**MAGNET HARDWARE**

- $B_0$ field from superconducting magnet
- RF transmit/receive
- Gradient coils

1. $B_0$ field
   - Superconducting coils in liquid helium
   - No power required after current injected to bring up field using induction
   - $1T = 10,000$ Gauss
   - $\frac{\nu}{2\pi} = 42.6$ MHz/T

2. Body gradient coils
   - Shim coil also embedded in here (not shown)
   - Usn. water cooled

3. RF transmit body coil
   - Circularly polarized $B_1$ field
   - Rotating $\perp$ to $B_0$ at Larmor freq
   - $B_1$ is several orders of magnitude smaller than $B_0$

4. RF receive-only
   - Head coils
   - Three one million watt amplifiers to add ramps to $B_0$ field
**Spin & Precession**

- nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers)
- moving charge creates magnetic field
  - classical picture
  - current loop from spinning charge (right-hand rule)
  - N.B.: classically this would cause EM radiation, spin down
- Stern-Gerlach experiment
  - pass nuclei through strong mag. field → split into just 2 beams

**Microscopic picture**
- no strong magnetic field $B\Phi = 0$
- magnetic field, $B\Phi = $
- strong $B\Phi$ plus oscillating $B_1$

**Macroscopic picture**
- all vectors same length, random directions
- slight excess of "up" (3 ppm)
- x-y components still random
- precessing vectors are "bunched" at any one moment around circle
- bulk magnetization precesses

**Precession**
- distinguish precession (slow) from spin (fast)
- treat classically, like spinning top

$$2\pi f_0 = \omega = \gamma B_0$$
  - Larmor freq. (eg, 63 MHz)
  - $\gamma$ in radians/sec
  - gyromagnetic ratio (eg, 1.5T)

left-hand rule: thumb = $B\Phi$
  - fingers = precession
  - like top: precession faster w/more gravity

- bulk equilibrium magnetization (parallel to $B_0$)

$$M_z^0 = |\vec{M}| = \frac{\gamma^2 h^2 B_0 N_s}{4 KT_s}$$
  - $I = \pm \frac{1}{2}$
  - $I(I+1)$
  - $4K T_s$

N.B.: compared to top & gravity
- magnetic relaxation is:
  - frictionless spin
  - signed gravity
  - can change precession dir
  - can stick under floor
  - neighbor bumping causes decay ($T_2$)
**Bloch Equation**

- Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation + relaxation)

\[
\text{(laboratory frame) } \frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} = \frac{M_x \hat{t} + M_y \hat{j} + M_z \hat{k}}{T_2} - \frac{(M_z - M_0^o) \hat{k}}{T_1}
\]

- In the Larmor-rotating coordinate system, a tilt \( \phi \) a phase shift \( \theta \) and a standard \( \vec{B}_0 \) excitation is rotation around \( x \)-axis

- Longitudinal and transverse relaxations

\[
\frac{dM_z'}{dt} = - \frac{M_z - M_0^o}{T_1}
\]

\[
\frac{dM_x'y'}{dt} = - \frac{M_x'y' + [M_z'(0) - M_0^o]}{T_2}
\]

- Solution to equations above: time-dependent free precession eqs.

\[
M_z'(t) = \left[ 1 - e^{-t/T_2} \right] + M_z'(0)e^{-t/T_1}
\]

\[
M_x'(t) = M_x'(0)e^{-t/T_2}
\]

- Reprinted from, re-growing from 0, recovered after pulse-decaying

\[
M_x'(12) = 63\% M_0^o
\]

\[
M_y'(12) = 37\% M_y'(0)
\]
**BLOCH EQUATIONS - MATRIX VERSION**

**Inner Product**
\( \cdot \) (dot product)
\( \cdot \) (scaled projection only)
\( \mathbf{a} \cdot \mathbf{b} = [a_x, a_y, a_z] [b_x, b_y, b_z] = a_x b_x + a_y b_y + a_z b_z \) (scalar)

**Outer Product**
\( \times \) (cross product)
\( \mathbf{a} \times \mathbf{b} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \) (vector)

**Just Precession**
\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B}_0
\]

\[
\frac{dM_x}{dt} = \frac{dM_y}{dt} = 0
\]
\[
\frac{dM_z}{dt} = -\gamma B_0 M_z
\]

**Solution:**
\[
\begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} =
\begin{bmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
M_x^0 \\
M_y^0 \\
M_z^0
\end{bmatrix} = R_z(\omega t) \mathbf{M}^0
\]

### Including Relaxation

\[
\frac{d\mathbf{M}}{dt} =
\begin{bmatrix}
-1/\tau_1 & -\gamma B_0 & 0 \\
-\gamma B_0 & -1/\tau_1 & 0 \\
0 & 0 & -1/\tau_1
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
M_0/\tau_1
\end{bmatrix}
\]

**Solution:**
\[
\begin{bmatrix}
M_x(t) \\
M_y(t) \\
M_z(t)
\end{bmatrix} =
\begin{bmatrix}
e^{-t/\tau_1} & 0 & 0 \\
0 & e^{-t/\tau_1} & 0 \\
0 & 0 & e^{-t/\tau_1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{R}_z(\omega t) \mathbf{M}^0 \\
0 \\
M_0(1-e^{-t/\tau_1})
\end{bmatrix}
\]
RF Field Polarization

- **polarization** (i.e., direction) of magnetic (not electric) field

- **linearly polarized field**

  \[ \vec{B}_{lin} = B_{lin} \cos(\omega t) \hat{x} \]

  - magnitude
  - phase angle
  - unit vector

  \[ \text{e.g., for } \vec{y} : \hat{y} \rightarrow \hat{t} \]

- **circularly polarized field** (quadrature)

  \[ \vec{B}_{cir} = \vec{B}_{1} (\hat{x} \cos(\omega t) - \hat{y} \sin(\omega t)) \]

  - made by adding two perpendicular linearly polarized fields
  - same freq/amp, 90° out of phase

  \[ \text{left-circularly polarized field} \]

- **rotates around } \vec{B}_{0} \text{ directions**

  \[ \vec{B}_{0} + \vec{B}_{1} \]

- **flipping around** X-axis versus

  - flipping around Y-axis is just a change in phase of RF stim

  \[ \text{across entire brain} \]

- **90° flip** (around X-axis)

- **180° flip** (around Y-axis)

  \[ \text{Typical} \]

  \[ \text{Typical} \]

  \[ \text{Typical} \]
**Signal Equation**

\[ \Phi(t) = \int_{\text{obj}} \mathbf{B}(r) \cdot \mathbf{M}(r,t) \, dr \]

- Magnetic flux through coil
- For a particular instant in time vs. Bloch M(t)
  which is change of M with time

\[ V(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int_{\text{obj}} \mathbf{B}(r) \cdot \mathbf{M}(r,t) \, dr \]

- Faraday law of induction
- Evaluate using free precession eq. (solution to Bloch)
- Neglect relaxation if complex notation of time-dependence from lab frame Bloch
- Ignore the change in the z-component of the magnetization
  Since it changes so slowly compared to the free precession of x- and y-components: \( \omega(r) \gg |f_1(r) | \)
- This is why we can only record transverse magnetization, \( M_{xy} \), but not longitudinal magnetization. (\( M_z \) changes too slowly, so \( V(t) = \frac{\partial \Phi}{\partial t} \approx 0 \))

\[ S(t) = \int_{\text{obj}} B_{xy}(r) M_{xy}(r,0) e^{-i \omega(r)t} \, dr \]

- Spatially dependent resonant freq in rotating frame - i.e. after subtraction of \( \omega_0 = \gamma B_0 \)
- Lab frame transverse magnetization at time \( t = 0 \)
- \( M_{xy} \) is new scaling direction in new frame
- Assume homogenes (ignore)
- Get from Y0

\[ S(t) = \int_{\text{obj}} M_{xy}(r,0) e^{-i \omega(t)\frac{\partial t}{\partial t}} \, dr \]

- Phase angle in rotating frame
- \( \omega = \text{radians/sec} = \text{radians} \cdot \frac{\text{sec}}{\text{sec}} \)
- Getting difference converts \( \text{lab} \rightarrow \text{rotating} \)
**Phase-Sensitive Detection**

how we get rotating frame

- Method for moving very high frequency Larmor oscillations down to tractable frequency range

- \( V(t) \approx 63 \text{ MHz} \)

- \( S(t) \approx 50 \text{ kHz} \)

- Reference signal (63 MHz)

- Demodulated signal \( \propto \) RF coil signal \( \propto \) reference (transmitter)

- \( \propto \sin(w_0 + \delta w)t \cdot \sin w_0 t \)

- \( \propto \frac{1}{2} \left[ \cos \delta w t - \cos (2w_0 + \delta w)t \right] \)

- This signal is digitized

-one freq - freq domain

-chirp - time domain

- Two signals are made from a single receiving RF coil

- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{\pi}{2} \) phase, then PSD)

- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \sqrt{2} \) better)
**FID** - free induction decay, $T_2^*$

- **FID (free-induction decay)** from an RF pulse w/ angle $\alpha$
  
  \[ S(t) = \sin \alpha \int_{-\infty}^{\infty} \rho(w) e^{-t/T_2(w)} e^{-i\omega t} dw \]

  - recorded signal (complex), not tip
  - spectral density function
  - frequency dependent
  - time dependent
  - intrinsic decay (spin-spin)

- For a single freq:
  
  \[ S(t) = M_z \sin \alpha e^{-t/T_2^*} e^{-i\omega t} \]

- Max FID amplitude at $t = 0$: $S(0) = M_z \sin \alpha$

- If field inhomogeneous (Lorentzian distribution)
  
  \[ S(t) = \pi M_z \Delta B_0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t} \]

  - where
    \[ \frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2^*} \]

  - $M_z$ is constant

- Spectral density for "Lorentzian" mag. field inhomogen.:
  
  \[ \rho(w) = \frac{M_z^2}{\Delta B_0^2 + (w - \omega_0)^2} \]

  - $\omega_0$ is bulk flip angle
  - $\Delta B_0$ is exponential fall off

  - In other dist, assume $T_2^*$ is exp. approx.
  - e.g., see bottom

**Basic FID envelope is proportional to**

\[ e^{-t/T_2^*} \]

*Both of these would be approximated by $T_2^*$

- Suggestive: actually a million cycles here
- Spin-spin fluctuations fast, not fixed w/ time
- These $B_0$ differ, so echo can cancel
- Decay rate $1/T_2$ is increased
- Typical time course 10's of nsec

- Continuum of frequencies (not lorentzian)
ECHOES - spin echo

Just after 90° x' pulse f_{e_0} + f_{hi} have same phase

Relaxation + phase dispersion of f_{e_0} + f_{hi}
(both from B>B_0)

Just after 180° y' pulse
(y' pulse like x' pulse but RF has +90° phase)

Echo caused by re-phasing of f_{e_0} + f_{hi}
(w/ decay due to T2)

- Remember [RF just tips vector(s) while retaining length]
  relaxation includes tips and shrinks (and grows for echo)

- 180° x' pulse works, too, but echo will be +π phase (left side in figs above)
- Echoes generated even if second pulse not 180° (see next)

- FID decay (and echo growth/decay)
described by T2*
from inhomogeneity

- Reduction in height of echo
compared to initial
described by T2,
  echo fixes the "star"
**ECHOES — spin echo**

- $\alpha_1 - \gamma - \alpha_2$ (both pulses along $y'$, for simplicity)

**effect of $\alpha$-pulse**

$M_x' \rightarrow M_x, \cos \alpha - M_z', \sin \alpha$

$M_y' \rightarrow M_y'$

$M_z' \rightarrow M_x', \sin \alpha - M_z', \cos \alpha$

**effect of $\gamma$ delay**

$M_x' \rightarrow (M_x \cos \omega \tau + M_y \sin \omega \tau) e^{-\gamma \tau / 2}$

$M_y' \rightarrow (-M_x \sin \omega \tau + M_y \cos \omega \tau) e^{-\gamma \tau / 2}$

$M_z' \rightarrow M_z (1 - e^{-\gamma \tau / T_2}) + M_z e^{-\gamma \tau / T_1}$

**immediately after $\alpha$ pulse**

$M_x' (w, 0) = -M_z (w) \sin \alpha$

$M_y' (w, 0) = 0$

$M_z' (w, 0) = M_z (w) \cos \alpha$

**after $\gamma$ delay**

$M_x' (w, \tau) = -M_z (w) \sin \alpha, \cos \omega \tau e^{-\gamma \tau / 2}$

$M_y' (w, \tau) = M_z (w) \sin \alpha, \sin \omega \tau e^{-\gamma \tau / 2}$

$M_z' (w, \tau) = M_z (w) \left[1 - (1 - \cos \alpha \gamma) e^{-\gamma \tau / T_1}\right] \sin \alpha$

**immediately after $\alpha_2$ pulse** (no effect on $M_y'$, rewrite $y'$; combine $x$ and $y$ eqs)

$M_x' (w, \tau) = M_z (w) \sin \alpha \left(\sin^2 \frac{x_2}{2} e^{i \omega \tau} - \cos^2 \frac{x_2}{2} e^{i \omega \tau}\right) e^{-\gamma \tau / 2}$

$-M_z (w) \left[1 - (1 - \cos \alpha \gamma) e^{-\gamma \tau / T_1}\right] \sin \alpha$

**time dependent free precession around $z'$ (rewrite $M_x' (w, \tau)$)**

$M_x' (w, \tau) = M_z (w) \sin \alpha, \sin^2 \frac{x_2}{2} e^{-\gamma \tau / 2} - \sin \omega \tau e^{-\gamma \tau / 2}$

$-M_z (w) \sin \alpha, \cos^2 \frac{x_2}{2} e^{-\gamma \tau / 2} - \sin \omega \tau e^{-\gamma \tau / 2}$

- For a large num of freq's:

$$M_x' (w, \tau) = M_z (w) \left[1 - (1 - \cos \alpha \gamma) e^{-\gamma \tau / T_1}\right] \sin \alpha \left(\sin^2 \frac{x_2}{2} e^{i \omega \tau} - \cos^2 \frac{x_2}{2} e^{i \omega \tau}\right) e^{-\gamma \tau / 2}$$

- For an echo, $t = 2\tau$:

$$S(t) = \sin \alpha, \sin^2 \frac{x_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t / T_2} e^{-i \omega (t - \tau)} dw$$

$$A_E = \sin \alpha, \sin^2 \frac{x_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T(t - \tau) / T_2} dw$$

- For $\alpha_1 = 90^\circ$, $\alpha_2 = 90^\circ$

$$S_1 (t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t / T_2} e^{-i \omega (t - \tau)} dw$$

- For $\alpha_1 = 90^\circ$, $\alpha_2 = 180^\circ$

$$S_2 (t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t / T_2} e^{-i \omega (t - \tau)} dw$$

- Multiply by $i$ (add $\pi / 2$ phase)

- $A_E$ etc.
Echo TRAINS — spin-echo trains

- it's (too) easy to make echoes...

\[ E_n = \frac{3^{(n-1)} - 1}{2} \]

2 echoes after end of nth pulse
3 RFs \(\rightarrow\) 4 echoes (here)
6 RFs \(\rightarrow\) 121 echoes (!)

\[ \alpha_1, \alpha_2 \quad FID_1, FID_2 \]

Secondary echo: \([SE_{1,2} \text{ acts like RF pulse} \]
\[ \alpha_3 \text{ makes an echo from it} \]

\[ SE_{1,2}, SE_{1,3} \]

- two more conventional two pulse spin echoes

Stimulated echo: combined effect of 3
\[ \alpha_1: M_L \rightarrow M_T \]
\[ \alpha_2: \text{leftover } M_T \text{ flipped to } M_L \text{ (saved)} \]
\[ \alpha_3: \text{ flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays} \]

(\text{after being held in limbo between 180° FID}_2 \text{ and } FID_3); acts like 2-pulse echo

- a useful multi-echo sequence (CPMG) is a
90° followed by 180° at 2\(\pi\) spacing

\[ e^{-t/\tau_1} \]

- typically, 90° and 180° applied in different axes (\(\chi, \theta, \eta, \gamma, \zeta, \ldots\)
which reduces phase errors due to imperfect 180° pulses
(since slightly-off rotation around \(\gamma\) affects phase less)
**Gradient Echoes**

- Initial negative gradient dephases spins.
- After $t = T$ of positive gradient, spins rephase.
- Does not correct for $T_2^*$ inhomogeneities. So echo amplitude is $A_E = e^{-t/T_2^*}$.
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay.

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not corrected. Hence, echoes are $T_2^*$-weighted, not $T_2$-weighted, more susceptible to inhomogeneities.

- Echo trains possible w/ gradient echo (CPMG-like).

- Faster the gradients are switched, more echoes you get.

