1 - $B_\Phi$ field from superconducting magnet

2 - gradient coils

3 - body RF transmit/receive

4 - RF receive-only

5 - shim coils (in gradients)

$B_\Phi \rightarrow z$ (longitudinal)

$B_1 \rightarrow x, y$ (transverse)

---

$max gradient: 
[80 \text{ mT/m}] 
[200 \text{ T/m/sec}]$

1T = \frac{10,000 \text{ Gauss}}{2 \pi} \approx 42 \text{ MHz/T}$

Earth: $0.25 - 0.65 \text{ G}

25 - 65 \text{ mT}$

---

(1) $B_\Phi$ field

(2) Body gradient coils

(3) RF transmit body coil

(4) RF receive-only head coils

RF transmitter (30 kW)

RF receiver

B1 is several orders of magnitude smaller than B0

---

Cylindrically polarized $B_1$ field rotating to $B_\Phi$ at Larmor frequency

3 - 1.5 million watt amplifiers to add ramps to $B_\Phi$ field

$B_1$ field

---

Superconducting coils in liquid helium

(no power required after current injected to bring up field using induction)

---

opposite direction $B_\Phi$ shield coils outside those not shown

---

Non-superconducting water-cooled, external shield

---

Hardware

MAGNET HARDWARE

---

Neuroimaging NOTES

Martin Sereno
18 Dec 2019
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field.
- Current loop from spinning charge (right-hand rule).
- N.B.: Classically, this would cause EM radiation, spin down.

**Stern-Gerlach experiment**

Pass silver atoms through strong magnetic field, split into just 2 beams.

**Microscopic picture**

- No strong magnetic field, \( B \phi = 0 \).
- Strong magnetic field, \( B \phi = \)...
- Strong \( B \phi \) plus oscillating \( B_1 \).

**Macroscopic picture**

- Bulk magnetization.
- Slight excess of "up" (3 ppm).
- Precessing vectors are "bunched" at any one moment around circle.

**Precession**

- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

- Bulk equilibrium magnetization (parallel to \( B_0 \))

\[
M_0^z = \frac{\gamma h B_0 N}{4KT_S}
\]

- Where \( I = \pm \frac{1}{2} \).

**N.B.:** Compared to top, gravity magnetic relaxation is:

- Frictionless spin, doesn't slow down.
- Signed gravity can change precession direction.
- Can stick under floor.
- Neighbor bumping causes decay (T2).

**Left-hand rule:** Thumb = \( B \phi \), fingers = precess.
**Bloch Equation**

- Time-dependent behavior of $\vec{M}$ in the presence of an applied magnetic field (excitation \(\leftrightarrow\) relaxation)
  - Precession: $\vec{B} = B_0$
  - Excitation: $\vec{B} = B_0 + \vec{B}_1$

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{yB} - \frac{M_x \vec{i} + M_y \vec{j}}{T_2} - \frac{(M_z - M_z^0) \vec{k}}{T_1}
\]

- In the Larmor-rotating coordinate system, a tilt with a phase shift in a standard $B_1$ excitation is rotation around $x$-axis

- Longitudinal and transverse relaxations
  - $\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_z^0}{T_1}$
  - $\frac{dM_x(t')}d{t'} = -\frac{M_x(t')}d{t'}$ [from Bloch equation after dropping applied field term]

- Solution to equations above: time-dependent free precession e.g.'s
  - Given $M_x, M_y$
    - $M_z(t) = M_z^0 \left(1 - e^{-t/T_2}\right) + M_z^0 \cdot e^{-t/T_1}$
    - $M_x(t') = M_x(t') \cdot e^{-t/T_2}$
    - $M_y(t') = M_y(t') \cdot e^{-t/T_2}$
Vector ADD, MULTIPLY

- adding vectors is easy
\[ \vec{c} = \vec{a} + \vec{b} = \left[ a_x + b_x, a_y + b_y \right] \]
- just add components (vector)
- applies to complex numbers
- generalizes to any \( D \)
\[ \| \vec{c} \| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- multiple ways to multiply vectors: here are 3

**Dot product**
(= inner product)
(= "scaled projection onto")
\[ c = \vec{a} \cdot \vec{b} = \left[ b_x b_y b_z \right] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = a_x b_x + a_y b_y + a_z b_z \]
- generalizes to any \( D \)
\[ c = \| \vec{a} \| \| \vec{b} \| \cos \theta \]
\[ c = p \| \vec{b} \| \] if \( \| \vec{b} \| = 1 \)
\[ p = \| \vec{a} \| \cos \theta \]

**Cross product**
(= outer product)
(can be generalized: see "geometric algebra")
\[ \vec{c} = \vec{a} \times \vec{b} = \left[ \begin{array}{ccc} 0 & b_z & -b_y \\ -b_z & 0 & b_x \\ b_y & -b_x & 0 \end{array} \right] \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} \]
(= geometrical algebra: bivector plane area)
 orthogonal specific to 3D
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \theta \]
\[ \| \vec{c} \| = \max \text{ if } \vec{a}, \vec{b} \text{ orthogonal} \]

cross product is skew-symmetric: \( A \times A = -A \times A \)

**Complex multiply**
(see also quaternions, geometric algebra generalization)
\[ \vec{c} = \vec{a} \cdot \vec{b} = \left[ \begin{array}{c} b_x - b_y \\ b_y b_x \end{array} \right] \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a_x b_x - a_y b_y \\ a_x b_y + a_y b_x \end{bmatrix} \]
- angles add
- magnitudes multiply
- specific to 2D
\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]
\[ \sum \text{ of angles: } \theta_1 + \theta_2 \]
**Effects of $\vec{M}$, $\vec{B}$, and $\Theta$ on Precession Freq.**

Bloch 1st term: $\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}$

- Cross prod. properties review:
  $$\| \frac{d\vec{M}}{dt} \| = \| \vec{M} \| \| \vec{B} \| \cdot \sin\Theta$$

**Starting condition**

$\Rightarrow$ now see effects of changing $\vec{M}$, $\vec{B}$, $\Theta$

**Change $\vec{M}$ length**

$\Rightarrow \frac{d\vec{M}}{dt}$ proportionally larger, so canals
  effect of larger $\vec{M}$

  $\Rightarrow$ same precession freq. as starting cond.

