)}(0) = M_z^{(0)} (1 - e^{-TR/T1}) + M_z^{(n)}(O_+) e^{-TR/T1}
\]

\(M_z^{(0)}\) before current pulse \(M_z^{(n)}\) "regrowth-from-zero" term

\(M_z^{(n)}\) "left-immediately after pulse" term (US. decaying)

- Given
  
  \[
  \begin{align*}
  &\text{(1) 90° pulse} \\
  &\text{(2) no } M_{xy} \text{ left}
  \end{align*}
  \]

\[
\rightarrow \text{ pure tip: } M_{xy} = M_z
\]

- Tip existing mag

\[
M_z^{(n)}(O_-) = M_{x'y'}^{(n)}(O_+) = M_z^0 (1 - e^{-TR/T1})
\]

\(M_z^{(n)}\) longitudinal mag just before pulse

\(M_{x'y'}(O_+)\) transverse we can record after pulse

\(M_z^0\) transverse mag depends on \(T1\)!

\[
\text{That is, the not-completely-regrown longitudinal magnetization, which depends on } T1, \text{ but which we cannot record, is completely converted to recordable transverse magnetization.}
\]

\[
I(r) = C \rho(r) \left( 1 - e^{-TR/T1(r)} \right)
\]

\(C\): rec. const. \(\rho(r)\) spectral dens, \(T1(r)\): equilib. \(M_z^0\)
IMAGE CONTRAST

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
  (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec
  TR images reached after 8 images
**IMAGE CONTRAST**

- Inversion recovery w/ no echo

\[ M_z^{(0)} = -M_z \]

- Recovery to end of first TI (from long. part of Bloch eq.)

\[ M_z' = M_z^{(0)} \left( 1 - 2 e^{-\frac{TI}{T1}} \right) \rightarrow \text{flipped into transverse by second pulse (180° 90°)} \]

- Longitudinal then regrows from zero (from first Bloch term only)

\[ M_z' = M_z^{(0)} \left( 1 - e^{-\frac{(TR-TI)}{T1}} \right) \]

- After second 180°, just change sign again

\[ M_z' = -M_z^{(0)} \left( 1 - e^{-\frac{(TR-TI)}{T1}} \right) \]

- Apply relaxation eq. again

\[ M_z' = M_z^{(0)} \left( 1 - e^{-\frac{TI}{T1}} \right) - M_z^{(0)} \left( 1 - e^{-\frac{(TR-TI)}{T1}} \right) e^{-\frac{TI}{T1}} \]

\[ M_z' = M_z^{(0)} \left( 1 - 2e^{-\frac{TI}{T1}} + e^{-\frac{TR}{T1}} \right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
**Image Contrast**

- Steady state mag just before 90°:
  \[ M_z^c(0) = M_z^o \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}}\right) + e^{-\frac{TR}{T_2}} \]

- The echo signal \( M_z^e \) unlike in simple saturation recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation:
  \[ A_E = M_z^o \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}}\right) + e^{-\frac{TR}{T_2}} \]

  If we assume \( TE \) much less than \( TR \), then we can simplify:
  \[ A_E = M_z^o \left(1 - e^{-\frac{TR}{T_1}}\right) e^{-\frac{TE}{T_2}} \]

- Similar equation for SE-IR:
  \[ A_E = M_z^o \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_2}}\right) e^{-\frac{TE}{T_2}} \]
**IMAGE CONTRAST**  
GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch $e$, again
- Assume $M_{z}^{(n)}(0_{-}) = 0$ → transverse dephased before next pulse

$$M_{z}^{(n)}(0_{-}) = M_{z}^{0}(1 - e^{-TR/T_1}) + M_{z}^{(n-1)}(0_{+}) e^{-TE/T_1}$$

- Assume we have a small tip angle:

$$M_{z}^{(n)}(0_{+}) = M_{z}^{(n)}(0_{-}) \cos \alpha$$

$$M_{z}^{(n)}(0_{-}) = M_{z}^{0}(1 - e^{-TR/T_1}) + M_{z}^{(n-1)}(0_{-}) \cos \alpha e^{-TR/T_1}$$

- Assume we are in steady state:

$$M_{z}^{(n)}(0_{-}) = M_{z}^{(n)}(0_{-}) = M_{z}^{ss}(0_{-})$$

- Prepulse or steady state longitudinal

$$M_{z}^{ss}(0_{-}) = \frac{M_{z}^{0}(1 - e^{-TR/T_1})}{1 - \cos \alpha e^{-TR/T_1}}$$

- Post-pulse or steady state transverse magnetization

$$M_{x,y}^{ss}(t) = \frac{M_{z}^{0}(1 - e^{-TR/T_1}) \cdot \sin \alpha e^{-TE/T_2} \sin \alpha}{1 - \cos \alpha e^{-TR/T_1}}$$

- Gradient echo amplitude

$$A_E = \frac{M_{z}^{0}(1 - e^{-TR/T_1}) \sin \alpha e^{-TE/T_2}}{1 - \cos \alpha e^{-TR/T_1}}$$

(T1 contrast mainly depends on flip angle, not TR)
Contrast - 4b

**IMAGE CONTRAST**  MDEPT / 3D FLASH

90° TD → 180° TI → R

90° TD → 180° TI → R

α (spiral phase) α

RF_in

RF_out

\

\[ G_x \]

\[ G_y \]

\[ G_z \]

Saturate, wait for contrast, invert, wait for contrast, FLASH (center out)

A) \[ M_2^z (\text{just after } 90°) = 0 \] (perfect 90°)

B) \[ M_2^z (\text{after } TD) = M_2^0 \left(1 - e^{-TD/T2}\right) \] (Bloch term *1)

C) \[ M_2^z (\text{just after invert}) = \cos \phi M_2^0 \left(1 - e^{-TD/T2}\right) \]

D) \[ M_2^z (\text{after } TI) = M_2^0 \left(1 - e^{-TI/T2}\right) + \left(\cos \phi M_2^0 \left(1 - e^{-TD/T2}\right)\right) e^{-TI/T2} \]

\[ = M_2^0 \left[1 - \left(1 - \cos \phi \left(1 - e^{-TD/T2}\right)\right) e^{-TI/T2}\right] \]

Special case TI = TD:

\[ = M_2^0 \left[1 - e^{-TI/T2}\right] \]

E) \[ M_2^z (\text{just after flip}) = M_2^0 \left[1 - \left(1 - \cos \phi \left(1 - e^{-TD/T2}\right)\right) e^{-TI/T2}\right] \sin \alpha \]

Using hard 130° inversion

Coronal hard alpha B1

Inhomogeneities (Thomas et al., '05)

- after the first RF pulse

\[ M_2^z (\text{after flip}) = M_2^0 \left[1 - \left(1 - \cos \phi \left(1 - e^{-TD/T2}\right)\right) e^{-TI/T2}\right] \sin \alpha \]
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules tied to membranes have wide range of resonant freqs ("bound") \[ T_2 = 1 \text{ msec} \]
- Free protons in blood, CSF, water have narrow range \[ T_2 = 50 \text{ msec} \]

- Mag transfer pulse sequence
  1) Off center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated
     \[ M_L \text{ of "bound" } \rightarrow M_L \text{ of "free"} \]
  3) Result of transfer \[ \rightarrow \text{ attenuation} \]

N.B. This always happens a little (cf. T1-weighted, T2-weighted)
  something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines

max \[ \rightarrow \text{ view as movie} \]
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: $\text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}}$
- Temporal SNR: $\text{tSNR} = \frac{\text{avg signal}}{\text{s.d. signal}}$
- "Contrast" is a difference
- Contrast-to-noise ratio:

$$\text{CNR}_{AB} = \frac{\text{C}_{AB}}{\text{C}_{N}} = \frac{S_A - S_B}{\text{SNR}_A - \text{SNR}_B}$$

**Spin-echo:**

$$A_e = M_0 \left(1 - e^{-\frac{TR}{T1}}\right) e^{-\frac{TE}{T2}}$$

**Gradient echo:**

$$A_e = \frac{M_0 \left(1 - e^{-\frac{TR}{T1}}\right)}{1 - \cos \alpha} e^{-\frac{TE}{T2T}}$$

**General rules:** Spin-echo, long TR GE

<table>
<thead>
<tr>
<th>proton density weighted</th>
<th>$TR \gg (no T1 diffs)$</th>
<th>$TE \gg (no T2 diffs)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T1$-weighted</td>
<td>$TR \simeq T1$ (big T1 diffs)</td>
<td>$TE \gg (no T2 diffs)$</td>
</tr>
<tr>
<td>$T2$-weighted</td>
<td>$TR \gg (no T1 diffs)$</td>
<td>$TE \simeq T2$ (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE $S/N$

- generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{N_{x} N_{y} N_{z}} \Delta t
\]

\[
\begin{array}{ccc}
\text{voxel size} & \text{num repeats} & \text{number of read timesteps} \\
(\text{the same number}) & & (\text{same size})
\end{array}
\]