- EPI hardware => 64 echoes.
**IMAGE CONTRAST**

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

  ![](image)

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[M_z\] before first pulse = \(M_z^0\)
  \[M_z = 0\] immed. after first pulse (i.e., 90° pulse)

- From Bloch eq, \(M_z\) just before second pulse:
  \[M_z^{(n)}(O_2) = \frac{M_z^0}{M_z \text{ before current pulse}} \left(1 - e^{-\frac{2\pi}{TR/T1}}\right) + \frac{M_z^{(o)}(O_+)}{M_z \text{ regrowth-from-zero term}} \left(1 + e^{-\frac{2\pi}{TR/T1}}\right) + \frac{M_z^{(o)\prime}(O_+)}{M_z \text{ left-imoed.-after-pulse term (i.e., decaying)}}\]

- Given \([1] 90° pulse \[2] \text{ no } M_{xy} \text{ left}\) → Pure tip: \(M_{xy} = M_z\)

- Tip existing mag
  \[M_z^{(n)}(O_2) = M_{xy}'(O_+) = M_z^0 \left(1 - e^{-\frac{2\pi}{TR/T1}}\right)\]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[I(r) = \mathcal{C} \rho(r) \left(1 - e^{-\frac{2\pi}{TR/T1}(r)}\right)\]

\(\mathcal{C}\): recov const.; spectral dens \(\rho(r)\)

Assume immediately recover of signal
**IMAGE CONTRAST**

IR (still just saturation-recovery — no echo)

- Inversion recovery w/ no echo

![RF diagram with TI, TD, TR, and M\textsubscript{z}](image)

"steady state" after here

- 180° deg pulse reverses longitudinal magnetization

\[ M_{z}' = -M_{z} \]

- Recovery to end of first TI from long. part of Bloch eq.

\[ M_{z}' = M_{z} \left( 1 - 2e^{-t_{1}/T_{1}} \right) \]

\[ \text{flipped into transverse by second pulse (180° 90°)} \]

- Longitudinal then regrows from zero from first Bloch term only

\[ M_{z}' = M_{z} \left( 1 - e^{-\left(\text{TR} - \text{TI}\right)/T_{1}} \right) \]

- After second 180°, just change sign again

\[ M_{z}' = -M_{z} \left( 1 - e^{-\left(\text{TR} - \text{TI}\right)/T_{1}} \right) \]

- Apply relaxation eq. again

\[ M_{z}' = M_{z} \left( 1 - e^{-\text{TI}/T_{1}} \right) - M_{z} \left( 1 - e^{-\left(\text{TR} - \text{TI}\right)/T_{1}} \right) e^{-\text{TI}/T_{1}} \]

\[ M_{z}' = M_{z} \left( 1 - 2e^{-\text{TI}/T_{1}} + e^{-\text{TR}/T_{1}} \right) \]

\[ \Rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
- Steady state mag (2nd TR) just before 90°:
  \[ M_z'(0^-) = M_0^z \left( 1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}} \right) \]

- The echo signal (\( M_z \)) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation:
  \[ A_E = M_0^z \left( 1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{TR}{T2}} \right) e^{-\frac{TE}{T2}} \]

- If we assume \( TE \) much less than \( TR \), then we can simplify:
  \[ A_E = M_0^z \left( 1 - e^{-\frac{TR}{T1}} \right) e^{-\frac{TE}{T2}} \]

- Similar equation for SE-1R:
  \[ A_E = M_0^z \left( 1 - 2e^{-\frac{TR}{T1}} + e^{-\frac{TR}{T2}} \right) e^{-\frac{TE}{T2}} \]
**IMAGE CONTRAST**  GRE w/ small tip angle

- Use basic longitudinal relaxation from Block 1, again
  \[ M_z^{(n)}(O_0) = 0 \]  transverse dephased before next pulse
  \[ M_z^{(n)}(O_0) = M_z^0 (1 - e^{-TR/T1}) + M_z^{(n-1)}(O_0) e^{-TR/T1} \]

- Assume we have a small tip angle:
  \[ M_z^0 \cos \alpha \Rightarrow M_z^{(n)}(O_+ \cos \alpha) = M_z^{(n)}(O_-) \cos \alpha \]
  \[ M_z^{(n)}(O_-) = M_z^0 (1 - e^{-TR/T1}) + M_z^{(n-1)}(O_-) \cos \alpha e^{-TR/T1} \]

- Assume we are in dynamic equilibrium:
  \[ M_z^{(n)}(O_-) = M_z^{(n-1)}(O_-) = M_z^{ss}(O_-) \]

**Pre-pulse**

- Steady state longitudinal:
  \[ M_z^{ss}(O_-) = \frac{M_z^0 (1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}} \]

**Post-pulse**

- Transverse magnetization:
  \[ M_z^{ss}(t) = \frac{M_z^0 (1 - e^{-TR/T1}) \cdot \sin \alpha e^{-t/T2*}}{1 - \cos \alpha e^{-TR/T1}} \]

**Gradient echo amplitude**

\[ A_E = \frac{M_z^0 (1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2*}}{1 - \cos \alpha e^{-TR/T1}} \]

T1 contrast mainly depends on flip angle, not TR \[ \cos 90^\circ = 1 \]  eliminates T1 weight since denom = numer
- Saturate, wait for contrast \(_1\), invert, wait for contrast \(_2\), FLASH (continued)

A) \( M_z' (\text{just after } 90^\circ) = 0 \) (perfect 90°)

B) \( M_z' (\text{after } TD) = M_z^0 (1 - e^{-TD/T_z}) \) (Bloch term 41)

C) \( M_z' (\text{just after invert}) = \cos \phi M_z^0 (1 - e^{-TD/T_z}) \)

D) \( M_z' (\text{after } TI) = M_z^0 (1 - e^{-TI/T_z}) + \left[ \cos \phi M_z^0 (1 - e^{-TD/T_z}) \right] e^{-TI/T_z} \)

Special case \( TI = TD \): \( M_z^0 \left[ 1 - e^{-TI/T_z} \right]^2 \)

- after the first \( \alpha \) pulse:

E) \( M_z' (\text{just after } \text{first } \alpha) = M_z^0 [1 - \cos \phi (1 - e^{-TD/T_z}) e^{-TI/T_z}] \sin \alpha \)
**CONTRAST-TO-NOISE**

- "contrast" defined as difference: \( C_{AB} = S_A - S_B \)
- contrast-to-noise ratio:
  \[
  \text{CNR}_{AB} = \frac{C_{AB}}{\sigma_n} = \frac{S_A - S_B}{\sigma_n} = \text{SNR}_A - \text{SNR}_B
  \]

- **spin-echo**: \( A_e = M_0^* (1 - e^{-TR/TE}) e^{-TE/2} \)

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<thead>
<tr>
<th>Tissue</th>
<th>( T_1(15T) )</th>
<th>( T_2 )</th>
<th>PD</th>
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<tbody>
<tr>
<td>GM</td>
<td>950</td>
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<th>Lauterbur. Tissue</th>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>760</td>
<td>77</td>
<td>0.69</td>
</tr>
<tr>
<td>WM</td>
<td>510</td>
<td>67</td>
<td>0.61</td>
</tr>
<tr>
<td>CSF</td>
<td>2650</td>
<td>280</td>
<td>1.00</td>
</tr>
</tbody>
</table>

---

- **gradient echo**: \( A_e = M_0^* \left( \frac{1 - e^{-TR/TE}}{1 - \cos \alpha e^{-TR/TE}} \right) \sin \alpha e^{-TE/T2^*} \)

---

- General rules: spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Proton density weighted</th>
<th>( TR \uparrow ) (no T1 diffs)</th>
<th>( TE \downarrow ) (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>( TR \approx T_1 ) (big T1 diffs)</td>
<td>( TE \downarrow ) (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>( TR \uparrow ) (no T1 diffs)</td>
<td>( T2 \approx T_2 ) (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{N_{x2} N_{y2} N_{z2} \Delta t}
\]

\[
\begin{array}{ll}
\text{voxel size} & \text{(of same number)} \\
\text{num repeats} & \text{number of voxels (of same size)} \\
\text{read timestep} & \\
\end{array}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[
  \rightarrow \text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}
  \]

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(T_n\) effect on S/N
  \[
  \rightarrow \text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{128 \times 128}{64 \times 64} = 2 \text{ times better S/N}
  \]

- # acquisitions \(T_n\) better S/N
  \[
  \rightarrow \text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{T_2}{T_1} = 1.41 \text{ times better S/N}
  \]

- Longer timestep during readout, \(\sqrt{\Delta t}\) better S/N
  \[
  \Delta t = \frac{1}{\text{BW}_{\text{read}}}, \quad \text{digitization timestep during echo acquisition}
  \]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog lowpass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- Must filter out freq's > \(f_{\text{max}} = \frac{1}{2 \Delta t}\) because they alias
**Complex Algebra**

- Real/Imaginary
  - Add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - Multi: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

- Amp/Phase
  - Add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - Multi: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)
  - Complex to Real Power: \((A, \phi)^n = (A^n, n\phi)\)

- Real to Complex Power
  - \(e^{j\phi} = \cos \phi + j \sin \phi\)
  - \(= \cos \phi, \sin \phi\)
  - Vector on Unit Circle
  - \(e^{j\phi} = (\cos \phi + j \sin \phi)^n\)
  - \(= \cos n\phi + j \sin n\phi\)

**Fourier Transform**

\[
H(f) = \int h(t) e^{-j2\pi ft} dt
\]

\[
F[g(x) \cdot h(x)] = G(k) \ast H(k)
\]