**Change $\Theta$ between $\vec{M}$ and $\vec{B}$**

$\Rightarrow \frac{d\vec{M}}{dt}$ goes up (then down) as $\sin\Theta$
  & cross prod., $-1$

  but circumference also
  goes up as $\sin\Theta$,
  cancelling again

  $\Rightarrow$ same precession freq.

**Change $\vec{B}$ length**

$\Rightarrow \frac{d\vec{M}}{dt}$ goes up, proportional to $\vec{B}$

  but circumference is same at starting cond.

  $\Rightarrow$ increased precession freq. $(\omega = \gamma \vec{B})$
**Simple Matrix Operations**

**Basic Idea**
- A matrix \([\text{rotates/scalates}]\) a vector \(\mathbf{b} = \mathbf{M} \mathbf{a}\)

**3D Example**
- \[
\begin{pmatrix}
    b_x \\
    b_y \\
    b_z
\end{pmatrix}
= \begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{21} & M_{22} & M_{23} \\
    M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{pmatrix}
    a_x \\
    a_y \\
    a_z
\end{pmatrix}
\]

**Add Translate** (after rotate/scale)
- Commonly used "hack" for aligning verts
- A 4D matrix \([\text{rotates/scalates}]\) a 3D vector \(\begin{pmatrix}
    b_x \\
    b_y \\
    b_z \\
    1
\end{pmatrix} = \begin{bmatrix}
    3 \times 3 \text{ rot/scale} & \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]
- N.B.: Have to keep track of order!!
  - Rotate/scale then trans \(\neq\) trans, then rot/scale changes rot component: untranslate, rot, retranslate

**3 Special Cases (3d):** Rotate around each major axis without changing length
(Scale = 1.0)

- **Rotate around x-axis:**
  \[
  R_x(\theta) = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & \sin \theta \\
  0 & -\sin \theta & \cos \theta
\end{bmatrix}
\]
  e.g., 90° flip

- **Rotate around y-axis:**
  \[
  R_y(\theta) = \begin{bmatrix}
  \cos \theta & 0 & -\sin \theta \\
  0 & 1 & 0 \\
  \sin \theta & 0 & \cos \theta
\end{bmatrix}
\]
  e.g., 180° flip to avoid adding 180° phase after 90° flip on x'

- **Rotate around z-axis:**
  \[
  R_z(\theta) = \begin{bmatrix}
  \cos \theta & \sin \theta & 0 \\
  -\sin \theta & \cos \theta & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]
  e.g., precession with 3D along z'

**General Case**
- Rotate around general z'-axis:
  \[
  R_{a'}(\theta) = R_z(-\theta) R_y(-\phi) R_z(\theta) R_y(\phi) R_z(\theta)
\]
  (Quaternions are more efficient)
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

**diff. eq.:** \( dM_{xy}(t) = -\frac{M_{xy}(t)}{T_2} \)

**Solution:** \( M_{xy}(t) = M_{xy}(O_+) \cdot e^{-t/T_2} \)

**Goal:**
1. Find eq. whose derivative satisfies diff. eq.
2. Also find soln. (one of many) that passes thru init condition
   \( \rightarrow \) since own diff. eq. is: derivative of funct. = const. funct.
   \( \rightarrow \) try exponential, since derivative \((e^x)' = e^x\)

\[
\begin{array}{c|c|c|c}
\text{diff. eq.} & \text{den. of var} & \text{constr} & \text{OK - have recovered orig. diff. eq.} \\
\hline
\text{M}(t) = -\frac{1}{T_2} & \text{M}(t) & \text{M}(t) & \text{N.B. this function is the unknown, like the x in x + 1 = 3} \\
\hline
\text{one soln} & \text{e}^{-t/T_2} & \text{e}^{-t/T_2} & \text{N.B.: same as M}(t) \\
\hline
\text{take deriv. to check} & \frac{-1}{T_2} & \frac{-1}{T_2} & \text{OK!} \\
\hline
\text{another soln} & \text{const} \cdot \text{e}^{-t/T_2} & \text{const} \cdot \text{e}^{-t/T_2} & \text{N.B. again same as M}(t) \\
\hline
\text{take deriv. to check} & \frac{-1}{T_2} \cdot \text{const} \cdot \text{e}^{-t/T_2} & \text{so: any const OK!} \\
\hline
\end{array}
\]

*initial condition*

\[
\begin{align*}
\text{information added to soln. (not from diff. eq.)} & : \\
\text{const} = M_{xy}(O_+) & \\
M'(t) = M_{xy}(O_+) \cdot e^{-t/T_2} & \\
magnetization immedi. after RF(B1) ends & \\
M_{xy}(O_+) & \\
\end{align*}
\]
**Bloch Eq. - Matrix Version**

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = R_z(\omega t)\vec{M}(0)
\]

Include Relaxation:

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_z^0) k}{T_1}
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix} e^{-\frac{t}{T_2}} & 0 & 0 \\ 0 & e^{-\frac{t}{T_2}} & 0 \\ 0 & 0 & e^{-\frac{t}{T_1}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1 - e^{-t/T_1}) \end{bmatrix}
\]
**EXCITATION IN THE ROTATING FRAME**