- size (volume) of voxels (with the number of voxels held constant), linear effect on $S/N$
  \[ \text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better } S/N \]

- more voxels (with size of voxels, $\Delta t$ per read step constant), $N^2$ effect on $S/N$
  \[ \text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{128 \times 128}{64 \times 64} = 2 \text{ times better } S/N \]

- # acquisitions $N^2$ better $S/N$
  \[ \text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{4}{1} = 4.1 \text{ times better } S/N \]

- longer time step during readout, $\sqrt{\Delta t}$ better $S/N$
  \[ \Delta t = \frac{1}{\text{BW}_{\text{read}}} \text{, digitization time step during echo acquisition} \]

- $\text{BW}_{\text{read}}$ determined by cutoff freq, analog low-pass filter

- $\Delta t$ controls $\text{BW}$ because low-pass cutoff has to be set higher for smaller (higher freq-detecting) $\Delta t$

- must filter out freq's $> f_{\text{max}} = \frac{1}{2\Delta t}$ because they alias
**COMPLEX ALGEBRA**

**real/imaginary**

*add:* \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)

*multiply:* \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

**angle/phase**

*add:* \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)

*multiply:* \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)

*divide:* \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1 / A_2, \phi_1 - \phi_2)\)

**complex to real power:** \((A, \phi)^n = (A^n, n\phi)\)

**e^{i\phi}**

*expansion as series*

*recognize cos, sin series*

*the real "e" to "purely imaginary power"*

= \(\cos \phi + i \sin \phi\)

= \(\cos \phi, \sin \phi\)

= vector on unit circle

\[e^{i\phi} = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi\]

**Fourier transform**

\[H(f) = \int h(t) e^{-2\pi i ft} dt\]

\[H(k) = \int h(t) e^{2\pi i kt} dt\]

**Convolution Theorem**

\[F[ g(x) \ast h(x) ] = G(k) \ast H(k)\]

\[\because \ \text{because of FFT, faster if kernel not small}\]

**Conjugation**

\[-\text{don't confuse freq, angle!}\]

\[\text{e}^{-i\omega t} = \text{freq \times \text{circular var.}}\]

\[\text{vector into circular variable \& angle (linear) at complex/circular var.}\]

**-how to convert:**

\[(r, i) \leftrightarrow (A, \phi)\]

\[A = \sqrt{r^2 + i^2}, \quad \phi = \arctan(i, r)\]

\[r = A \cos \phi, \quad i = A \sin \phi\]

| N.B.: 2nd kind of vector multiply different than dot product and cross product (and C.A.
non-commutative pseudoscalar multiply) |

| shorthand for a unit vector(\(2\hat{b}\)) |

| pointing in the direction of \(\phi\) |

| -for arbitrary amplitude, multiply \(A e^{i\phi}\) |

| -phase is integral \(\phi = \int \omega \ dt\) |

**Fourier transform and Convolution**

\[f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) \, dz\]

| Cross-conclusion |

| \(E(x) = g(x) \otimes h(x) = \int g(z) \cdot h(x+z) \, dz\) |

| the Fourier transform of two functions multiplied by each other equals the convolution of each function |

| \(\because \ \text{the Fourier transform of each function} \) |
Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-i\frac{2\pi ft}{T}} dt \]

- How to calculate \( H(f) \) for one \( f \) (\( f=3 \)):
  (real signal: only need 2 correlations)

\[ h(t) \]

real signal

imaginary signal (zero)

\[ \cos(2\pi \cdot 3t) \]

\[ e^{-i\cdot3t} \]

\[ \sin(2\pi \cdot 3t) \]

\[ \text{i.e., two correlations} \]

real frequency domain

imaginary frequency domain

Cartesian (A, \( \phi \))

Polar coords (A, \( \phi \))

Amplitude frequency domain

Phase frequency domain

like correlating with \( \sin \) and \( \cos \) (at each freq) so we get phase (at each freq.)
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]

\[ e^{-i\phi} = e^{i(-\phi)} \]

\[ = \cos (-\phi) + i \sin (-\phi) \]

\[ = \cos \phi - i \sin \phi \]

- \( \cos \) is an "even" function, \( \sin \) is an "odd" function

\[ \cos(x) \quad \text{even} \]

\[ \downarrow \text{equals} \]

\[ \cos(-x) \]

\[ \downarrow \text{flips} \]

\[ -\cos(x) \]

\[ \sin(x) \quad \text{odd} \]

\[ \downarrow \text{flips} \]

\[ \sin(-x) \]

\[ \downarrow \text{equals} \]

\[ -\sin(x) \]

- \text{An orthogonal decomposition}
- think of discretely sampled \( \sin(bx), \cos(bx) \) as vectors
- \( \text{Corr} (\vec{v}_1, \vec{v}_2) \equiv \text{projection of } \vec{v}_1 \text{ onto } \vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2 \)

\[
\begin{align*}
\text{Corr}(\cos b_1 x, \sin b_1 x) &= 0 \\
&= \sin \text{ & cos of same frequency are orthogonal} \\
&= \sin 2x \quad \cos 2x \\
\text{Corr}(\sin b_1 x, \sin b_2 x) &= 0 \\
&= \text{different integer frqys of } \sin \text{ & cos are orthogonal} \\
&= \sin 2x \quad \sin 3x \\
\text{Corr}(\cos b_1 x, \sin b_2 x) &= 0 \\
&= \text{as above}
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[ \int_{x=-\infty}^{x=\infty} f(x) g(x) \, dx = 0 \]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- start with spike in image domain
- take example of spike at $x = 0$
  \[
  \cos(x), \cos(2x), \cos(kx) \text{ all equal 1 there.}
  \]
  all freq's correlate w/ spike at $x = 0$

- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-2\pi j k x}$ cos and sin at location of spike

- pos, pair (real) spikes same dist from origin
  \[
  \begin{align*}
  1 & \Rightarrow \text{pick just "even" cos's (multiple freqs)} \\
  -1 & \Rightarrow \text{pick just "odd" sin's (multiple freqs)} \\
  \end{align*}
  \]
  \[
  \text{one spike at distance from origin}
  \]
  \[
  \text{this is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)}
  \]
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image \(\rightarrow\) Imaginary image

2. Amplitude image \(\rightarrow\) Phase image

3. Complex vectors

Equivalent representations of image & spatial frequency space.
FOURIER TRANSFORM OF REAL IMAGE (2)

- what a single k-space point looks like
  for real image (polar coordinates $A, \psi$ instead of $r, \theta$)

- Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ $\sin, \cos$ — don't confuse $k_x, k_y$ w/ $\sin, \cos$!
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: cosine in image space, then shifted in x-dir

REAL IMAGE

\[ I(x, y) = \cos(x) \]

FT OF REAL IMAGE

\[ \text{FT of } I(x, y) \]

\[ k_x \rightarrow \quad k_y \rightarrow \]

Same

\[ \text{FT} \quad \text{FT}^{-1} \]

Complex

zero

zero

zero

real component less than above because rot:

N.B.: an example of the "Fourier Shift Theorem" (see below)

45° rot compared to complex above
FOURIER TRANSFORM OF IMAGE (S)

(cont) center of k-space (real image)
complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

center of k-space:

\[ H(k) = \int_i h(x) \cdot e^{-2\pi i k_x x} \ dx \]

positive center k-space

avg image brightness \[ \Rightarrow \]

1 (real)

FT OF REAL IMAGE

FT

FT \[ \Rightarrow \]

[ the center of k-space is zero w/ pure sin or cos image b/c avg. brightness = 0 ]

COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) \]

complex

FT, FT \[ \Rightarrow \]

"missing" spike results in single spike correlating with \cos and \sin

N.B.: this k-space is non-Hermitian: k-space will only have Hermitian symmetry if image is real:

Hermitian symm. when complex conjugate (= complex num w/ sign flipped in imag. part) is equal to funct value w/ imag arg:

1D: \[ H(k) = \overline{H(k)} \]

2D: \[ H(-k_x, k_y) = H^*(k_x, k_y) \]

FT OF COMPLEX IMAGE

spike only on one side of k-space

N.B. this is like what an artifact "spike" does tho it would have rand. phase

N.B. this is also exactly what a gradient does to image space !]
**Gradient Coils**

- Gradient coils for $x, y, z$ generate approximately a linear gradient in the strength of the $z$-component of the magnetic field $B_z$.

- For example, the $x$ gradient coil induces a ramp in $z$-component of the magnetic field when moving in the $x$-direction:

$$B_{g,x} = G_x x$$

- Since a pure linear gradient of $B_{g,z}$ in only the $x$, $y$, or $z$ directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the $x$- and $y$-direction ($B_{g,x}$ and $B_{g,y}$).