**Convolution**

\[
f(x) = g(x) \ast h(x) = \int_{-\infty}^{\infty} g(x) \cdot h(x-z) dz
\]

\[
\text{Cross-correlation}
\]

\[
G(x) = g(x) \otimes h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x+z) dz
\]

- Shorthand for a unit vector \((\hat{e})\) pointing in the direction of \(\phi\)
- For arbitrary amplitude, multiply \(A e^{j\phi}\)
- Phase is integral of freq variable \(\phi = \int \omega dt\)

- Change in Phase is freq

\[
\frac{d\phi}{dt} = \omega
\]

- because of FFT, faster than convolution for small kernel

**Convolution Theorem**

\[
\text{F} [g(x) \cdot h(x)] = G(k) \ast H(k)
\]

- the Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function

- doesn't confuse freq, angle

- makes freq is linear x time

- angle is linear!
Fourier transform (1)

$$ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi}{T} t} dt $$

- How to calculate $H(f)$ for one $f$ ($f=3$):

- **Real signal**
  - $h(t)$
  - $t \rightarrow$

- **Imaginary signal**
  - $e^{-i\omega t}$
  - $t \rightarrow$

- **Cos**
  - $\cos(2\pi ft)$
  - $t \rightarrow$

- **Sin**
  - $\sin(2\pi ft)$
  - $t \rightarrow$

- **Complex multiply**

- **Integrate/sum**

- **3**

- **Frequency domain**
  - $3 \rightarrow f$

- **Cartesian ($r,i$)**
  - Polar coords ($A,\phi$)

- **Amplitude**
  - $r$ $\rightarrow f$

- **Phase**
  - $\phi$ $\rightarrow f$

- **Like correlating with sin and cos (at each freq) so we get phase (at each freq)**
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos (-\phi) + i \sin (-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \cos \phi \text{ is an "even" function, } \sin \phi \text{ is an "odd" function}

\[ \cos(x) \quad \downarrow \text{equals} \quad \cos(-x) \]
\[ \quad \downarrow \text{flips} \quad -\cos(x) \]

\[ \sin(x) \quad \downarrow \text{flips} \quad -\sin(x) \]

An orthogonal decomposition

- think of discretely sampled \( \sin(bx), \cos(bx) \) as vectors
- \( \text{Corr}(\mathbf{v_1}, \mathbf{v_2}) \equiv \text{projection of } \mathbf{v_1} \text{ onto } \mathbf{v_2} \equiv \mathbf{v_1} \cdot \mathbf{v_2} \)

\[
\begin{align*}
\text{Corr}(\cos(b_1x, \sin(b_1x)) &= 0 \\
&= \sin \& \cos \text{ of same frequency are orthogonal} \\
\text{Corr}(\sin(b_1x, \sin(b_2x)) &= 0 \\
&= \text{different integer freqs of } \sin \text{ or } \cos \text{ are orthogonal} \\
\text{Corr}(\cos(b_1x, \sin(b_2x)) &= 0 \\
&= \text{as above}
\end{align*}
\]

- In the continuous case, orthogonal functions defined as:

\[ \int_{x=0}^{x=h} f(x) g(x) \, dx = 0 \]
Fourier Transform of an Image (2)

1. Real image → imaginary image
   - (zero) → (zero)

2. Fourier Transform → Inverse Fourier Transform
   - Amplitude image → Phase image
   - Amplitude spat. freq. → Phase spat. freq.
   - View complex vectors directly

3. Complex vectors
   - Zero vectors

- 3 equivalent representations of image & spat. freq. space
FOURIER TRANSFORM OF AN IMAGE (3)

- what a single k-space point looks like in image space (polar coordinates $A, \phi$ instead of $r, \theta$)

- Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin $\cdot$ cos — don't confuse $k_x, k_y$ w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
Fourier Transform of Image (4)

- 3 equivalent representations of one k-space point in image space

**Image Space**

**Real/Imag.**

**Amplitude/Phase**

Arrows

The arrow length (const) is k-space amplitude

The final summed imaginary part of inverse FT of a real-valued image should sum to zero

\[ \Sigma i = 0 \]

\[ \Sigma r = \text{real image amp} \]

- If all we want in image space reconstruction is real component, can just add real components of complex vectors at each image space point for every image corresponding to each k-space pit.

- The k-space phase will offset phase of real-valued image space cosine/sine

- Therefore, for real-valued image, can visualize inverse FT as real-valued sum of real-valued cosine/sines

- N.B.: can not do this w/MRI k-space data since phase errors (e.g., multiple wraps) will mess up sum of real component \( \rightarrow \) must use amplitude image

**Same**

- \( k_x = 2 \)
- \( k_y = 0 \)
GRADIENT COILS

- gradient coils for $x, y, z$ generate approximately * a linear gradient in the strength of the $z$-component of the magnetic field $B_z$. 

- For example, the $x$ gradient coil induces a ramp in $z$-component of the magnetic field when moving in the $x$-direction:

$$B_{g,z} = G_x x$$

* Since a pure linear gradient of $B_{g,z}$ in only the $x, y, z$ directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the $x$- and $y$-direction ($B_{g,x}$ and $B_{g,y}$).

- The other magnetic field components are usually ignored because they are so small relative to $B_{g,z}$. Since $B_{g,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{g,x}$, $B_{g,y}$, and $B_{g,z}$.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms $B_{g,x}$ and $B_{g,y}$ are known; they can be included in the reconstruction process.
**SLICE SELECTION (G_2)**

- Slice select gradient on during RF stim

\[ B_z = f \left( \frac{x}{2\pi}, G_{z,2} \right) \]

- Protons here can only be excited by a narrow band of radio frequencies

- To apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

  \[ \text{in practice, Gaussian pulse envelope good too} \]

- This excites protons in a narrow slab

- Since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing \( z \)-gradient

  \[ \text{RF} \quad \text{slice select} \quad \text{equal area} \]

  \[ \text{G}_2 \quad \text{re-phasing} \]

  \[ \text{approximation from assuming} \quad \text{tip occurs instantaneously in middle} \]

  \[ \text{valid for small tip: } 90^\circ \rightarrow 52^\circ \]

  \[ \text{in practice: adjust to max, use} \quad \text{crusher to kill spurious transverse on} \ 180^\circ \]
PULSES FOR SLICE SELECTION

Fourier approach details

- Fourier transform approach to slice-selective pulse (linear approx. even though tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} df \]

- Time dependent
- RF stimulation (complex)

Solve with:

\[ p(f) = \text{frequency band} \]

\[ B_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) e^{-i2\pi f_0 t} \]

- Amplitude controlling flip angle
- Frequency width (controls slice width)
- Sinc envelope determined by freq. width \( f_w \)
- Modulation (complex) at center freq. \( f_c \)

- Larger oscillations at center freq.

\[ \text{sinc} \text{ envelope width inversely proportional to } f_w \]

Fourier Transform Pair, Rules

- Convolution in one domain is multiplication in the other
- Convolution with delta function impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{\pi}{6} \)
FREQUENCY ENCODING (1)

- frequency encode gradient \( G_x \) causes precession rates to vary linearly in \( x \)-direction

\( \uparrow \text{precession rate} \quad \uparrow \text{frequency} \quad \uparrow B_z \quad \text{in} \quad x \)-direction

\[ x \rightarrow \]

\[ \Rightarrow \text{correct} \quad (\text{remember that strength of } G_x \text{ causes variation of slope of } B_z \text{ in } x \)-direction) \]

- different frequency signals are mixed together and recorded as a 1-D signal over time

\[ \Rightarrow \text{correct, but remember, we are recording the summed phase angle after de-modulation} \]

- a Fourier transform, which converts back and forth between \( x \)-position \( (\text{or time}) \) and spatial frequency \( (\text{or temporal freq}) \) is done on signal

\[ \Rightarrow \text{correct} \]

- spatial frequencies get confused/conflicted with precession frequencies

\[ \Rightarrow \text{wrong}!! \]

- therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[ \Rightarrow \text{wrong}!! \quad \rightarrow \text{it actually converts spatial frequencies to spatial position} \]

\[ \rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \rightarrow \text{the precession freq ramp is constant each timestep} \]
FREQUENCY ENCODING (2) - correct intuition - why phase critical

- "frequency"-encode gradient ($G_x$) turned on during
  during echo causes precession rates
to immediately vary with $x$-position

- at beginning of gradient on, the phase of
  signal coming from each $x$-position is the same

  Summed phase angle is what we measure

- early after gradient on, phase advances (because
  of faster precession frequency) unlike with greatest
  phase advance at largest $x$-position

  ![Diagram showing phase advance]

  One cycle of
  spatial frequency
  of phase angle
  ($= \text{low spatial freq}$)

- later during gradient on, phase advances cause
  multiple wraps around of phase angle across space

  ![Diagram showing multiple wraps]