- Original Bloch eq. in laboratory frame:
  \[
  \frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{B} \tag{21}
  \]
  gradients

- Add on-resonance $B_1$ to $\mathbf{M}$
  \[
  \mathbf{B} = B_1(t) (\cos \omega_0 t \hat{\mathbf{i}} - \sin \omega_0 t \hat{\mathbf{j}}) + \mathbf{B}_0 \hat{\mathbf{k}}
  \]

- Matrix version

- Substitution to convert to the rotating frame

- After substitution any off-resonance appears as residual $B_0$ ($B_2$)
  (see off-res notes page)
  \[
  \left[ \begin{array}{c}
  \mathbf{M} \\
  \mathbf{B}
  \end{array} \right] = \begin{bmatrix}
  \mathbf{R}_2(\omega_0 t) & \mathbf{M}_{\text{rot}} \\
  \mathbf{R}_2(\omega_0 t) & \mathbf{B}_{\text{rot}}
  \end{bmatrix}
  \]

- Rotating frame < on-resonance
  * basic excite
  \[ \frac{d\mathbf{M}_{\text{rot}}}{dt} = \mathbf{M}_{\text{rot}} \times \mathbf{B}_{\text{off}} \]

- Rotating frame < off-resonance
  * general
  \[ \frac{d\mathbf{M}_{\text{rot}}}{dt} = \mathbf{M}_{\text{rot}} \times \mathbf{B}_{\text{off}} \]

- Gradient: $\omega(t) = \gamma_0 \mathbf{M}_z$
  off-res: appears as residual $B_0$, lifting $B_1$ vect.
  out of $x-y$ plane

- Rotating frame < on-resonance
  * small tip approx.
**BLOCH EQ. SUMMARY**

\[
d\vec{M}/dt = \vec{M} \times \vec{B} - \frac{M_x i + M_y j}{T_2} - \frac{(M_z + M_0)k}{T_1}
\]

(lab-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \( d\vec{M}/dt \) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \( T_2, T_1 \) decay
  - \( \vec{B}_1 \) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1) **lab frame**
   - Just precession

2) **rotating frame**
   - \( \vec{M} \) stopped
     - That is, \( B_\phi = 0 \)

3) **add \( \vec{B}_1 \)**
   - \( \vec{B}_1 \) also stopped!
     - But \( \vec{M} \times \vec{B}_1 \) still works!!
     - "Precess" around \( \vec{B}_1 \) axis

4) **off-resonance**
   - Slow precess, now around tilted \( \vec{B}_{\text{eff}} \)
   - Tilting plane of apparent \( B_z \) comp. from residual precess. around \( Z \) from off-resonance
RF FIELD POLARIZATION

- polarization (change of direction) of magnetic field (vs. electric field)

- linearly polarized field, \( \overline{B_1(t)} = B_1 \cdot \cos \omega t \hat{\mathbf{z}} \) for magnet strength \( \{ \pm 1, 1 \} \cdot 1 \)

- N.B.: \( \overline{B_1} \) adds to much larger \( \overline{B_0} \)

- circularly polarized field (quadrature), \( \overline{B_{1c}(t)} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \) = \( B_1 \cdot e^{-i \omega t} \)

- in the rotating coordinate system flipping around x-axis vs. y-axis is just difference in phase of RF

- typical 90° flip (around x-axis)

- typical 180° flip (around y-axis)

- 180° flip regenerates 6X power at 90°
**SIGNAL EQUATION**

\[ \Phi(t) = \oint_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- **magnetic flux**
  - thru coil \( \rightarrow \) scalar
  - (integral mag. field perpendicular to area)
- **local magnetization**
  - generated by coil geometry
  - at each point
  - in object
- **position**
  - \( \mathbf{r} \rightarrow x, y, z \)

**Faraday's Law of Induction**

\[ V(t) = -\frac{d\Phi(t)}{dt} = \frac{d}{dt} \oint_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- ignore change in z-comp. \( \mathbf{M} \) because so slow \( \rightarrow \) i.e., we only see \( M_{xy} \), not \( M_z \)
- substitute \( \mathbf{M}(t) \) with lab frame \( M_{xy}(t) = M_{xy}(0) e^{-i\omega t} \)
- simplify:
  1. ignore decay (assume this \( t=0 \))
  2. assume phase-sensitive detection \( \left[ \text{Sw (difference from } W_0 \right] \rightarrow \text{rotating frame!} \)

**Laboratory frame Bloch solutions:**

\[ M_z \rightarrow \text{same} \]
\[ M_T = M_{xy}(0) e^{-i\omega t} \]

- spatially dependent resonant freq. in rotating frame \( \rightarrow \) i.e., after subtraction \( \omega_0 = \gamma B_0 \)

**Standard Signal Expression**

\[ \mathbf{S}(t) = \int_{\text{obj}} M_{xy}(\mathbf{r},0) e^{-i \frac{\omega(t)}{\omega_0} \mathbf{r}} \, d\mathbf{r} \]

- **phase angle in rotating frame**
  - \( \omega t = \text{radians/sec} \)
  - \( \rho = \omega t \mathbf{w} \)

- **signal is vector sum across object of local transverse magnetization vectors**

- **evaluation using free precession e.g.** (solution to Bloch) ignoring relaxation
  - rewrite w/ complex notation in time-dependence from lab frame Bloch

- **Omit receive and init excite flip phase offsets**

- **demodulate**

- **get difference from \( W_0 \) (value needed)**

- **sum across object**

- **complex: \( 2 \mathbf{M}(0) \)**
**PHASE-SENSITIVE DETECTION**

How we get rotating frame

\[ V(t) \xrightarrow{\text{multiply}} \text{Low-Pass Filter} \xrightarrow{\text{PSD}} S(t) \]

- Method for moving very high-frequency Larmor oscillations down to tractable frequency range

Demodulated signal \( \alpha \) RF coil signal \( \cdot \) reference (transmitter)

\[ \sin[(\omega_0 + \delta \omega)t] \cdot \sin[\omega_0 t] \]

\[ \frac{1}{2} \left[ \cos \delta \omega t - \cos (2\omega_0 + \delta \omega)t \right] \]

This signal is digitized

**One freq - freq domain**

Signal

Reference

Demodulated

After filter (rotating frame!!)