- The other magnetic field components are usually ignored because they are so small relative to $B_{g,z}$, since $B_{g,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{g,z}$, $B_{g,y}$, and $B_{g,x}$.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms $B_{g,x}$ $B_{g,z}$ are known; can be included in the recon. process.
**SLICE SELECTION \((G_z)\)**

- Slice select gradient on during RF stim

\[
B_z = \frac{z}{\frac{\pi f}{G_z}}
\]

- Protons here can only be excited by a narrow band of radio frequencies

- To apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

- This excites protons in a narrow slab

- Since the slice-selection gradient introduces (space-dependent) phase shifts (see freq. encode) these have to be removed by a post-excitation rephasing \(z\)-gradient

- Approximation from assuming tip occurs instantaneously in middle
- Valid for small tip: \(90^\circ \rightarrow 52\%\)
- In practice: adjust to max, use crusher to kill spurious transverse on \(180^\circ\)
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} P(f) e^{-i 2\pi ft} df \]

Time-dependent RF stimulation (complex)

Solve with: \( P(f) = \) frequency band

\[ B_1(t) = A \cdot f_w \text{sinc}(\pi f_w t) e^{-i 2\pi f_c t} \]

Frequency

- Fourier Transform Pair, Rules
- Convolution in one domain is multiplication in the other
- Convolution with delta function, impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{1}{t} \)

- Frequency
- Convolution
- Multiply
- Equals
- Frequency
- Time
- Convolution
- Multiply
- Equals
**SLICE SELECT RF PULSES**

Interleaved Acquisition $\Rightarrow$ better S/N b/c imperfect slice profile

Common RF pulses

- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian

$pulse$ $envelope$ $\Rightarrow$ FT

Freq spectrum $\Rightarrow$ "everything"

$\Rightarrow$ need $\approx \frac{1}{3}$ gap to avoid overlap

$\Rightarrow$ pulses need to be "apodized" (have "foot" removed)

$\Rightarrow$ multiply by function so begin/end of pulse is differentiable

**Fat Saturation**

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
  $\Rightarrow$ need high quality (narrow-freq) pulse to avoid saturate water!

**How To**

1. Fat sat pulse
2. wait T2 so fat signal decays, but no T1 regrowth of fat
3. RF stim for water "protons-of-interest"

**Adding Another Gradient Tilts Slice**

$G_z, G_x, G_y$

$z, y, x$

Plane of constant $B_z$

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude
  or RF freq band: $\gamma$
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq  \Rightarrow gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF  \Rightarrow same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs  \Rightarrow same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum  \Rightarrow FT of FID to get $\Delta x$ offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading

  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"

  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies

  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

  - spectroscopy
    - samples of oscillations in time-domain
    - FT \rightarrow frequency-domain spectrum of shifts
  - MRI
    - samples of spatial freq in freq-domain
    - FT \rightarrow spatial object (like a time-domain signal)

- the original analogy only 'works' because $FT \approx FT^{-1}$ (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient \( G_x \) causes precession rates to vary linearly in \( x \)-direction.

\[ \text{Precession} \quad \uparrow B_\parallel \quad \downarrow \text{Frequency} \quad \text{in} \quad \text{x-direction} \]

\[ \Rightarrow \text{Correct} \quad (\text{remember that strength of} \ G_x \text{causes variation of slope of} \ B_\parallel \text{in} \quad \text{x-direction}) \]

- Different frequency signals are mixed together and recorded as a 1-D signal over time.

\[ \Rightarrow \text{Correct}, \quad \text{but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which can convert back and forth between \( x \)-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal.

\[ \Rightarrow \text{Correct} \]

- Spatial frequencies get confused/conflicted with precession frequencies.

\[ \Rightarrow \text{Wrong!!} \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position.

\[ \Rightarrow \text{Conceptually Wrong!!} \]

\[ \Rightarrow \text{FT actually converts spatial frequencies to spatial position} \]

\[ \Rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \Rightarrow \text{the precession freq. ramp is constant each timestep} \]

N.B.:

- Gradient ramp does not need to be exactly the same as recorded.

\[ \Rightarrow \text{Wrong!!} \]
FREQUENCY ENCODING (2) connect intuition — why phase critical

"Frequency"-encode gradient ($G_x$) turned on during

during echo causes precession rates
to immediately vary with x-position

↑ $B_z$ in x-direction

$G_x$ levels
(= slope)

actually

- at beginning of gradient on, the phase of
  signal coming from each x-position is the same
  Summed phase angle is what we measure

- early after gradient on, phase advances (because
  of faster precession frequency) arise with greatest
  phase advance at largest x-position

Single
timepoint
early

360
0
0
x

$\varphi$

One cycle of
spatial frequency
(= low spatial freq)

- later during gradient on, phase advances cause
  multiple wraparounds of phase angle across space

Single
timepoint
later

360
0
0
x

Multiple cycles of
spatial frequency
(= hi spatial freq)

- in practice, the lowest spatial frequency (= 0)
  occurs in the middle of the gradient on time
  because the phase is "wrapped" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

$\delta$ is spatial
frequency

$G_x$

$\delta$ = max neg
$\delta$ = 0
$\delta$ = max positive

Individual
RF data
Samples (after
demodulation)
FREQUENCY encoding (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (Temporal freq ↔ time)

\[ H(\beta) = \int_{-\infty}^{\infty} h(t) e^{-i 2\pi ft} dt \]

"k" is often used instead of "f" for the frequency variable

Imaging equation: (Spatial freq ↔ space)

\[ S(\beta) = \int_{-\infty}^{\infty} I(x) e^{-i 2\pi \beta x} dx \]

Sum across x of object

This is done by RF coil recording sum

Oscillations come from readout phase wrapping, where \( f \) is single spatial freq (e.g. 5) and \( x \) goes across object

\[ \text{END: } G_{xt} \]
\[ \text{SE: } G_x (t-TE) \]

\( f = G_{xt} \), that is, spatial freq depend on amount of time gradient was on (this \( f \) increases with time!)

do\n't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each \( x \) position)

To make image, do inverse Fourier transform of recorded signal \( S(\beta) \)

Gx

RF

get this single readout point by summing signal across x-position (RF coil records sum)

even though variable is \( \beta \), it represents one time period during readout

one data point (i.e., one spatial freq) during readout (2 components)
ALTERNATE DERIVATION (incl. effects of \( G_x \)) SIGNAL Eq

- oscillators at \( w = xB \) at each position (just \( x \) for now)

\[
S(t) = M(x) e^{-i \Phi(x)} dx
\]

- by definition, freq, \( w \) is rate of change of phase, \( \Phi \)

\[
\frac{d \Phi(x,t)}{dt} = w(x,t) = xB(x,t)
\]

and \( \Phi(x,t) = \int_0^t w(x,t) dt = \int_0^t B(x,t) dt \)

- assuming phase initially 0, \( B \) affected by gradients

\[
B(x,t) = B_0 + G_x(t)x
\]

so

\[
\Phi(x,t) = \oint B_0 dt + \left[ \oint G_x(t) dt \right] x
\]

\[
= w_0 t + 2\pi K_x(t)x
\]

\( K \) is time integral of gradient waveform

- demodulation removes the \( B_0 \)-caused carrier frequency \( e^{-i w_0 t} \) from the first equation

\[
S(t) = \int_x m(x) e^{-i 2\pi K_x(t)x} dx
\]

amplitude of each oscillator

- gradient-controlled phase
Phase-encode gradient \( G_y \)

- Turn on gradient after excitation but before readout

- Different levels of \( G_y \)

- Higher levels of \( G_y \) (slope of \( B_z \) in \( y \)-direction!)
  - Higher spatial freq. (more phase wraps) in \( y \)-direction

- Phase wraps persist after phase-encode gradient off

- Read-out gradient \( (G_x) \) phase wraps then add to phase-encode phase

2D Imaging Equation

\[
S(k_x, k_y) = \int \int I(x, y) \cdot e^{-i 2\pi (k_x x + k_y y)} \, dx \, dy
\]

- Signal recorded at single time point (one readout point)
- Complex signal (from phase-sensitive detection)
- Done by RF coil
- Scalar (what we try to reconstruct)
- Phase angle (of scalar magnetization) in rotating frame, set by gradients

Ignoring relaxation, spatial frequency \( k_x \) and \( k_y \) have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING — Two phase-encode gradients

- Use $z$-gradient for 2nd phase-encoding instead of slice selection
- Excitation of whole slab (slice-select is whole brain)
- Simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

$$S(k_x, k_y, k_z) = \text{signal recorded at single point of readout}$$

$$I(x, y, z) = e^{-i2\pi(k_xx + k_yy + k_zz)} dx\,dy\,dz$$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each excited volume contributes to signal from each pulse instead of just slice
  - Phase stripes created throughout volume vs. slice:

N.B., this ignores relaxation affects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoidal sine of from 2-comp quadrature).

Stripes here represent complex value.

Phase of whole image summed to one (complex) number by RF coils.