  Multiple cycles of
  spatial frequency
  of phase angle
  ($= \text{hi spatial freq}$)

- in practice, the lowest spatial frequency ($= 0$)
  occurs in the middle of the gradient on time
  because the phase is "wound" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

  ![Diagram showing spatial frequency]

  6 is spatial frequency
  $G_x$
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (temporal freq $\leftrightarrow$ time)

$$ H(\delta) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i2\pi\delta t} \, dt $$

$k$ is often used instead of $\delta$ for the frequency variable

Imaging equation: (spatial freq $\leftrightarrow$ space)

$$ S(\delta) = \int_{-\infty}^{\infty} I(x) \cdot e^{-i2\pi\delta x} \, dx $$

Sum across $x$ of object

Signal strength at one $x$-position (brightness of image point)

Spin density (spectral density)

Oscillations come from readout phase wrapping, where $f$ is single spatial freq (e.g., 5) and $x$ goes across object

$G_x$ RF

Even though variable is $\delta$, it represents one time point during readout

To make image, do inverse Fourier transform of recorded signal $S(\delta)$

don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each $x$ position)
**ALTERNATE DERIVATION (incl. effects of G_x) SIGNAL EQ**

- Oscillators at \( w = \gamma B \) at each position (just \( x \) for now)

\[
S(t) = m(x) e^{-i \phi(x)} dx
\]

- By definition, freq, \( w \) is rate of change of phase, \( \phi \)

\[
\frac{d\phi(x,t)}{dt} = w(x,t) = \gamma B(x,t) \quad \text{and} \quad \phi(x,t) = \int_{0}^{t} w(x,t) dt = \int_{0}^{t} B(x,t) dt
\]

- Assuming phase initially 0, \( B \) affected by gradients

\[
B(x,t) = B_0 + G_x(t) x
\]

\[
\phi(x,t) = \gamma \int_{0}^{t} B_0 dt + \left[ \gamma \int_{0}^{t} G_x(t) dt \right] x
\]

\[
= \omega_0 t + 2\pi K_x(t) x
\]

- Demodulation removes the \( B_0 \)-caused carrier frequency \( e^{-i \omega_0 t} \) from the first equation

\[
S(t) = \int_{x} m(x) e^{-i 2\pi K_x(t) x} dx
\]

- Amplitude of each oscillator gradient-controlled phase
- turn on gradient after excitation but before readout
- different levels of \( G_y \)
- higher levels of \( G_y \) (slope of \( B_z \) in y-direction)
  \( \Rightarrow \) higher spatial freq. (more phase wraps) in y-direction
- phase wraps persist after phase-encode gradient off
- read-out gradient (\( G_x \)) phase wraps then add to phase-encode phase

**2D Imaging Equation**

\[
S(k_x, k_y) = \int \int I(x, y) e^{-j2\pi(k_x x + k_y y)} \, dx \, dy
\]

- signal recorded at single time point (one readout point)
- complex signal (from phase-sensitive detection)
- sum across x-y plane
- done by RF coil
- image (strength of magnetization at each x-y point)
- scalar (what we try to reconstruct)
- phase (vector of unit length and particular angle which is function of \( G_x \) and \( G_y \))
- phase angle (of scalar magnetization) in rotating frame, set by gradients

- ignoring relaxation, spatial frequency \( k_x \) and \( k_y \) have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING — two phase-encode gradients

- use 2-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)

- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

\[ S(k_x, k_y, k_z) = \square \quad I(x, y, z) = e^{-j2\pi(k_x x + k_y y + k_z z)} \, dx \, dy \, dz \]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)

- SNR much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

\[ \text{phase stripes created throughout volume vs. slice:} \]

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

- Since the phase-encoded gradient and the freq. encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since $\phi$ from 2-comp quadrature!)

Stripes here represent complex value

Phase of whole image summed to one (complex) number by RF coils

[more rotation higher spatial freq.]

- Successive read-out steps:

  - Small phase encode $G_y$
  - Large phase encode $G_y$

- 3D phase encode w/ $G_y$ and $G_z$ starts rotated in y-z plane

  - Large phase encode $G_z$
  - Small phase encode $G_y$
**GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT**

- K-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ K = \gamma \int_{\text{record time}} G(t) \, dt \]

spatial freq.
recorded at
\( t = \text{record time} \)

- all of the following gradients end up at the same point in k-space:

- Frequency-encode field

  \[ RF \rightarrow 90^\circ \rightarrow G_x \rightarrow \text{samples} \rightarrow K_y \rightarrow K_x \]

- Frequency-encode gradient echo

  \[ RF \rightarrow 180^\circ \rightarrow G_x \rightarrow \text{samples} \rightarrow K_y \rightarrow K_x \]

- Frequency-encode spin-echo (plus gradient echo !!)

  \[ RF \rightarrow 90^\circ \rightarrow G_x \rightarrow \text{samples} \rightarrow K_y \rightarrow K_x \]

- Phase-encode then frequency encode gradient echo

  \[ RF \rightarrow 90^\circ \rightarrow G_y \rightarrow G_x \rightarrow K_y \rightarrow K_x \]

The figure illustrates the movement of k-space points due to gradient applications, with specific examples of how different types of gradients affect the k-space locations. The diagrams show the progression of k-space vectors under various gradient and RF pulse sequences.
**Image Reconstruction**

\[
S(k_x, k_y) = \int \int \int I(x, y) \, dk_x \, dk_y \, e^{-i 2\pi (k_x x + k_y y)}
\]

\[
I(x, y) = \int \int S(k_x, k_y) \, dk_x \, dk_y \, e^{i 2\pi (k_x x + k_y y)}
\]

Adding exponents same as multiplying two \( e^{i 2\pi k_x} \)’s

Same as two sequential 1D FFTs (actual code)

- In practice, finite number of samples, \( N \) and \( M \), are collected
  - \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \rightarrow \) discrete sum)

\[
I(x, y) = \sum_{m=-M/2}^{M/2-1} \sum_{n=-N/2}^{N/2-1} S(n,m) \, e^{i 2\pi n \frac{\Delta k_x}{\Delta k_x} \frac{x}{\Delta k_x}} \, e^{i 2\pi m \frac{\Delta k_y}{\Delta k_y} \frac{y}{\Delta k_y}}
\]
**Sampling**

- Consider effects of limited points in k-space
  - Limited in range of frequencies sampled (k_min → k_max)
  - Limited in rate of sampling (Δk)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

**Space**

- Correct reconstruction

**Spatial Frequency**

- Infinite frequency range
  - Infinitely fine sampling

- Finite frequency range
  - Finite spacing of samples

- Aliasing occurs in spatial domain
  - Replicas overlap, causing wraparound

- Finite frequency range
  - Top-wide spacing of samples

Thus, finer sampling of same range of spat. freqs increases FOV
UNDER/OVER SAMPLE

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ \delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

More examples

FOM (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

3 more examples (not incl. less samples to same spat. freq. [bottom last page])

Basic Image

Spatial freq. K-space

Some num samp. to 2X spat. freq. (i.e., gradients stronger or time ON longer)

2X num. samples to same spat. freq. (i.e., gradients weaker or time ON shorter)

2X number samples to 2X spat. freq. (i.e., gradients stronger or time ON longer)

- Basic image
- Square pix
- X-pix half width
- Replicas intrude
- Scanner makes square image "wrap" occurs
- Square pix
- Twice x-pix count so FOV = 2X
- This is "phase oversamp"
- Scanner crops to square replicas move out
- X-pix half width
- Twice x-pix count
- Same FoV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Sample data freq. (spatial freq.)

2. Result in image space

\[ \leftrightarrow \]

\[ \times \text{ multiply} \]

\[ \ast \text{ convolution} \]

\[ \leftrightarrow \]

\[ = \text{ equals} \]

\[ \leftrightarrow \]

\[ = \text{ equals} \]

Useful FT's

- Rect
  \[ \text{Rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \text{sinc}(\pi wx) \]

- Gaussian
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n/w) \xrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \delta(k - nw) \]

\[ \text{Fov} = \frac{1}{\Delta k} \]
\[ \Delta k = \frac{1}{\text{Fov}} \]
**Point-Spread Function**

\[ \hat{I}(x) = \Delta k \sum_{n: \Delta k, \Delta k} = S(n \Delta k) e^{i2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]
- Substitute into recon to get PSF:
  \[ h(x) = \Delta k \sum_{n: \Delta k, \Delta k} = e^{i2\pi n \Delta k x} \]
- Simplify
  \[ h(x) = \Delta k \frac{\sin \left( \pi N \Delta k x \right)}{\sin \left( \pi \Delta k x \right)} \]  
- That is, image is reconstructed from a sum of sinc's, because
  - the FT of a boxcar pixel in \( k \)-space is an image sinc
  - all outside center at sinc zero-crossing if \( k_{max} = x_{max} \)