**Chirp - time domain**

Chirp

Center

Demodulated

No freq signal

- Two signals are made from a single receiving RF coil

- A quadrature coil can be treated the same way (OK to combine after adding T/2 phase, then PSD)

- Quadrature coil has better S/N since noise in each part is uncorrelated (T/2 better)
- Echoes

**FID - FREE INDUCTION DECAY, T2**

- Signal (FID) resulting from RF pulse w/ angle \( \alpha \)

\[
\hat{S}(t) = \sin \alpha \sqrt{\frac{\rho(w)}{w \to \infty}} e^{-t/T_2} e^{-i\omega t} \, dw
\]

- recorded complex signal

- spectral density funct.

- time-dep. decay

- rapid incr. oscillations freq.

- all atoms assumed to interact in same way in freq. range under curve

- An example spectral density ("Lorentzian inhomogeneity")

\[
\rho(w) = \frac{M_0^2}{(\Delta B)^2 + (w - \omega_0)^2}
\]

- \( \omega = \gamma B \) (Black)

- \( \Delta w = \gamma \Delta B \)

- \( \rho(w) \) behavior: \( y = \frac{C}{c+x^2} \) (fixed height)

- [subt \( \rho(w) \), rearrange to extract \( \omega_0 \), take integral]

\[
\tilde{S}(t) = \frac{\pi \cdot M_0^2 \cdot \gamma \Delta B \phi \cdot \sin \alpha}{C^2 + C^2 + w^2} e^{-t/T_2} e^{-i\omega_0 t}
\]

- complex signal

- from integral of \( \phi \)

- \( \Delta w \) (big \( \Delta \) big sig.)

- (big \( \Delta \) big decay)

- regular T2 decay

- oscillations (complex) (one freq.)

- [combine T2 + static terms]

\[
\tilde{S}(t) = \pi \cdot \frac{M_0^2 \cdot \gamma \Delta B \phi \cdot \sin \alpha}{C^2 + C^2 + w^2} e^{-t/T_2} e^{-i\omega_0 t}
\]

- overall decay rate including inhomogeneous \( B \phi \)

\[
\frac{1}{T_2^{*}} = \frac{1}{T_2} + \frac{1}{T_2'}
\]

- unrecognizable/recognizable "intrinsic" "static"

- e.g. extra decay from RF

- echo can cancel fix!

- suggestively, since 10^6 cycles/sec

- N.B. complex \( \rho \), not integrable

- only approximated by \( e^{-t/T_2^{*}} \)
**ECHOES** — spin echo

1. **Just after 90° x' pulse**
   - $f_{lo} + f_{hi}$ have same phase
   - Relaxation + phase dispersion of $f_{lo} + f_{hi}$ (both from $B > B_0$)

2. **Just after 180° y' pulse**
   - y' pulse like x' pulse but RF has +90° phase
   - Echo caused by re-phasing of $f_{lo} + f_{hi}$ (w/ decay due to $T_2$)

- Remember brief RF just tip vectors while retaining length
- Relaxation includes tips and shrinks ($M_r$) and grows ($M_z$, echo)
- 180° x' pulse works, too, but echo will have +π phase (left side in Figs above)

- Echo generated even if second pulse not 180° (see next)

---

**Diagram**

- FID decay (and echo growth/decay)
  - Described by $T_2^*$
  - From inhomogeneity

- Reduction in height of echo compared to initial described by $T_2$
  - Echo fixes $T_2^*$
**ECHOES — spin echo**

\[ \alpha_1 - T - \alpha_2 - T \]  (both pulses along \( y' \) for simplicity)

**effect of \( y \) pulse**

\[
\begin{align*}
M_x' &\rightarrow M_x' \cos \alpha - M_z' \sin \alpha \\
M_y' &\rightarrow M_y' \\
M_z' &\rightarrow M_x' \sin \alpha - M_z' \cos \alpha \\
\Leftrightarrow \text{(etc for } \alpha_x, \alpha_z) &
\end{align*}
\]

**General transforms (operations)**

**effect of \( T \) delay**

\[
\begin{align*}
M_x' &\rightarrow (M_x' \cos \omega T + M_y' \sin \omega T) e^{-T/T_2} \\
M_y' &\rightarrow (-M_x' \sin \omega T + M_y' \cos \omega T) e^{-T/T_2} \\
M_z' &\rightarrow M_z' (1 - e^{-T/T_1}) + M_z' e^{-T/T_1}
\end{align*}
\]

**immediately after \( \alpha_1 \) pulse**

\[
\begin{align*}
M_x'(w, 0_1) &= -M_z'(w) \sin \alpha_1 \\
M_y'(w, 0_1) &= 0 \\
M_z'(w, 0_1) &= M_z'(w) \cos \alpha_1
\end{align*}
\]

**for one isochromat of freq. \( w \)**

**after \( T \) delay**

\[
\begin{align*}
M_x'(w, T) &= -M_z'(w) \sin \alpha, \cos \omega T e^{-T/T_2} \\
M_y'(w, T) &= M_z'(w) \sin \alpha, \sin \omega T T e^{-T/T_2} \\
M_z'(w, T) &= M_z'(w) \left[ 1 - \left( -1 \cos \alpha \right) e^{-T/T_1} \right]
\end{align*}
\]