Successive readout steps:

More rotation
Higher spatial freq

E.g., after y-gradient, spins at a point might be 2 cycles ahead while after x-gradient spins at same point 8 cycles ahead; but counting wraps in y-direction, still only 2 ahead.

3D phase encode w/ G_y and G_z starts rotated in y-z plane
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ \mathbf{k} = \mathbf{G}t \]

(k is area under curve)

all of the following gradients end up at the same point in k-space:

Frequency-encode FID

Frequency-encode gradient echo

Frequency-encode spin-echo (plus gradient echo!!)

Phase-encode then frequency encode gradient echo

\[ \mathbf{G}_y \]

\[ \mathbf{G}_x \]
**IMAGE RECONSTRUCTION**

\[
S(k_x, k_y) = \iiint \omega(2\pi(k_x x + k_y y)) \, dx \, dy
\]

\[
I(x, y) = \iiint S(k_x, k_y) e^{i2\pi(k_x x + yk_y)} \, dk_x \, dk_y
\]

\[\text{Ideally image is real, in practice complex} \rightarrow \text{use amplitude image:} \quad |A| \rightarrow |\phi| \rightarrow \frac{A}{e^{i\phi}}\]

Adding exponents same as multiplying two \(e^{i2\pi k_x x}\)’s

Same as two sequential 1D FFT’s (actual code)

\[
I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i2\pi n \Delta k_x x} \Delta k_x \right] e^{i2\pi m \Delta k_y y} \Delta k_y
\]

In practice, finite number of samples, \(N\) and \(M\), are collected. \(k_x\) and \(k_y\) directions of \(K\)-space (integral \(\rightarrow\) discrete sum).
SAMPLING  aliasing, FOV  aliasing from insufficient samples in time domain

must consider effects sampling limited points in k-space
limited in range of frequencies sampled \((k_{\text{min}} \rightarrow k_{\text{max}})\)
limited in rate of sampling \((\Delta k)\)

N.B. aliasing less familiar when result of limited frequency
domain sampling than limited space or time domain sampling

[as above w/ blurring, ringing]

- as above w/ blurring, ringing

- aliasesing occurs in spatial domain
- replicas overlap, causing wraparound

- finite frequency range
- too-wide spacing of samples

thus, finer sampling of same range of spat. freqs increases FOV
**UNDER/OVER SAMPLE**

More examples

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]
\[ \delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

\text{FOV (distance to repeat) is reciprocal of spatial frequency sampling interval}

\text{Pixel size is FOV divided by K-space sample count}

3 more examples (not incl. less samples to same spat. freq [bottom last page])

- **Basic Image**
- **Same num samp. to 2x spat. freq.** (i.e., gradients stronger or time ON longer)
- **2x num. samples to same spat. freq.** (i.e., gradients weaker or time ON shorter)
- **2x number samples to 2x spat. freq.** (i.e., gradients stronger or time ON longer)

- **Space**
- **Pix**

- Basic image
- Square pix
- X-pix half width
- Replicas intrude
- [Scanner makes square image "wrap" occurs
- Square pix
- Twice X-pix count so FOV = 2X
- This is "phase oversamp"
- [Scanner crops to square
- Replicas move out
- X-pix half width
- Twice X-pix count
- Same FOV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. image/brain space
   - convolve

2. sampled data spatial frequency
   - multiply

Useful FTs:

- Rect: \( \text{Rect}(x) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi Wk) \)
- Gaussian (special case): \( e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \)
- Gaussian (adj width): \( e^{-ax^2} \xrightarrow{\mathcal{F}} \frac{\sqrt{\pi/a}}{a} e^{-\frac{mk^2}{a}} \)
- Comb: \( \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k}) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \)

Fov = \frac{1}{\Delta k}
\Delta k = \frac{1}{\text{Fov}}
**Point-Spread Function**

\[ \hat{I}(x) = \Delta k \sum_{n} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(n \Delta k) = 1 \]

- Substitute into \( \text{Recon} \) to get PSF:
  \[ h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x} \]

- Simplify
  \[ h(x) = \Delta k \frac{\sin \left( \pi N \Delta k x \right)}{\sin \left( \pi \Delta k x \right)} \Rightarrow \text{periodic} \]

- That is, image is reconstructed from a sum of \( \text{sinc} \)’s, because the FT of a boxcar pixel in \( k \)-space is an \( \text{sinc} \).

**Diagram:**

- Image
- FT
- Frequency
- How PSF modifies ideal (infinite \( k \)) image
- Convolve
- Ringing
- Multiplying
- Acquisition window (truncates high spatial

**Note:**
- Central replica same as sinc.
**GENERAL LINEAR INVERSE RECON FOR MRI**

\[ S(k_x) = \int \frac{I(x) e^{-i 2\pi k_x x}}{dx} \quad \text{Signal eq. \xrightarrow{} fwd problem} \]

\[ I(x) = \int S(k_x) e^{i 2\pi k_x x} dk_x \quad \text{Recon eq. \xrightarrow{} inv. problem} \]

\[ s = F^T i \]

\[ s = \begin{bmatrix} s_{xy} \\ s_{ky} \end{bmatrix}, \quad F = \begin{bmatrix} F_{xy} \\ F_{ky} \end{bmatrix} \]

\[ F_{xy} = g(x,y) e^{-i \phi(x,y)} e^{-\left(\frac{3.8 T_2}{2}ight)} \]

\[ F_{ky} = g(x,y) e^{i \phi(x,y)} e^{\left(\frac{3.8 T_2}{2}ight)} \]

\[ T_2 \text{ decay} \quad \text{Error (x,y dep.)} \quad \text{Freq + phase} \]

**Multi-coil**

\[ s = \begin{bmatrix} s_{xy} \\ s_{ky} \end{bmatrix} = \begin{bmatrix} F_{xy} \\ F_{ky} \end{bmatrix} i \]

\[ F^+ = (F^T F)^{-1} F^T \quad \text{Min norm inverse} \]

\[ i = \begin{bmatrix} i_{xy} \\ i_{ky} \end{bmatrix} = \begin{bmatrix} F_{xy} \\ F_{ky} \end{bmatrix} \]

\[ F^+ = \begin{bmatrix} F^T (F F^T)^{-1} F^T \end{bmatrix} \]

\[ \text{Over-determined} \]

\[ i = \begin{bmatrix} (F^T F)^{-1} F^T \end{bmatrix} s \]

\[ \text{slice-by-slice} \]

\[ \text{assumes slice select swamps noise} \]
**FAST SPIN ECHO (FSE)**

- one 90° pulse followed by multiple 180° pulses (e.g., 8)
  - each with a different phase-encode gradient
  - each phase “winder” is “unwound” because leftover phase would be refocused away by 180° (i.e., EPI where it persists between blips)

- the effective TE is the TE when center of k-space is collected (largest effect on contrast, largest echo)
- each subsequent echo has more T2 decay: \( E_n = e^{-n\text{TE}/T_2} \) \( n = 1, 2, \ldots, M \)

- by arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted
  - different prominence of T2 decay in contrast controlling \( k_y = 0 \) echo

- possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train

- 3DFSE — like 2D except
  - wind/unwind added to thick slice select (w/ emuskers on 180°)

- largest amp: echo from center of k-space because low gradient induces low dephasing
  - N.B.: only one read phase
  - subsequent 180°s reset from right to left

- N.B.: all those 180° pulses deposit a lot of RF power: 90° + 180° = 45x power 30°
MULTI-SLAB 3DFSE, PROBLEMS

- echoes die out quickly \( \propto e^{-t/T_2} \)
- since echoes after 90° limited to \( \leq 30 \), can't fill 3-D k-space in a reasonable time
- SAR constraint

\[
SAR \propto B_0^2 \theta^2 \Delta B
\]

\( \Rightarrow \) 180° pulses deposit 4-6x power of 90°
- "multi-slab" is halfway between slices and single-slab
- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses \( \Rightarrow \) longer than non-selective 'hard' pulses

\( \text{Echo train e.g. 20} \)
- etc to fill 3D k-space
\[ \text{\( G_z \) is "partition"} \]
\[ \text{\( Gy \) is "phase encode"} \]
\[ \text{\( G_x \) readout needs no pre-wind since 180° does it} \]

\( \text{TE} \) is time from 90° to echo thru center of k-space

- hard to get under 8 msec inter-echo spacing
- limits speed of covering k-space
SINGLE-SLAB 3D FSE

---

**Regular FSE (180° pulse train)**

- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
- This "storage" in Z-axis preserves magnetization for longer time
- Smaller flip angles allow much longer echo trains
- Enough to collect whole plane of 3-D k-space

**Different than hyper-echoes (not symmetric)**

- Contrast must consider STE

\[ SE = \sin \alpha, \sin^2 \beta e^{-2\pi T/\tau} \]
\[ STE = \frac{1}{T} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-2\pi T/\tau} \]