---

**Image**

- How PSF modifies ideal (infinite \( k \)) image
  - \( \ast \) convolve
  - \( = \) ringing

**Frequency**

- FT
  - \( \times \) multiply
  - Acquisition window (truncates high spatial)

**Central replica same as sinc**
**GENERAL LINEAR INVERSE RECON FOR MRI**

\[ S(k_x) = \int_{x} I(x) e^{-i 2\pi k_x x} \, dx \]  
**Signal eq. \rightarrow fWd problem**

\[ I(x) = \int_{k_x} S(k_x) e^{+i 2\pi k_x x} \, dk_x \]  
**Recon eq. \rightarrow inv. problem**

\[ s = Fi \]

\[
\begin{bmatrix}
F_{x,y,z} = g(x,y) \\
\text{coil gain at this location}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{mult.coil} \\
\text{coil 1} \\
\text{coil 2}
\end{bmatrix}
\]

\[ c = F^+ s \quad \text{over-determined} \]

\[
F^+ = \left( F^T F \right)^{-1} F^T \quad \text{More Penrose inverse}
\]

\[
\begin{bmatrix}
(F^T F)^{-1} F^T \\
(F^T F)^{-1} F^T \\
(F^T F)^{-1} F^T \\
(F^T F)^{-1} F^T \\
\end{bmatrix}
\]

\[ \text{Slice-by-slice} \quad \text{assume slice select swamps dBs} \]
FAST SPIN ECHO (FSE)  

- One 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.
- Each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips).

**2DFSE**

- The "effective TE" is the TE when center of k-space is collected (largest effect on contrast, largest echo).
- Each subsequent echo has more T2 decay: \( E_n = e^{-n\frac{TE}{T2}} \) \( n = 1, 2, \ldots M \).
- By arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted.

**3DFSE** - Like 2D except wind/unwind added to thick slice select (w/ crushers on 180°)

- N.B. all those 180° pulses deposit a lot of RF power: 90° + 180° = 45° x power 30°.
FAST GRADIENT ECHO (GRASS, FLASH, MP RAGE)

- small tip so TR can be greatly reduced (e.g., 10 msecs, less than T2)
- ‘leakage’ undecayed transverse magnetization unwound and re-used spoiled before next shot

STEADY-STATE COHERENT (GRASS, FISP)

- unwind phase from phase-encode MT before next pulse (here because TR < TE)
- unwind read gradient, too
- $T_2/T_1$-weighted (CSF very bright)

STEADY-STATE, SPOILED (SPGR, FLASH)

- spoil with random gradient (but this still allows some x refocusing)
- spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- good gray-white contrast (T1-weighted)

NON-STEADY STATE, MAGNETIZATION PREP

- longitudinal mag.
- not affected much by low angle pulses

- preparation pulse → strong T1-weighting
- contrast varies in spatial-frequency-dependent way
Echo Planar Imaging (EPI)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (kxy) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved, without changing the size of the Gy "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**SPIN ECHO EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxyhemoglobin (probably as the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo cancels static $T_2^*$ ($T_2'$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing!)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For $TE \approx 100$ ms, spins diffuse 10's of $\mu m$, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in mediums and larger vessels
**SPIN ECHO EPI**

- EPI is a multi-gradient echo pulse sequence
- "spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space
- "asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2* weighting (for ky=0 echo)

- the 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used
- at higher fields, spin echo EPI is more promising

[signal to noise is higher so we can take spin echo hit]
[contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording]
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform
from resonant circuit w/non-uniform sampling to get constant $\Delta k_x$

- sinusoids in both $G_x$ and $G_y$ give spiral $k$-space trajectory

- constant angular velocity goes too fast at large $k_x$, $k_y$
- constant linear velocity better but impractical near $k_x=0$, $k_y=0$
- compromise: start constant angular, end constant linear →

**Constant angular velocity**

- $w(t) = w_0 t$
- $k(t) = A t e^{i w_0 t}$
- $G(t) = \frac{d}{dt} k(t)$
  - $= A e^{i w_0 t} + i A t w_0 e^{i w_0 t}$
- $G_x(t) = A \cos w_0 t - A t w_0 \sin w_0 t$
- $G_y(t) = A \sin w_0 t + A t w_0 \cos w_0 t$

**Constant linear velocity**

- $w(t) = w_0 T E$
- $k(t) = A T E e^{i w_0 T E}$
- $G(t) = \frac{d}{dt} k(t)$
  - $= \frac{A}{2 T} e^{i w_0 T E} + \frac{A}{2} w_0 e^{i w_0 T E}$
- $G_x(t) = \frac{A}{2 T} \cos w_0 T E + \frac{A}{2} w_0 \cos w_0 T E$
- $G_y(t) = \frac{A}{2 T} \sin w_0 T E + \frac{A}{2} w_0 \sin w_0 T E$
**Spiral 3D IR FSE** (From Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)
  - All echoes after 90° derive from magnetization with some T1 contrast (vs. non-steady-state)
- Possible to preserve sign
- High, uniform contrast, but lots of waiting (T1), high BW

\[180°(\text{prep}) \quad T1 \approx 700 \text{ msec}\]

**RF**

\[180° \quad 90° \quad 180° \quad 180° \times 16 \quad 180°(\text{prep})\]

\[G_2\]

\[G_y\]

\[G_x\]

**Sig.**

**Loop order**

3D **k-space**

(“Stack of Spirals”)

**Spiral interleaves**

\[k_x\] echoes

\[k_z\] interleaves

Echoes \(\rightarrow\) (after one 90°)
**Phase Errors & Echo-Centering Errors**

- Anything that causes a deviation of the $B_z$ field strength from the expected value ($B_{0,z} + G_{x,z}X + G_{y,z}Y + G_{z,z}Z$) changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

- Correct with shimming and $B_0$-mapping/phase unwrapping before reconstruction.

**Fourier Shift Theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int e^{-i2\pi k_x x_0} S(k_x) e^{i2\pi k_x x} dk_x,$$

- x-offset in image, x-offset in signal, k-space signal.

**Echo Centering Error**

- If realignment of all spins ($k_x=k_y=0$) doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction, magnitude image unchanged.

**Fourier freq. shift theorem**

Freq. shift in freq. domain causes phase shift in spatial freq. space.

$$e^{i2\pi k_x x} I(x) = \int S(k_x-k_x_0) e^{i2\pi k_x x} dk_x.$$
FAST SCAN ARTIFACTS  EPI vs. Spiral

brain-induced field defects lead to phase errors

**EPI**
- $G_x$ readout gradient strong $\Rightarrow$ field defects smaller percentage less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weak and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the x-direction, for example, wraps and unwraps phase as a function of x-position
- but $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

- for a given x-position, the strength of the spurious gradient is constant, so the accumulation of phase error results in a shift in the y-direction \(=k_x\)-space spin-stripe disp. \(k_x = \text{more phase err}\) (darker)
- the phase error causes a shift in the y-direction proportional to $x$-gradient strength \(=\) shear but no blurring \(k_y\)
  (N.B. shift varies w/x-position) $y \xrightarrow{\text{x}} y$  

**Spiral**
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error \(=\) more shearing
- for spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift, because higher frequency phase stripes misaligned relative to low spatial freq. 

- for defects with more complex contours in the y-direction (than linear, as above) the vertical shifts (in EPI) will vary with $y$-position, and may result in signals from different $y$-positions being reconstructed on top of each other
**Image-Space View of Localized Bφ Defect, Effect on Recon**

- Localized Bφ defects often arise from air pockets embedded in tissue:
  - Air in middle/outer ear → indentation in interior temporal lobe
  - Air under olfactory epithelium → orbitofrontal cys, ant, thal. compression

---

**Collect one data (k-space) point**

\[
\text{4 cycles of phase in y-dir (y-gradient)} \quad + \quad \text{Localized } \Delta \text{Bφ defect} \quad + \quad \text{Brain structure sampled with distorted stripes} \quad = \quad \text{one complex number}
\]

**Reconstruction from distorted data points**

\[
\text{...} \quad + \quad [\text{amplitude and phase of this component}] \quad + \quad [\text{Same for 5 cycles}] \quad + \quad \text{...} = \quad \text{Local upward displacement of image (phase encode dir)}
\]

**N.B.:** Image shift only occurs if diff. spot frequencies sampled in successively later echotimes (see next page)

**Spatial information is lost when continuous changes in phase are flattened by Bφ defect**

\[
\text{shifts can pile multiple pixels on top of each other into one bright pixel}
\]

**Local estimates of ΔBφ needed to correct images**

1) Fieldmap method: < multiple TEs to est. local ΔBφ from T1/TE slope
2) Point-spread-function: < extra phase encode to estimate PSF (should be f-function) deconvolve distorted image in phase-encode direction
- when local B₀ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded within same echo time (e.g., with single line phase encoding) this will correspond to constant spatial freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/ EPI, static B₀ defect causes more and more local displacement of image phase stripes for each additional ky line
  - that is, later lines have greater spatial freq. offset
  - effectively stretches k-space in ky direction
  - same num samples to higher spatial freq.
    - shrinks FOV (squishes voxels – see FOV page)

- when image is reconstructed, region with local B₀ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) static local ΔB₀ defect
  2) successive increases in phase error for successive spatial freq. measurements during long EPI readout
  3) small size of ky phase encode blips relative to B₀ defect (much less of this effect in freq. encode direction)

- Respiration (which affect B₀) in 3D FLASH might cause similar effect within k₂ partition (if successive spatial freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the $z$-component of the $B$ field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (e.g., printed circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.