**immediately after \( \alpha_2 \) pulse (no effect on \( M_y' \); rewrite \( y' \); combine \( x \) and \( y \) eqn's)**

\[
\begin{align*}
M_x'(w, T) &= M_z'(w) \sin \alpha_1 \left( \sin^2 \omega T e^{-i\omega T} - \cos^2 \omega T e^{i\omega T} \right) e^{-T/T_2} \\
&\quad - M_z'(w) \left[ 1 - \left( 1 \cos \alpha \right) e^{-T/T_1} \right] \sin \alpha_2
\end{align*}
\]

**time dependent**

- Free precession around \( z' \) (rewrite \( M_x', M_y', \ldots \))

- For a large num of freq's:

\[
\begin{align*}
\text{terms } 1 &\text{ are dephasing} \rightarrow \text{FID of echo} \\
\text{terms } 2 &\text{ are dephasing} \rightarrow \text{FID of echo} \\
\text{term } 3 &\text{ is a nesphasing} \rightarrow \text{nephase at } t = 2T
\end{align*}
\]

\[
\begin{align*}
\sigma(t) &= \sin \alpha_1 \sin^2 \omega_2 \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/T_2} e^{-i\omega(t-TE)} \, dw \\
A_E &= \sin \alpha_1 \sin^2 \omega_2 \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-TE/T_2} \, dw
\end{align*}
\]

**Peak amp**

\[
\begin{align*}
\sigma_1(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/T_2} e^{-i\omega(t-TE)} \, dw \\
\sigma_2(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/T_2} e^{-i\omega(t-TE)} \, dw \\
\sigma_3(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/T_2} e^{-i\omega(t-TE)} \, dw
\end{align*}
\]

\[
\begin{align*}
\sigma_1(t) &= 100 \% \\
\sigma_2(t) &= \text{no } \frac{1}{2} \text{ factor} \\
\sigma_3(t) &= \text{multiply by } i \rightarrow \text{add } \frac{1}{2} \text{ phase}
\end{align*}
\]

\[
\begin{align*}
M_z \left( \frac{\sin \alpha_1 \sin^2 \omega_2}{2} \right) e^{-TE/T_2}
\end{align*}
\]

\[
\begin{align*}
\text{etc for } A_E \ldots \text{like a}
\end{align*}
\]
**Echo TRAINS** - spin-echo trains

- It's (too) easy to make echos...

\[
E_n = \frac{3^{n-1} - 1}{2}
\]

- 3 RFs \(\rightarrow\) 4 echos (here)
- 6 RFs \(\rightarrow\) 12! echos (!!)

- **Typically**, 90° and 180° applied in different axes \((x', \text{ then } y', \text{ etc.})\)
- which reduces phase errors due to imperfect 180° pulses
- (since slightly off rotation around \(y'\) affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize 90°, 180°)
- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_z$
- Problem #2: Third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

\[ \text{Rule for effect of } \alpha \text{ RF pulse on transverse mag} \]

\[ \text{Rule for effect of } \alpha \text{ RF pulse on longitudinal mag} \]

\[ \text{Echo when phase path crosses zero} \]
3 - Pulse Echo Amplitudes

- Assume $M_0^z = 1$

RF transmit

RF receive

Echo | Time | Amplitude
--- | --- | ---
$SE_{1,2}$ ($t = 2T_1$) | $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2T_1/T_2}$ | $\alpha_1 = 90^\circ, \alpha_2 = 180^\circ$ or $e^{-2T_1/T_2}$
| | | $2\alpha_1 = \alpha_2$ or $\sin^3 \alpha_1 \cdot e^{-2T_1/T_2}$

$SE_{2,3}$ ($t = T_1 + 2T_2$) | $\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-T_3/T_1} e^{-2T_1/T_2}$ | (N.B. T1)

$SE_{1,3}$ ($t = 2(T_1 + T_2)$) | $\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + T_2)/T_2}$ | (N.B. T1)

- $T_1$-dependence in STE (but also SE$_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**Hyper Echoes**

1. **Echoes**
   - **N.B.: Coord start to put z horiz vs. Bloch notes**
   - **Start M phase evol.**
   - **3 Symmetries**
   - **Ignore amplitude**
   - **This is spin echo!**
   - **2 solid lines 1 dashed line**

2. **Rotation of M around tilted axis in transverse x-y plane by RF with flip, α, and phase, φ = P(α, φ)**
   - **1) vect. rotation of M around tilted axis in transverse x-y plane by RF with flip, α, and phase, φ = P(α, φ)**
   - **2) Rotation around z by phase evolution due to freq. offset, ω (Bo offset) and time, t = ω(t)**
   - **Three symmetries:**
     - **Solid lines:** phase evol. at RF flip or RF flip end or RF flip
     - **Dashed lines:** just 180° echo again

3. **By combining long sequences observing these symmetries, can generate as strong echo even w/ many inserted 90° pulses in between**

**Practical use**

- Multi-echo example
- Can also use to prepare, then separate read-out
- Practical prob: 180° pulses deposit a lot of RF (6x 90°)
- Prob at high fields
- By arranging to get big echo in middle of k-space, can get by with much less RF power

- Hennig & Scheffler (2001)
- Normalize M amplitude = 1
- Sphere surface defines 2D space for M moved by

---

*Note:* The image contains handwritten text and diagrams related to hyper echo sequences in magnetic resonance imaging (MRI). The text explains the principles behind hyper echo sequences, including the rotation of the magnetization vector (M) around a tilted axis in the transverse x-y plane using a RF pulse, and the effect of phase evolution due to frequency offsets and time. The practical applications and considerations for using such sequences are also discussed.
**Gradient Echoes** - T2*, GE chains