**Single-slab 3D FSE pulse seq.**

- Variable flip angle (<1 msec)
- Hard (non-selective) pulse not 180°

**NB:** time to scan k-space is \( \propto S \sqrt{X} \)
- Apparent contrast time b/c of "storage"
- (e.g. TE\(\text{eff} = 585\) ms looks like FSE TE=140 ms)
FAST GRADIENT ECHO (GRASS, FLASH, SPGR, MPRAGE)

- Small tip so TR can be greatly reduced (e.g., 10 msec, less than T2)
- 'Leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

STEADY-STATE COHERENT (GRASS, FISP)

- Unwind phase from phase-encode M_T before next pulse (there because TR < TE)
- Unwind read gradient, too
- Signal decay exponential:
  \[ S = k_s \sin x \left( 1 + \cos x \right) \frac{1}{T2/T1} e^{-T/T2} \]
  \( k_s = \text{tissue T2/11} \)
  \( \text{brain 0.1}
  \( \text{fat 0.3}
  \( \text{CSF 0.7}

STEADY-STATE SPOILED (SPGR, FLASH)

- Spoil with random gradient (but this still allows some x refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

NON-STEADY STATE, MAGNETIZATION-PREP

NOMINAL INVERSION TIME

effect TI actually time to TR that records signal

K-space center

MPRAGE

- Preparation pulse \( \rightarrow \) strong T1-weighting
- Contrast varies in spatial and freq-dependent way
**Quantitative T1 — Intro, Methods**

**Motivation**

- Image values are arbitrary/relative (diff segs, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - At 3T, B1 can vary by 25% across the brain
  - This can invert contrast in a fast gradient echo

**Pre-scan normalise**

- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set params to get low GM/WM contrast
- Collect data scan (e.g., MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

**T1 divided by T2**

- MPRAGE → strong T1-Contrast
- SPACE → T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems < noise in regions of low signal

**MP2RAGE**

1st volume → PD-weighted
2nd copy of volume → max T1-weighted

- N.B. SSFP-like in partition, phase-encode dir
- Convert to -0.5 to 0.5 image: $S = \text{real} \left[ \frac{\tilde{S}_{\text{T1}} \cdot \tilde{S}_{\text{T2}}}{\| \tilde{S}_{\text{T1}} \|^2 + \| \tilde{S}_{\text{T2}} \|^2} \right]$
QUANTITATIVE T1 — HELMS 2-FLIP ANGLE METHOD

- Start with gradient echo signal e.g., dropping T2-decay → $e^{-TE/T2}$

$$S_{\text{Ernst}} = \frac{A \cdot \sin \alpha \cdot 1 - e^{-TR/T1}}{1 - \cos \alpha \cdot e^{-TR/T1}}$$

Ernst eq.

(max: $\cos \alpha e^{-TR/T1}$) 

"Ernst angle"

- Simplify/linearize/estimate

TR ≪ T1

linear approx. of exponentials

Taylor expansion simplification of $\sin, \cos, \text{drop small term}$

- Solve for TD and $A$ (proton density) given signals from 2 diff flip angles

$$T1_{\text{est}} = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1}$$

$$A_{\text{est}} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2 \alpha_2 - S_1 \alpha_1}$$

- Problem: flip angle varies a lot at 3T (e.g., 25%) from nominal/requested (e.g., flip series)

- Collect spin-echo and stimulated echo (EVI) estimate T1/T2

$$S_{\text{SE}} = k \cdot \sin^3 \alpha \cdot e^{-TE/T2}$$

$$S_{\text{STE}} = \frac{1}{2} \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2} \cdot \sin \frac{T1}{2}$$

$$\alpha = \arccos \left( \frac{S_{\text{STE}} \cdot e^{-TM/2}}{S_{\text{SE}}} \right)$$

Jiru & Klose (2006)
- Single shot EPI collects all k-space lines (e.g. 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (∆t) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the Gy "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**SPIN ECHO EPI** why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of deoxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static $T_2^*$ ($T_2'$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For $TE \approx 100$ ms, spins diffuse 10's of mm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion is less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extramural bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

- Over half of SE-BOLD at LST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.
- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.
- "Asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo).

- The 180°-pulse rephasing reduces the T2* signal, which is why the partially replaced asymmetric spin echo has been more commonly used.
- At higher fields, spin echo EPI is more promising:
  - Signal to noise is higher so we can take spin echo hit
  - Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- Coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff fall-offs
  -> but what does this look like in k-space?

- Slow variation in RF field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space (N.B. not addition!)

- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space— at all spatial frequencies!!!

- Simple example w/ "brain" consisting of one spatial freq:

  **Image domain**

  \[
  \text{image domain} \quad \xrightarrow{\text{FT}} \quad \text{spatial freq. domain}
  \]

  - Image ("brain")
  - \( \times \) (multiply)
  - FT
  - \( (\text{equals}) \)
  - Acquired image

- N.B. inverse FT of k-space data "smeared" in spatial freq.

  Space is sharp image w/ fall-off (not blurred image).

- "Smeared" means normally orthogonal spatial freq’s "leak" to adj freqs.

[GRAPPA — construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center — across mult coils]

[SENSE — general linear inverse approach]

- N.B.: neither would work unless normally orthog. spatial freqs, blurred!
- Excite multiple slices at once
- Function of $G_z$ blips is to shift slices in $G_y$ direction
- This occurs because for given slice, a phase pedestal is added to the entire slice
  - This "Fourier Shift Theorem"
  - [N.B.: different than $B_0$ defect-induced incremented phase errors]
- Problem w/ all up $G_z$ blips -> phase error builds up
  - Trick #1: Start w/ 2 slices, one at $z=0$, other above
    - If $180^\circ$ phase shift used, blip up/down same! (no effect at $z=0$)
    - I.e., move top or bottom replica
  - Trick #2: For multiple slices not all at $z=0$, phase no longer same for even/odd
    - But can add phase to equilibrate to k-space before recon.
  - Trick #3: For more than 2 slices:
    - 1st, even, odd, even, odd
MULTI BAND/BLIPPED CHIP (cont.)

- relation between leave-one-out aliasing and nominally fully-sampled SMS

- Leave alternate lines out, wraps image
- SENSE/GRAPPA to fix block coil view swears K-space data
- Nominally, w/ SMS we record every line of K-space
- But equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- Slice GRAPPA
  - reg GRAPPA → recon missing lines
  - slice GRAPPA → recon multiple K-spaces, i.e., not for each overlapped slice
    by training on fully-sampled data at beginning of scan

- Inter-slice 'leakage block'
  - when training GRAPPA kernel on fully-sampled data, also minimize inter-slice leakage (split-slice-GRAPPA)
  - can also do regular GRAPPA on top of this
  - reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout
  - gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- center 4 $k_x$, $k_y$ for this $k_z$ partition: TE off
- finished the first partition
- entire k-space must be filled before 3D image is reconstructed
- since entire volume is excited each shot, potentially higher S/N
- must use smaller flip angle to avoid killing $M_L$ since entire volume excited every partition (e.g. every 80 ms sec)

- main issue is movement artifact since data assembled from many shots over several secs
- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- By using smoothly changing gradients (sinusoids) less gradient power required than with trapezoids (less eddy currents)

  Earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant \( \Delta k_x \)

- Sinusoids in both \( G_x \) and \( G_y \) give spiral \( k \)-space trajectory

- \( \text{RF} \)

  \[ k_y \]

  \[ k_x \]

  \[ \text{Sample all orientations of each spatial frequency while slowly increasing spatial freq.} \]

- Constant angular velocity goes too fast at large \( k_x, k_y \)

- Constant linear velocity better but impractical near \( k_x = 0, k_y = 0 \)

- Compromise: start constant angular, end constant linear

<table>
<thead>
<tr>
<th>Constant angular velocity</th>
<th>Constant linear velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w(t) = \omega_0 t )</td>
<td>( w(t) = \omega_0 T_e )</td>
</tr>
<tr>
<td>( k(t) = A t e^{i \omega_0 t} )</td>
<td>( k(t) = A T_e e^{i \omega_0 T_e} )</td>
</tr>
<tr>
<td>( G(t) = \frac{1}{A} \frac{d}{dt} k(t) )</td>
<td>( G(t) = \frac{1}{A T_e} \frac{d}{dt} k(t) )</td>
</tr>
<tr>
<td>( G_x(t) = A \cos \omega_0 t - \frac{A}{2} \omega_0 \sin \omega_0 t )</td>
<td>( G_x(t) = \frac{A}{2} \cos \omega_0 T_e + \frac{A}{2} \omega_0 \cos \omega_0 T_e )</td>
</tr>
<tr>
<td>( G_y(t) = A \sin \omega_0 t + \frac{A}{2} \omega_0 \cos \omega_0 t )</td>
<td>( G_y(t) = \frac{A}{2} \sin \omega_0 T_e + \frac{A}{2} \omega_0 \sin \omega_0 T_e )</td>
</tr>
</tbody>
</table>

\( G_x \) | \( G_y \)
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)
  - All echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
  - Possible to present sign
  - High, uniform contrast, but lots of waiting (TI), high BW

RF

180° (prep1) TI = 700 msec

180° <--- 90°

180°

180° x 16

180° (prep2)

G_2

G_y

G_x

Sig.