  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching 1 cm in functional overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

These effects do not build up over time in phase encode direction. Since they only occur when gradients are turned on. These distortions are predictable and can be corrected.
SHIMMING AND \( B_0 \)-MAPPING

- Passive iron shims inserted to flatten \( B_0 \) field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the \( B_0 \) field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

[Linear shim coils impose gradients in \( x \), \( y \), and \( z \)]

[Higher order shims impose higher order spherical harmonic field components (e.g., \( z^2 \)]

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the \( B_0 \) field

- Local resonance offsets caused by \( B_0 \) defects estimated from images
  - e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - This only corrects spatially gradual field defects
  - Local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
**NAVIGATOR Echos**

for motion correction

- **1D navigator**: excite beam of spins, detect movement from k-space shift of echo (e.g., from diaphragm)

- **3D navigator**: collect 3D sphere in k-space

  Rotation of object $\rightarrow$ rotation of k-space amplitude pattern

  Translation of object $\rightarrow$ phase shift of k-space phase (Fourier shift)

  - sample at sufficient k-space radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

  Welch et al. (2002) MRM

  $\uparrow$ equator $\rightarrow$ up, equator $\rightarrow$ down

  $z(n) = \frac{2n - N - 1}{N}$

  $y(n) = \cos(T \pi \sin^{-1} z(n)) \sqrt{1 - z^2(n)}$

  $x(n) = \sin(T \pi \sin^{-1} z(n)) \sqrt{1 - z^2(n)}$

  (skip poles — slew rate to high)

  - can be used for prospective motion correction (rotate, translate w/ gradients)
  - better estimate, because of speed, than Full TR & EPI images (27 ms vs. 2-4 sec)
  - may need to smooth rot,trans estimates across time (e.g., Kalman filter)
RF FIELD INHOMOGENEITIES $B_1$ inhomogeneities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way. Variations can be used (cf. SMASH, SENSE) or corrected.

- transmit coil inhomogeneities affect the flip angle in a spatially varying way: potentially worse (why local transmit is rare). Usually fixed by using a large transmit coil (e.g. body coil).

- RF penetration at higher fields (higher RF frequencies) is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field.

- the advantage of the falloff in signal seen with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain).

- different sensitivity functions from different coils can be used to scan less lines in k-space.

- SMASH - use weighted sensitivity profiles to form sinusoidal harmonics, which are then used to fill in missing k-space lines.

- SENSE - subsample k-space (like SMASH) then unfold aliased images in image domain after estimating sensitivity profile.

- SPACE RIP - simple linear inverse after 1D FT (more general).
Coil fall-off allows k-space under sampling.

Slow variation in RF field fall-off (e.g., 1-3 c/s/cycle/ FOV) causes a blur in k-space as if scanning with a filter at all spatial freqs!

To see this, a multiplication in image space (with a low spatial freq) equals a convolution in k-space (at all freqs!)

For example:

\[
\begin{array}{c}
\text{in image space} \\
\text{at single freq}
\end{array}
\]

\[
\begin{array}{c}
\text{FOV}
\end{array}
\]

N.B., inverse FT of k-space data that has been smeared in freq space is simply a sharp image, w/ a fall off \(\Rightarrow\) i.e., k-space blur does not blur the reconstructed image.

Because some image data is spread in k-space, can undersample.

- **GRAPPA** — construct k-space kernel to fill in missing data + other coils
  - train it w/ fully sampled data near k-space center

- **SENSE** — gen linear inverse
  - wouldn’t work unless spatial freq’s had been blurred!
**Diffusion - Weighted Imaging**

- Simple diffusion weighting
  - RF
  - \( G_z \)
  - \( G_y \)
  - \( G_x \)

- "Apparent diffusion coefficient" map
  - Calculate D image from \( b = 0 \)
  - To get large \( b \), need \( G_y, G_z \) (need big \( G_z \))

- Long \( G_z \) gives spurious T2-weighting
- Use stimulated echoes:
  - 90° RF — \( G_z \) — 90° RF — 90° RF

**Anisotropic Diffusion (Gaussian)**

- Measure D along multiple axes
- Have to measure tensor, not scalar
  - Even for determining one primary direction

\[
D = \begin{bmatrix}
U_x & U_y & U_z \\
D_{xx} & D_{xy} & D_{xz} \\
D_{yx} & D_{yy} & D_{yz} \\
D_{zx} & D_{zy} & D_{zz}
\end{bmatrix}
\begin{bmatrix}
U_x \\
U_y \\
U_z
\end{bmatrix}
\]

- Since \( D \) is symmetric, only need 6 measurements

**Diffusion Surface (non-Gaussian)**

- Need to measure diffusion in many directions
- E.g., icosahedral subselection (126)
- Required for even two fiber directions

**Fiber tract mapping**

- An ill-posed problem
- E.g., constrain both ends and center point (?!)
- Need vis. areas test
PERFUSION - ARTERIAL SPIN LABEL

- Basic idea: tag blood below area of interest, collect control & tagged image assuming directional input flow!

  | Continuous ASL (CASL) | tag a plane continuously → greatest on, blood gets adiabatically inverted as it passes through location w/convolved resonance freq.
  | Pulsed ASL (PASL) | e.g., EPISAR, FAIR, PICORE, QUIPSS II
  | tag block of tissue below slice(s) →

- Small diffs between control and tag (~1%)
  - requires accurate balancing of control & tag images

- Contrast problems:
  - transit delays → biggest confounding factor
  - relaxation rate diffs, venous clearance (vs. microspheres, which get stuck!)

  | solutions for quantitative perfusion |
  | in set delay so all spins arrive into flow velocity capillaries |
  | kill end of tag to reduce spatial variation of tag |

QUISPSS II - quantitative perfusion

1) Pre-saturate spins in target slices
2) Tag 180° pulse below slices
3) Saturate tagged block to end tag (TI)
4) EPI of spiral images of target slices

\[ \Delta M = \text{flow} \times \left[ 2M_0 \frac{\text{TE}}{\text{TI}_1} e^{-\text{TI}_2/\text{TI}_1} \right] \]

Can extract flow and BOLD adjacent subtraction minimize movement artifact

1) Alternate tag and control, GRE TE=30 ms
2) Dual echo spiral
   - Control-tag → flow
   - Control+tag → BOLD
   - BOLD-weighted
   - Tag-control-tag-control...
Phase-Encoded Stimulus & Analysis

Periodic stimuli (phase-encoded) — e.g., 8 cycles at 64 sec/cycle

Calculate significance
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, no freq (= movement)

Smooth
- Vector average of complex significance (A, φ) with that at nearest neighbor surface points

Display
- Plot phase using hue and saturation to indicate significance

Delay correction
- Record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- Vector average after reversing angle of one & penalizes inconsistent more than just avg of angles

FFT, convert to A, φ

Typically 0.5-5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend
**Convolution**

\[ f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) \, dz \]

- Definition of convolution

**Graphically:**
- Place kernel at \( x \)
- Reverse it and:
  - Multiply
  - Sum
  - Current \( x \)

**Why we reverse:**

**Impulse response function (HRF):**

\[ \text{Impulses (ERP design)} \]

- \( a \) occurred a while ago
- \( b \) occurred recently
- \( c \) large effect

**How to calculate convolution for this time point:**
- Only 3 terms in integral — all other zero

**Note:** Cross-correlation same as convolution except no reversal, \( h(x+z) \) instead of \( h(x-z) \)
**GENERAL LINEAR MODEL**

\[ \vec{y} = \vec{X} \hat{h} + \vec{S} \vec{b} + \vec{n} \]

- Goal is to solve for the hemodynamic response functions, \( h \)