- Initial negative gradient dephases spins
- After $t = T$ of positive gradient, spins rephase
- Does not correct for T2* inhomogeneities, so echo amplitude is $A_E = e^{-t/T2*}$
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

**Key Difference** between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled

- Hence, echoes are T2*-weighted, not T2-weighted => more susceptible to inhomogeneities

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get

- EPI hardaware => 64 echoes
IMAGE CONTRAST

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

RF

\[ \begin{array}{c}
1 \quad 70^\circ \quad \text{TR} \\
2 \quad 3 \quad 4 \\
\end{array} \]

\[ M_z \text{ longitudinal magnetization} \]

\[ M_z^0 \]

\[ M_z^0 (1 - e^{-\text{TR}/T1}) + [\text{ignored}] \]

- "steady state" after here

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ immed. after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, \( M_z \) just before second pulse:

\[ M_z'(0^-) = M_z^0 (1 - e^{-\text{TR}/T1}) + M_z^0 (0^-) e^{-\text{TR}/T1} \]

\[ M_z \text{ before current pulse} \]

\[ M_z^0 \text{ "regrowth-from-zero" term} \]

\[ M_z^0 \text{ "left-immed.-after-pulse" term (i.e., decaying)} \]

- Given:
  \[ (1) \ 90^\circ \text{ pulse} \]
  \[ (2) \text{ no } M_{xy} \text{ left} \]

\[ \text{pure tip: } M_{xy} = M_z \]

- Tip existing mag

\[ M_z'(0^-) = M_z^0 \]

\[ M_{xy}'(0^-) = M_z^0 (1 - e^{-\text{TR}/T1}) \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(\mathbf{r}) = C \int \rho(\mathbf{r}) (1 - e^{-\text{TR/T1}(\mathbf{r})}) \text{ rem const., spectral dens} \]

\[ \rho(\mathbf{r}) \text{ p. density, underlies eqilib. } M_z^0 \]
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
  - because they are brighter than all the rest
  - because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur (e.g. at 3T, flip angle varies almost 25% across brain)

---

- at 3T, steady state
  - for typical 1-2 sec
  - TR images reached after 8 images
**IMAGE CONTRAST**

IR (still just saturation-recovery — no echo)

- inversion recovery w/ no echo

\[
\begin{align*}
&\text{RF} \\
&T = T_1, 90^\circ, T_D, 180^\circ, 90^\circ, 180^\circ, 90^\circ \\
&\text{"steady state" after here}
\end{align*}
\]

- 180° pulse reverses longitudinal magnetization

\[ M_z'' = -M_z^0 \]

- recovery to end of first TI from long. part of Bloch eq.

\[ M_z' = M_z^0 \left(1 - 2e^{-\frac{TI}{T_1}}\right) \rightarrow \text{flipped into transverse by second pulse (180°)} \]

- longitudinal then regrows from zero

\[ M_z' = M_z^0 \left(1 - e^{-\left(\frac{TR - T_1}{T_1}\right)}\right) \]

- after second 180°, just change sign again

\[ M_z' = -M_z^0 \left(1 - e^{-\left(\frac{TR - T_1}{T_1}\right)}\right) \]

- apply relaxation eq. again

\[ M_z' = M_z^0 \left(1 - e^{-\frac{TI}{T_1}}\right) - M_z^0 \left(1 - e^{-\left(\frac{TR-TI}{T_1}\right)}\right) e^{-\frac{TI}{T_1}} \]

\[ M_z' = M_z^0 \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_1}}\right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
IMAGE CONTRAST

\[ \text{SE, IR-SE} \]

- steady state mag \((2^\text{nd TR})\) just before 90°
  \[ M_z, (t) = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}} + e^{-\frac{TR}{T_2}} \right) \]

- the echo signal \((M_z^t)\) unlike in simple saturation-recovery FID
  has an additional delay before it is recorded, so
  we have to take account of transverse mag relaxation
  \[ A_E = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}} + e^{-\frac{TR}{T_2}} \right) e^{-\frac{TE}{T_2}} \]

- if we assume \(TE\) much less than \(TR\), then we can simplify:
  \[ A_E = M_z^0 \left(1 - e^{-\frac{TR}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

- similar equation for SE-IR
  \[ A_E = M_z^0 \left(1 - 2e^{-\frac{TR}{T_1}} + e^{-\frac{TR}{T_2}} \right) e^{-\frac{TE}{T_2}} \]
Image Contrast

GRE w/ small tip angle

\[ \alpha_{n-1} (< 90^\circ) \]

1. RE = slice select
2. TE = refocus slice select
3. Gy = phase encode
4. Gz = make echo + center of echo
5. TR = for IR

\[ M_{\perp}^{(n)}(O_-) = M_{\perp}^0 (1 - e^{-TR/T_2}) + M_{\perp}^{(n-1)}(O_+) e^{-TR/T_1} \]

- Use basic longitudinal relaxation from Bloch eq, again
- Assume \( M_{\perp}^{(n)}(O_-) = 0 \) → transverse dephased before next pulse

\[ M_{\perp}^{(n)}(O_-) = M_{\perp}^0 (1 - e^{-TR/T_2}) + M_{\perp}^{(n-1)}(O_+) e^{-TR/T_1} \]

Assume we have a small tip angle:

\[ M_{\perp}^{(n)}(O_+) = M_{\perp}^0 (O_-) \cos \alpha \]

\[ M_{\perp}^{(n)}(O_-) = M_{\perp}^0 (1 - e^{-TR/T_2}) + M_{\perp}^{(n-1)}(O_-) \cos \alpha e^{-TR/T_1} \]