FID

Echo_1

Echo_2

Loop order

3D k-space

("stack of spirals")

k_x

k_y

k_z

Spiral interleaves

k_x interleaves

k_y echoes

k_z echoes (after one 90°)
**Phase Errors & Echo-Centering Errors**

- Anything that causes a deviation of the $B_2$ field strength from the expected value ($B_{0,z} + G_{x,z}x + G_{y,z}y + G_{z,z}z$) changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

- Fourier shift theorem:
  
  $$I(x - x_0) = \int_{k_x} e^{-i2\pi k_x(x-x_0)} S(k_x) e^{i2\pi k_x x} \, dk_x$$

  - Correct with shimming and $B_0$-mapping/phase unwrapping before reconstruction.

**Echo Centering Error**

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction 
  
  
  Magnitude image unchanged.

- Fourier freq. shift theorem:
  
  $$S(k_x - k_{x_0}) e^{-i2\pi k_x x} \, dk_x$$

  Offset in spatial freq. space.
FAST SCAN ARTIFACTS

EPI vs. Spiral

brain-induced field defects lead to phase errors

**EPI**
- $G_x$ readout gradient strong \(\rightarrow\) field defects smaller percentage, less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weak and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- An extra gradient in the $x$-direction, for example, maps and unmaps phase as a function of $x$-position
- But $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

the lack of blurring has lead to a preference for EPI, despite the substantial image shifts

- For a given $x$-position, the strength of the spurious gradient is constant, so the accumulation of phase error results in a shift in the $y$-direction ($k_x$-space spin stripe displ.)
- The phase error causes a shift in the $y$-direction proportional to $x$-gradient strength (= shear) but no blurring
  (N.B. Shift varies with $x$-position) $y \rightarrow y$

**Spiral**
- With center-out spirals, phase errors accumulate in a radial direction
- Thus, higher spatial frequencies have more error (= more shearing)
- For spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq. $y \rightarrow y$

- For defects with more complex contours in the $y$-direction (than linear, as above) the vertical shifts (in EPI) will vary with $y$-position, and may result in signals from different $y$-positions being reconstructed on top of each other
Localized $B\phi$ defects often arise from air pockets embedded in tissue
- Air in middle/outer ear $\rightarrow$ indentation in inferior temporal lobe
- Air under olfactory epithelium $\rightarrow$ orbitofrontal cx, ant, thal. compression

Collect one data (k-space) point

4 cycles of phase in y-dir (x-direction) $+ \rightarrow$ localized $B\phi$ defect

Complex multiply ($= \mathrm{correlate} \sin, \cos$ with brain)

Brain structure sampled with distorted stripes

= one complex number

Reconstruction from distorted data points

... $+$ [undistorted stripes used by inverse FFT] $\times$ amplitude and phase of this component $+$ [same for 5 cycles] $+$ ... $=$

Local upward displacement of image phase (phase encode dir)

**N.B.: image shift only occurs if shift spatial trend sampled w/ successively later echoes (see next page)

Same defect makes leftward dent in vertical phase stripes

Spatial information can be lost when continuous changes in phase are flattened by $B\phi$ defect

Shifts can pile multiple pixels on top of each other into one bright pixel

Local estimates of $\Delta B\phi$ needed to correct images

1) Fieldmap method: $<$ multiple TE's to estimate $\Delta B\phi$ from $\gamma/T_E$ slope
2) Point-spread-function: $<$ extra phase encode to estimate P.S.F. (should be E-function)

Deconvolve distorted image in phase-encode direction
LOCALIZED $B\phi$ DEFECT, EFFECT ON RECON (2)

- When local $B\phi$ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- If each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- A k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- However, with w/EPI, static $B\phi$ defect causes more and more local displacement of image phase stripes for each additional ky line

```
- That is, later lines have greater spat. freq. offset
- Effectively stretches k-space in ky direction
- Same num samples to higher spatial freq.
  shrinks FOV (squishes voxels - see FOV page)
```

- When image is reconstructed, region with local $B\phi$ defect shifted oppositely

Thus, local shift effect due to combination of 3 things:

1) Static local $\Delta B\phi$ defect

2) Successive increases in phase error for successive spat. freq. measurements during long EPI readout

3) Small size of ky phase encode blips relative to $B\phi$ defect (much less of this effect in freq. encode direction)

- Respiration (which affect $B\phi$) in 3D FLASH might cause similar effect within $k_z$ partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the $z$ component of the $B$ field $- B_z$ — in the $x$, $y$, and $z$ directions.

- In practice, gradient coils are non-linear (e.g., printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

- A non-linear slice-select gradient will excite a curved slice.

- Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching 1 cm in function-structure overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- These effects do not build up over time in phase-encode direction since they only occur when gradients are turned on.

- These distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction...
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)
- Linear shim coils impose gradients in $x$, $y$, and $z$
- Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)
- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field
- Local resonance offsets caused by $B_0$ defects estimated from images
  \[ \Rightarrow \text{e.g.: sample phase at multiple echo times} \]
- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  \[ \Rightarrow \text{this only corrects spatially gradual field defects} \]
  \[ \Rightarrow \text{local defects due to air in sinuses much higher order than shims} \]
- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un-warp the reconstructed magnitude image
- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
**NAVIGATOR ECHOS**

- **1D navigator**
  - **Bφ drift problem**
    - Slow up/down drifts in Bφ continuously occur
    - A pedestal in Bφ is pedestal in phase (not gradient)
      - Which causes spatial shift (Fourier shift theorem)
    - In EPI, mainly affects phase-encode dir b/c small slip
      - Result is successive volumes drift in phase-encode dir
  
  **Gradient balance problem**
  - Unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
    - Causing N/2 (Nyquist) ghosting ≥ another phase error

- **3D navigator**: collect 3D sphere in k-space
  
  Rotation of object → rotation of k-space amplitude pattern
  Translation of object → phase shift of k-space phase (Fourier shift)

  - Sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - Do N/S hemispheres separately (less T2*, cancel EPI-like error accumulation)

  Welch et al. (2002) MRM

  $z(n) = \frac{2n - N - 1}{N}$

  $y(n) = \cos(TN\pi \sin^{-1} z(n)) \sqrt{1 - z^2(n)}$

  $x(n) = \sin(TN\pi \sin^{-1} z(n)) \sqrt{1 - z^2(n)}$

  (skip poles – slew rate too high)

  - Can be used for prospective motion correction (rotate, translate w/ gradients)
  - Better estimate, because of speed, than full TR of EPI images (24 ms vs. 2.4 sec)
  - May need to smooth rot, trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES $B_0$ inhomogeneities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way.
  - variations can be used (c.f. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
- potentially worse (why local transmit is still in progress)
- usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields ($\leq$ higher RF frequencies)
  is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

- record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)

- divide small coil body coil at each voxel to determine receive field

- use receive field to normalize main image(s)

[see also: eT1, MP2RAGE, T1/T2]
**DIFFUSION-WEIGHTED IMAGING**

- Simple diffusion weighting
  - $\text{RF}$ $\rightarrow$ $90^\circ$ $\rightarrow$ $T_1$ $\rightarrow$ $\text{G}_x$ $\rightarrow$ $\text{G}_y$ $\rightarrow$ $\text{G}_z$ $\rightarrow$ $T_2$ $\rightarrow$ $\text{readout}$

- "Apparent diffusion coefficient" map
  - To get large $b$, need $G$ $\uparrow$ $T_1$ (need big $G$'s)
  - Long $T_1$ gives spurious $T_2$-weighting
  - Can use stimulated echoes:
    - $90^\circ$ RF $\rightarrow$ $\text{G}_x$ $\rightarrow$ $90^\circ$ RF $\rightarrow$ $90^\circ$ RF $\rightarrow$ $\text{G}_z$

1) Anisotropic Diffusion (Gaussian)
   - Measure $D$ along multiple axes
   - Have to measure tensor, not scalar
   - Even for 1 primary direction:
     - $D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{xy} & D_{yy} & D_{yz} \\ D_{xz} & D_{yz} & D_{zz} \end{bmatrix}$
   - Since $D$ is symmetric, only need 6 measurements
   - $D = u^T D u$ (scalar diffusion)
   - $D_{ij} = D_{ji}$ (diffusion tensor measurement direction)

2) Length Scale by multiple $b$-values
   - Fit line to semilog signal as function of $b$
   - Anti-straight line: multi-exponential, e.g., $S = S_0 e^{-bD_1} + S_0 e^{-bD_2}$
   - $D_1 / D_2$ ratio measures tissue size

- Classical diffusion coefficient
  - $D = \frac{\partial \langle x^2 \rangle}{\partial t}$
  - e.g., brain $D = 0.001 \text{ mm}^2/\text{sec}$

- Spins acquire phase during first $\Delta T$
  - If spins diffuse (= move) along gradient by time $T$, signal is lost because negative $\Delta T$ doesn't re-phase
  - Attenuation: $A(b) = \frac{S_w}{S_0} e^{-bD}$
  - $b = Y^2 G^2 T^2 (1 - \frac{T}{T_L/3})$

- Summarizes gradient amp, length, delay, $T_L$
  - Calculate $D$ image from $b = 0$
  - $b$ large => add more $b$'s

- An ill-posed problem
  - Constraints on both ends and center point (!)