**Data = design \cdot HDR + drifts \cdot weights + noise**

\[ \begin{bmatrix} \text{data} \\ \text{texp} \end{bmatrix} = \begin{bmatrix} \text{hemo} \\ \text{exp} \end{bmatrix} \begin{bmatrix} \text{experimental design} \\ \text{unknowns} \end{bmatrix} \begin{bmatrix} \text{hemo} \\ \text{exp} \end{bmatrix} + \begin{bmatrix} \text{lin poly} \\ \text{lin poly} \end{bmatrix} \begin{bmatrix} \text{factors} \\ \text{factors} \end{bmatrix} + \begin{bmatrix} \text{assumed white} \\ \text{data} \end{bmatrix} \]

\[ y \]  

\[ \begin{bmatrix} \text{y} \\ \text{n} \end{bmatrix} = \begin{bmatrix} \text{X} \\ \text{N} \end{bmatrix} + \begin{bmatrix} \text{S} \\ \text{N} \end{bmatrix} + \begin{bmatrix} \text{n} \\ \text{N} \end{bmatrix} \]

\[ \downarrow \text{multiple conditions} \]

\[ \begin{bmatrix} \text{cond1 occurs} \\ \text{cond2 occurs} \\ \text{cond1 re-occurs} \end{bmatrix} \]

\[ \begin{bmatrix} \text{h}_1 \text{cond1} \\ \text{h}_2 \text{cond2} \end{bmatrix} \]

\[ \text{matrix notation for discrete convolution of stimulus pattern with hemodynamic resp. funs.} \]

\[ \text{maximum likelihood estimate} \]

1) Assume white noise, solve for \( \hat{h} \)

\[ \hat{h} = (X^T P_{s}^T X)^{-1} X^T P_{s}^T y \]

or

\[ \hat{h} = (X_1^T X_1)^{-1} X_1^T y \]

where \( P_{s}^T = I - S (S^T S)^{-1} S^T \)

\( \implies \) projection matrix that removes part of vector that lies in \( S \) space

2) Significance (how to construct F-ratio)

\[ F = \frac{N-K-L}{K} \left[ \frac{y^T (P_{Xs} - P_{s}) y}{y^T (I - P_{Xs}) y} \right] \]

\[ \implies \text{see diagram next page for geometric interp} \]
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance

\[ \hat{y} = X\hat{h} + \hat{e} \]

projection matrix \( P_x \) operates on \( \hat{y} \) to give projection of data into experiment space \( X \)

- when nuisance functions \( S \) are considered, problem: \( S \) may not be orthogonal to \( X \)

\[ \Rightarrow \text{for example: linear trend not orthogonal to std. block design} \]

\[ \Rightarrow \text{remember: "orthogonal" means [dot prod. = 0]} \]

\[ \text{block design} \]

\[ \text{linear trend} \]

\[ \Rightarrow \text{sum: not 0} \]

Geometric Picture

(Liu et al. 2001, Neuroimage)

\( X_S \): space of data modeled by all reference and nuisance

orthogonal projection onto nuisance \( (P_Sy) \)

(data explained by reference and nuisance)

orthogonal projection onto reference \( (P_r y) \)

(data explained by reference)

oblique projection onto reference \( (E_{xy}) \)

oblique projection onto nuisance \( (E_{sy}) \)

data \( (\hat{y}) \)

error \( (e) \) not explained by reference and nuisance \( (F \text{ denom.}) \)

\[ (I - P_Sy)y \]

same as projection onto reference only in special case where \( S \perp X \)
1) MNI auto-Talairach → generates 4×4 matrix
   - make average brain target (blurry)
   - blur target further, blur single brain (a lot), gradient descent on xcorr
   - repeat w/ less blurring of avg target and current brain
   - problem: variable neck cut-offs
     only 2 points near center of brain!
     C → but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices
     C → interpolates scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate by interpolating in 3D

   find points in 5×5×5 within 10% of WM, set near scale for them
   build Voronoi to interpolate scales between above
   soap-bubble smooth Voronoi boundaries (3 iterations)
   re-scale each voxel

3) Skull Stripping (output: "brain")
   "shrink-wrap" algorithm
   - start with ellipsoidal template
   - minimize brain penetration and curvature
     - curvature: spring force
       (from center-to-neighbor vector sum)
     - brain penetration
       apply force along surface normal that prevents surface from entering gray matter
implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")

more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t)
\]

rule for each vertex, \( \mathbf{r}_{\text{center}} \)

\[
\mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}})
\]

identities 3x3

\begin{align*}
\text{stronger than normal} & (0.5) \\
\text{expand by} & \text{distribution to:} \\
\text{projection onto} & \text{vector to} \left\{ \\
\text{minus projection} & \text{of neighbor onto} \\
\text{normal} = \text{tangential}! & \text{neighbor vertex} \\
\end{align*}

\[
\lambda_{\text{normal}} \left[ \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\#\text{vertices}} \sum_{\text{neigh}} (\mathbf{n}_{\text{neigh}} \mathbf{n}_{\text{neigh}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \right]
\]

vector, subtract off

\[
\mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \frac{30}{d} \max \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{threshold}} \right] \right]
\]

\[
\text{d sample points into brain along the direction of normal}
\]

\[
\text{outside (dark)} \\
\text{skin (light)} \\
\text{skull (dark-light-dark)} \\
\text{GM} \\
\text{WM} \\
\text{ideal skull strip}
\]

Snapshot of surface and "core sample" from one vertex
4) Non-isotropic filtering (output: "wm")
- Preliminary hard threshold: output is GM/WM blood vessels (bright) also truncated to black orig brightness
- Find ambiguous/boundary voxels
  \[ \Rightarrow \text{20\% or more of 26 immediate neighbors different} \]
- Find plane of least variance
  For each direction (from icosahedral supersimulation)
  Consider 5x5x5 volume around 1 voxel
  Find plane of least variance in this Hemisphere
  Median filter w/ hysteresis
  \[ \Rightarrow \text{if 60\% of within-slab differ, reverse classification} \]
  \[ \Rightarrow \text{"flosses" sulci without blurring} \]

5) Find cutting planes
- Midbrain → [Callosum, to separate hemispheres (SAG)] → Talairach to start
  Fill WM in SAG or T1R till midline
- Midbrain, to avoid fill into cerebellum (T1R)

6) Region-growing to define connected parts (output: "filled")
- Inside-out, outside-in, inside-out — for each hemisphere
- Up/down cycles within each plane
- Plane-by-plane
- "Wormhole filter:" (3x3x3 = center + 26)
  \[ \Rightarrow \text{fill (unfilled) voxel if 66\% neighbors differ} \]
  \[ \Rightarrow \text{eliminates structures within 1-D structure} \]
7) Surface Tessellation (output: rh.orig, lh.orig)

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  -> 30 cross-products oriented properly

- long list of values associated with each numbered vertex
  e.g. [position (orig, morphed),
       area (orig, morphed),
       curvature (intrinsic, Gaussian),
       "sulcanness" (summed ± movement during unfolding),
       cortical thickness,
       fMRI data, EEG/MEG dipole strength]

- separate fMRI data set must be aligned, sampled

- fMRI voxels larger
  Sample at each surface vertex
  nearest-neighbor "soap bubble" smoothing
  to interpolate data onto hi-res mesh

- some quantities only well-defined on surface
  -> gradient of magnitude of
cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON

- smoothing/infilation/WM/pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- scalar error (fixed by reducing curvature)
- smaller scalar error (fixed by moving toward target image value)

\[ J_{\text{normal}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right]^2 \]

- across all vertices
- \( \frac{1}{2} \) so no centroid or derivative
- across all vertices
- unit normal vector
- vector from current center to one neighbor (position vector diff)
- \( \mathbf{x} \)-direction in tangent plane
- \( \mathbf{y} \)-direction in tangent plane

\[ J_{\text{tangential}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right]^2 \]

- across all vertices
- \( \frac{1}{2} \), no centroid or derivative
- across one vertex
- \( \mathbf{t} \)-vector
- \( \mathbf{x} \)-direction in tangent plane
- \( \mathbf{y} \)-direction in tangent plane

\[ J_{\text{image}} = \frac{1}{2} \sum_{\text{centers}} \left[ I_{\text{targ}}(\mathbf{r}_{\text{center}}) - I(\mathbf{r}_{\text{center}}) \right]^2 \]

- for WM: mean of voxels labeled WM in 5 mm neighborhood
- for pia: global — small num for CSF-like

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{targ}}(\mathbf{r}_{\text{center}}) - I(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

- go the opposite direction (vector) of the direction of largest scalar error for one vertex
- \( \nabla I(\mathbf{r}_{\text{center}}) \)
- calculate gradient on blurred-by-Gaussian image

\[ \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] \mathbf{n}_{\text{center}} \]

- unit normal vector
- \( \mathbf{x} \)-component of tangential
- \( \mathbf{y} \)-component

\[ + \sum_{\text{neighbors}} \left[ \mathbf{t}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] \mathbf{t}_{\text{center}} + \left[ \mathbf{t}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] \mathbf{t}_{\text{center}} \]
- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \( (S_{\text{surf}} - S_{\text{tang}})^2 \)
- Bootstrap: Morph to one brain, make avg target, remorph to avg tang

Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (clashed lines)
- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (to average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\( \text{N.B.}: \) morph will have changed local vertex density compared to more uniform native mesh (use native for sing. sub.)