Assume we are in dynamic equilibrium:

\[ M_{\perp}^{(n)}(O_-) = M_{\perp}^{(n-1)}(O_-) = M_{\perp}^{ss}(O_-) \]

Pre-pulse steady state longitudinal

\[ M_{\perp}^{ss}(O_-) = \frac{M_{\perp}^0 (1 - e^{-TR/T_1})}{1 - \cos \alpha e^{-TR/T_1}} \]

Post-pulse transverse magnetization

\[ M_{\perp}^{ss}(t) = \frac{M_{\perp}^0 (1 - e^{-TR/T_1}) \cdot \sin \alpha e^{-TE/T_2}}{1 - \cos \alpha e^{-TR/T_1}} \]

Gradient echo amplitude

\[ A_E = \frac{M_{\perp}^0 (1 - e^{-TR/T_1}) \sin \alpha e^{-TE/T_2}}{1 - \cos \alpha e^{-TR/T_1}} \]

T1 contrast mainly depends on flip angle, not TR → cos \( \theta = 1 \) → eliminates T1 weight since denom = numerator
**IMAGE CONTRAST**

- **MDEFT / 3D FLASH**

- Saturate, wait for contrast, invert, wait for contrast, **FLASH** (center out)

A) \( M_z' \) (just after 90°) = 0 (perfect 90°)

B) \( M_z' \) (after TD) = \( M_z^0 \left(1 - e^{-\text{TD}/T_2}\right) \)

C) \( M_z' \) (just after invert) = \( \cos \phi M_z^0 \left(1 - e^{-\text{TD}/T_2}\right) \)

D) \( M_z' \) (after TI) = \( M_z^0 \left(1 - e^{-\text{TI}/T_2}\right) + \left[\cos \phi M_z^0 \left(1 - e^{-\text{TD}/T_2}\right)\right] e^{-\text{TI}/T_2} \)

- Special case TI = TD:
  \( M_z' \) = \( M_z^0 \left[1 - e^{-\text{TI}/T_2}\right]^2 \)

- After the first pulse:
  \( M_z' \) (just after the first pulse) = \( M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-\text{TD}/T_2}\right)\right] e^{-\text{TI}/T_2}\right] \sin \alpha \)
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules & bond to membranes have wide range of resonant freqs ("bound") \( \Rightarrow T_2 = 1 \text{ msec} \) i.e., signal not visible w/ usual TE

- Free protons in blood, CSF, water have narrow range of resonant freqs ("free") \( \Rightarrow T_2 = 50 \text{ msec} \)

- Mag transfer pulse sequence
  1) Off-center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal \( M_L \) of "bound" \( \Rightarrow M_L \) of "free"
  3) Result of transfer \( \rightarrow \) attenuation

\( \Rightarrow \) NB. this always happens a little (cf. T1-weighted, T2-weighted)
  something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines

\[ \max \rightarrow \text{view as movie} \]
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\text{avg obj signal}}{\text{avg s.d. non-object region}} \)
- Temporal SNR: \( \Delta \text{SNR} = \frac{\Delta S}{\sigma_s} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:
  \[
  \text{CNR}_{AB} = \frac{S_A - S_B}{\sqrt{\sigma_s^2}} = \text{SNR}_A - \text{SNR}_B
  \]

### Spin-Echo:

\[
A_e = M_e^0 (1 - e^{-TR/T1}) e^{-TE/T2}
\]

### Gradient Echo:

\[
A_e = \frac{M_e^0 (1 - e^{-TR/T1})}{1 - \cos \alpha} e^{-TE/T2\alpha}
\]

**MRI Tissue Parameters**

<table>
<thead>
<tr>
<th>Tissue</th>
<th>T1</th>
<th>T2</th>
<th>PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>950</td>
<td>100</td>
<td>0.89</td>
</tr>
<tr>
<td>WM</td>
<td>600</td>
<td>80</td>
<td>0.45</td>
</tr>
<tr>
<td>CSF</td>
<td>2500</td>
<td>2200</td>
<td>1.60</td>
</tr>
<tr>
<td>blood</td>
<td>1200</td>
<td>100</td>
<td>2.00</td>
</tr>
</tbody>
</table>

**Last page**

- General rules: spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Proton Density Weighted</th>
<th>TR HH (no T1 diffs)</th>
<th>TE HH (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR HH (big T1 diffs)</td>
<td>TE HH (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR HH (no T1 diffs)</td>
<td>TE HH (big T2 diffs)</td>
</tr>
</tbody>
</table>
**SIGNAL-TO-NOISE S/N**

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{N_a N_x N_y N_z \Delta t}
\]

- Size (volume) of voxels (with the number of voxels held constant)
  - Linear effect on S/N
  - \( \Rightarrow \) e.g., \(3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N} \)

- More voxels (with size of voxels, \( \Delta t \) per read step constant)
  - \( T_n \) effect on S/N
  - \( \Rightarrow \) e.g., \(64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{128 \times 128}{64 \times 64} = 2 \text{ times better S/N} \)

- \# acquisitions \( T_n \) better S/N
  - \( \Rightarrow \) e.g., \(1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{72}{11} = 1.41 \text{ times better S/N} \)

- Larger timestep during readout, \( \sqrt{\Delta t} \) better S/N

\( \Delta t = \frac{1}{\text{BW}_{\text{read}}} \), digitization timestep during echo acquisition

- \( \text{BW}_{\text{read}} \) determined by cutoff freq, analog low-pass filter
- \( \Delta t \) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \( \Delta t \)
- Must filter out freq's \( > f_{\text{max}} = \frac{1}{2 \Delta t} \) because they alias
COMPLEX ALGEBRA

real/imaginary

add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)

mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

angle/phase

add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)

commute: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)

divide: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1 / A_2, \phi_1 - \phi_2)\)

complex to real power: \((A, \phi)^n = (A^n, n\phi)\)

\[ e^{i\phi} = [\text{expand as series}]
\]

recourage cos, sin series

\[ = \cos \phi + i \sin \phi \]