- Need vis. areas test fcr tracts
Practical Diffusion-Weighted Pulse Seq

- Spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - Allows longer TE
  - Flips $M_2$ so rephase gradient (same sign as de-phase)

- Eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "Doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)
  - Also, keep crushers orthogonal to diffusion-encoding gradients

Nagy et al. (2014) MRM

$T_{1RSE} = 0 = T_1 - T_2 - T_3 + T_4$

Phase dispersion (90 echo)

Twice-refocused Spin-echo (for center k-space)
**PERFUSION - ARTERIAL SPIN LABEL**

- **Basic idea:** Tag blood below area of interest, collect control & tagged image to assess directional input flow!
- Continuous ASL (CASL): Continuously tag a plane, gradient on, blood gets adiabatically inverted as it passes through location w/combined resonant freq.
- Pulsed ASL (PASL): e.g., EPiSTAR, FAIR, PICORE, QUIPSS II
- Tag block of tissue below slice(s)

- Small diffs between control and tag (~1%)
  - **Requirements:** Accurate balancing of control & tag images
- **Contrast problems:** Transit delays → biggest confounding factor
  - Relaxation rate diff
  - Venous clearance (vs. microspheres, which get stuck!)

**QUIPSS II - Quantitative perfusion**

1. **Pre-saturate spins in target slices**
2. **Tag - 180° pulse below slices**
3. **Control - 180° pulse above slices** (to control off-resonance)
4. **Saturate tagged block to end tag (TI)**
   - Both tag and control
   - Can use train of thin slices
   - Pulses at top of tag band

**ΔM = flow × [2M_0(TI, e^{-T2/T1})]**

<table>
<thead>
<tr>
<th>TI</th>
<th>500</th>
<th>1000</th>
<th>1500 msec</th>
</tr>
</thead>
<tbody>
<tr>
<td>Par.</td>
<td>Frontal</td>
<td>Temporal</td>
<td>Occipital</td>
</tr>
</tbody>
</table>

**Solution for quantitative**
- Insert delay so all spins arrive into low velocity capillaries
- Kill end of tag to reduce spatial variation of tag

**Turbo ASL**
- Use TI longer than TR
- Omit QUIPSS tag ending
  - Tag pulse
  - Control image
  - Control pulse
  - Tag image
  - Tag pulse
  - Control image

2X faster but limited slice size

**Alternate tag and control, GRE**

| Control - tag → flow
| Control + tag → BOLD

**Dual echo spiral**

- TE = 30 ms: BOLD-weighted
- k = 0 early → high S/N flow
- TE = 30 ms → BOLD

**Con extract flow and BOLD**

**Adjacent substructures minimize movement artifact**
OFF RESONANCE EXCITATION

- main idea: examine evolution of $\vec{M}$ vector in rotating coord syst set to "off-resonance" $\vec{B}_1$ field freq ($\omega_{RF}$), not Larmor freq of $\vec{M}$ ($\omega_0$)

- normally, if rotating coord syst freq set to Larmor freq ($\omega_{RF} = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) $\implies$ implies effective $B_z = 0$ in rotating

- now, move $\vec{M}$ to rotating coord syst at $\vec{B}_1$ freq lower than $\omega_0$ (assume $\vec{B}_1 = 0 = \Phi$): existing $\vec{M}$ will now appear to precess around $z$-axis:

N.B.: this is precession in already rotating coordinate system!
(slow relative to $\omega_0$)

$\Delta\omega_0 = \omega_0 - \omega_{RF}$

freq of precession in rotating coord syst of $\vec{M}$
Larmor freq of $\vec{B}_1$ = "incorrectly set" rotating coord syst freq

- thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $\vec{B}_z$ field is causing "extra" precession

"extra" $\vec{B}_z$ component is proportional to $\Delta\omega_0$ offset $\implies$ can be pos or neg: not coord syst freq too low $\implies$ pos $\vec{B}_z$, too high $\implies$ neg $\vec{B}_z$

- extra $\vec{B}_z$ adds to $\vec{B}_1$ resulting in slow precession around tipped axis: $\vec{B}_{eff}$ (effective)

- extra $x$-gradient can have same effect on $\Delta\omega_0$ (changes $\omega_0$ instead of changing $\omega_{RF}$)

$\vec{B}_{eff} = \left(\frac{\Delta\omega_0}{\gamma}\right)\hat{k} + B_z\hat{k} + B_x\hat{i}$

adiabatic RF pulse $\approx$ flow-driven CASL tag

RF: sweep freq $\omega_0$: constant
W0: $\omega_0$: sweeps because spins flow along gradient direction
SPECTROSCOPY + IMAGE

- **chemical shift**: small displacement resonant freq due to shielding of target nucleus (e.g., $^1H$) by surrounding electron orbitals

  - e.g., acetic acid:
    - oxygen attracts electron so less shielding of target nucleus

  - how we get chemical shift spectrum:
    - Larmor oscillations are multiplied (PSD) by center freq to obtain $\Delta f$ (not MHz high freq)
    - data before FT is a series of time-domain samples of the mix of shifted-freq offsets
    - FT turns data into "shift-spectrum"

  - Pulse Sequence:
    - since we are already using phase (freq) encoding for space, we need an "extra dimension" with all gradients OFF!
    - use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal $S$ and FT-it like chemists do!

- $\text{N.B.: opposite "direction" of FTs!}$

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
</table>
| NMR    | Oscillation $\rightarrow$ Shift freq. spectrum (ft)
| MRI    | Spatial $\rightarrow$ Spatial object (like time domain signal)
**PHASE-ENCODED STIMULUS & ANALYSIS**

- Map polar angle
- Map frequency
- Map eccentricity
- Map prox/distal axis, Road maps

**Periodic stimuli (phase-encoded)** - e.g., 8 cycles at 64 sec/cycle

**Calculate significance**
- Ratio between amplitude at stimulus frequency (=signal) and average of amplitudes at other frequencies (=noise)
- Ignore harmonics, low freq (=movement)

**Smooth**
- Vector average of complex significance \((A, \phi)\) with that at nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay correction**
- Record responses to opposite directions of stimulus (ccw/lcw, inf/out, up/down)
- Vector average after reversing angle of one
  - Penalizes inconsistent more than just avg of angles

**Typically 0.5-5% amplitude**

**Strongly periodically activated single voxel time course**

**Remove constant (avg) and linear trend**

**Real**

**Imaginary**

\(\text{FFT, convert to } A, \phi\)

**Freq = total TR's/2**

**Reversed CCW**

**CCW significance (complex)**
CONVOLUTION

\[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{+\infty} g(z) \cdot h(x-z) \, dz \]

- definition of convolution

```
\[ \text{graphically: place kernel at } x \]
\[ \text{reverse it} \]
\[ \Rightarrow \text{mult, sum} \]
\[ \text{current } x \]
```

- how to calculate one term
- sum across all \( z \) to get the value of the convolution at \( x \)
- move kernel to calculate next \( x \)

**Why we reverse**

Impulse response function (HRF)

```
\[ t \]
```

Impulses (ERP design)

```
a b c
\[ t \]
```

- impulse occurred a while ago
- small effect

- impulse occurred recently
- large effect

how to calculate convolution for this time point (only 3 terms in integral – all other zero)

N.B. cross-correlation same as convolution except no reversal \( h(x+z) \) instead of \( h(x-z) \)
**GENERAL LINEAR MODEL**

\[
\hat{y} = X\hat{h} + Sb + \hat{n}
\]

- goal is to solve for the hemodynamic response functions, \(h\)

\[
\begin{bmatrix}
\vec{y}
\end{bmatrix}_N = \begin{bmatrix}
X
\end{bmatrix}_N \begin{bmatrix}
\hat{h}
\end{bmatrix}_k + \begin{bmatrix}
S
\end{bmatrix}_N \begin{bmatrix}
b
\end{bmatrix}_N + \begin{bmatrix}
\hat{n}
\end{bmatrix}_N
\]

\(t.exp\) \rightarrow \(t.hemo\) \rightarrow \(\begin{bmatrix}
\hat{h}
\end{bmatrix}_k\)

\[
\text{factors} \rightarrow t.exp \rightarrow \begin{bmatrix}
\text{lin poly}...
\end{bmatrix}_k \rightarrow \begin{bmatrix}
\text{lin poly}...
\end{bmatrix}_N
\]

\[
\text{unknowns} \rightarrow \begin{bmatrix}
\text{exp}
\end{bmatrix}_N
\]

\(\downarrow\) multiple conditions

\[
\begin{bmatrix}
h_1 \\
h_2
\end{bmatrix}
\]

\(\text{cond}_1 \text{ occurs}\) \(\boxed{\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}}\) \(\text{cond}_2 \text{ occurs}\)

\(\text{cond}_1 \text{ re-occurs}\) \(\boxed{\begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}}\)

\(\text{cond}_2 \text{ re-occurs}\) \(\boxed{\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}}\)

\(\text{matrix notation for discrete convolution of stimulus pattern with hemodynamic resp. fun.}\)

\(\text{basic vectors of signal space}\)

\(\text{basis vectors of interference space}\)

**Maximum likelihood estimate**

1) assume white noise, solve for \(\hat{h}\)

2) \(\hat{h} = \left( X^T P_s^\perp X \right)^{-1} X^T P_s^\perp y \) where \(P_s^\perp = I - S (S^T S)^{-1} S^T\)

   or

   \(\hat{h} = \left( X_\perp^T X_\perp \right)^{-1} X_\perp^T y \) where \(X_\perp = P_s^\perp X\)

3) Significance (how to construct F-ratio) \(\rightarrow\) design matrix nuisance effects removed from cols

\[
F = \frac{N-K-L}{K} \left[ \frac{y^T (P_{XS} - P_s) y}{y^T (I - P_{XS}) y} \right] \rightarrow \text{see diagram next page for geometric interp}
\]
**Geometric Interpretation of General Linear Model**

- With no nuisance functions \( S \), we could look at **orthogonal projection** of data onto experimental design and compare that to error to determine significance.

\[
\hat{y} = X\hat{h} + \hat{e}
\]

\[
X\hat{h} = P_x \hat{y}
\]

Projection matrix, \( P_x \), operates on \( \hat{y} \) to give projection of data into experiment space, \( X \).

- When nuisance functions \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \).

\[ \Rightarrow \text{for example: linear trend not orthogonal to std. block design} \]

\[ \begin{align*}
\text{orthogonal projection} & \quad \downarrow \text{dot prod.} = 0 \\
\text{oblique projection} & \quad \Rightarrow \text{sum: not 0}
\end{align*} \]

**Geometric Picture**

\( X_S \): space of data modeled by all reference and nuisance

\( E_s \): space of data explained by reference and nuisance

\( P_{xs} \): orthogonal projection onto reference + nuisance

\( P_s \): orthogonal projection onto nuisance

\( E_y \): oblique projection onto reference

\( P_{ys} \): orthogonal projection onto nuisance

Error (S) not explained by reference and nuisance

\[ [(I - P_{xs})y] \]

\[ (P_{xs} - P_s)y \]

Data orthogonal to nuisance

Same as projection onto reference only in special case where \( S \perp X \)
SEGMENTATION & SURFACE RECON

1) MNI auto-Talairach → generates 4x4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a lot), gradient descent on xcorr
   - repeat w/ less blurring of avg target and current brain
   - problems: variable neck cut off
     -> but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
   - interpolated scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
   - find points in 5x5x5 within 10% of WM, get new scale for them
   - build Voronoi to interpolate scales over above soap bubble smooth Voronoi boundaries (3 iterations)
   - re-scale each voxel

3) Skull Stripping (output: "brain")
   - "shrink-wrap" algorithm
   - start with ellipsoidal template
   - minimize brain penetration and curvature
   - curvature: spring force
     (from center-to-neighbor vect sum)
   - brain penetration
     apply force along surface normal that prevents surface from entering gray matter
SEGMENTATION & SURFACE RECON

- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something')
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t)
\]

\[\text{for one vertex, previous local quantities}\]

\[
\mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

\[\text{id identity 3x3}\]

\[\text{vector to neighbor vertex}\]

\[\text{vector onto normal}\]

\[\text{project second onto first}\]

\[\text{expand by distribution to neighbor vertex}\]

\[\text{stronger than normal}\]

\[\text{weaker than tangential}\]

\[\text{projection of a neighbor or vertex vector onto normal, in the direction of the normal (n) is squared (as shown) so we get a vector out (not a scalar)}\]

\[\text{inferior normal component}\]

\[\text{average normal component}\]

\[\lambda_{\text{MRI}}\]

\[\text{intensity MRI data}\]

\[\mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \max \left[ 0, \tanh \left[ I(\mathbf{r}_{\text{center}} - d\mathbf{n}_{\text{center}}) - I_{\text{mesh}} \right] \right]
\]

\[\text{max force at 1.0} \quad \text{max force at 0} \quad \text{max force at 1.0} \quad \text{max force at 0}\]

\[\text{d sample points into brain along the direction of normal}\]

\[\text{d sample points into brain along the direction of normal}\]

\[\text{outside (dark)}\]

\[\text{skin (light)}\]

\[\text{skull (dark-light-dark)}\]

\[\text{GPI} \quad \text{ideal skull strip}\]

\[\text{WM} \quad \text{etc}\]

\[\text{Snapshot of surface and "core sample" from one vertex}\]
SEGMENTATION & SURFACE RECON

4) Non-isotropic filtering (output: "win")
- Preliminary hard thresholds
- Find ambiguous/boundary voxels
  \( \Rightarrow 20\% \) or more of 26 immediate neighbors different
- Find plane of least variance
  For each voxel (from icosahedral supersampling)
  Consider 5x5x5 volume around voxel
  Find plane of least variance in this hemisphere
  Medium filter with hysteresis
  \( \Rightarrow \) if 60\% of within-slab differ, reverse classification
  \( \Rightarrow \) "flosses" sulci without blurring

5) Find cutting planes

midbrain

[callosum, to separate hemispheres (SAG)]
[midbrain, to avoid fill into cerebellum (HOR)]

[Talairach to start; fill WM in SAG or HOR till min area]

6) Region-growing to define connected parts (output: "filled")
- inside-out, outside-in, inside-out -- for each hemisphere
- up/down cycles within each plane
- plane-by-plane
- "wormhole filter" (3x3x3 = center + 26)
  \( \Rightarrow \) fill (unfilled) voxel if 66\% neighbors differ -- eliminates structures within, 1-D structure
7) **Surface Tessellation** (output: rh.orig, lh.orig)

- Variable num neighbors possible!
- Quads to triangles

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  - so cross-products oriented properly

- Long list of values associated with each numbered vertex
  - position (orig, morphed)
  - area (orig, morphed)
  - Curvature (intrinsic, Gaussian)
  - Sulcussness (summed L movement during unfolding)
  - Cortical thickness
  - fMRI data
  - EEG/MEG dipole strength

- Separate fMRI data set must be aligned, sampled

  - fMRI voxels larger
  - Sample at each surface vertex
  - Nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh

- Some quantities only well-defined on surface
  - Gradient of magnitude of cortical map measure (e.g., eccentricity)

3D data set \( \rightarrow I(x,y,z) \)

2D surface

- Vertex list (coords)
  - \( x_1, y_1, z_1, x_2, y_2, z_2, \ldots \)
  - Vertex nums implicit

- Face list (vertex nums)
  - \( a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, \ldots \)
  - Face nums implicit

\[ \text{fMRI} \rightarrow \text{surface initial sample same value here} \]

\[ \text{interpolate on surface} \]
SEGMENTATION & SURFACE RECON

- smoothing/inflation/WM, pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

- scalar tangential error (fixed by redistributing vertices)
- scalar curvature error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[ J_{\text{normal}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 \]

\[ J_{\text{tangential}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 \]

\[ J_{\text{image}} = \frac{1}{2 \#\text{vert}} \sum_{\text{centers}} \left[ I_{\text{center}}^{\text{targ}} - I(r_{\text{center}}) \right]^2 \]

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction with self-interest test

\[ - \frac{\partial J}{\partial r_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{center}}^{\text{targ}} - I(r_{\text{center}}) \right] \nabla I(r_{\text{center}}) \]

\[ + \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right] n_{\text{center}} \]

\[ + \sum_{\text{neighbors}} \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right] t^x_{\text{center}} + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right] t^y_{\text{center}} \]

N.B.: eq. 9 in data, Fischl & Sereno different — and incorrect!
**SULCUS-BASED CROSS-SUB. ALIGN**

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \((S_{\text{real}} - S_{\text{target}})^2\)
- Bootstrap: morph to one brain, make avg target, remorph to avg target

---

**Smooth wm** \(\rightarrow\) **inflated** \(\rightarrow\) **sphere** \(\rightarrow\) **registered sphere**

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crunches (from average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\(\Rightarrow\) N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)