= vector on unit circle

\[ e^{i\phi} = (\cos \phi + i \sin \phi)^n \]

= \cos n\phi + i \sin n\phi

- how to convert:

\((r, i) \leftrightarrow (A, \phi)\)

\[ A = \sqrt{r^2 + i^2}, \phi = \arctan(i/r) \]

N.B.: 2nd kind of vector mult.

different than dot product and cross product (and G.A.
non-commutative pseudoscalar multiply)

short-hand for a unit vector(2b)

= in the direction of \(\phi\)

- phase is freq. variable

\[ \phi = \int \omega \, dt \]

Fourier transform

\[ H(f) = \int h(t) e^{-2\pi if t} \, dt \]

\[ H(\hat{f}) = \int h(t) e^{2\pi if t} \, dt \]

Convolution

\[ f(x) = g(x) * h(x) = \int g(z) \cdot h(x-z) \, dz \]

\[ \phi(x) = g(x) \otimes h(x) = \int g(z) \cdot h(x+z) \, dz \]

Convolution Theorem

\[ F[g(x) \cdot h(x)] = G(k) \ast H(k) \]

\[ \text{because of FFT, faster if kernel not small} \]
Fourier transform (1)

For one $f$

$H(f) = \int_{-\infty}^{\infty} h(t) e^{-i \frac{2\pi ft}{\text{cyc/sect}} \cdot t} dt$

Sum across $t$ (complex add)

How to calculate $H(f)$ for one $f$ ($f=3$):

(real signal: only need 2 correlations)

$H(t)$

$\cos (2\pi 3t)$

$\sin (2\pi 3t)$

Complex multiply

Real frequency domain

Imaginary frequency domain

Cartesian (r, $\phi$)

Amplitude frequency domain

Phase frequency domain

like correlating with $\sin$ and $\cos$ (at each freq) so we get phase (at each freq.)
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos(\phi) + i \sin(\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \( \cos \) is an "even" function, \( \sin \) is an "odd" function

An orthogonal decomposition

- think of discretely sampled \( \sin(bx) \), \( \cos(bx) \) as vectors
- \( \text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \) projection of \( \vec{v}_1 \) onto \( \vec{v}_2 \)

\[
\begin{align*}
\text{Corr}(\cos b_1 x, \sin b_1 x) &= 0 \\
&= \text{Sin} \neq \text{Cos \& same frequency are orthogonal} \\
&= \text{Sin} 2x \quad \text{Cos} 2x \\
\text{Corr}(\sin b_1 x, \sin b_2 x) &= 0 \\
&= \text{different integer freqs of Sin \& Cos are orthogonal} \\
&= \text{Sin} 2x \quad \text{Sin} 3x \\
\text{Corr}(\cos b_1 x, \sin b_2 x) &= 0 \\
&= \text{as above}
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{x=h_i}^{x=h_0} f(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR w/ COS, SIN

- Start with spike in image domain
- Take example of spike at $x = 0$
  \[
  \begin{bmatrix}
  \cos(x), \cos(2x), \cos(kx)
  \end{bmatrix}
  \text{all freq's correlate w/ spike at } x = 0
  \]

- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates.

- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-ixk}$ cos and sin at location of spike.

- Pos, pair (real) spikes same dist from origin
- Pos/neg, pair (imaginary) spikes, same dist orig.
- One spike at distance from origin

- This is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image \(\rightarrow\) imaginary image
   \[\text{Real spac. freq.} \rightarrow \text{Imag. spac. freq.}\]
   \[\text{Inverse Fourier Transform}\]

2. Amplitude image \(\rightarrow\) phase image
   \[\text{Ampl. spac. freq.} \rightarrow \text{Phase spac. freq.}\]
   \[\text{View complex vectors directly}\]

3. Complex vectors
   \[\text{Zero vectors}\]
   \[\text{View complex vectors directly}\]

- 3 equivalent representations of image & spac. freq. space
FOURIER TRANSFORM OF REAL IMAGE (3)

- What a single k-space point looks like
  for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

Image Space

K-space (spatial freq. space)

Amplitude

Brightness of stripes proportional to k-space amplitude

Phase

Distance from center is stripe spacing

Angle of point perpendicular to angle of stripes

Inverse Fourier transform (image recon.)

N.B.: not same as "stripe phase" above

Cartesian dimension of k-space — x- and y- spatial freq.

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin$t$ cos — don't confuse $k_x, k_y$ w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq. of the 2D wave and rotates it
- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)
- Example: Cosinusoid in image space, then shifted in x-dir

**REAL IMAGE**

\[ I(x, y) = \cos(x) \]

**FT OF REAL IMAGE**

\[ FT(I(x, y)) \]

**I(x, y) = \cos(x - \pi/4)** → halfway between cos and sin (shifted 45° to right)

**N.B.: an example of the "Fourier Shift Theorem"** (see below)

**45° rot compared to complex above**

**Phase now 45° at**\[ k_x = 1 \]
\[ k_y = -1 \]

**Complex**

**Real component less than above because rot:**
FOURIER TRANSFORM OF IMAGE (S)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

\[ \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{R} \\
  \text{zero} & \rightarrow \text{zero}
\end{align*} \]

\[ \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{C} \\
  \text{max} & \rightarrow \text{complex}
\end{align*} \]

FT OF REAL IMAGE

\[ H(k) = \int \overline{h(x)} e^{-i\pi k x} dx \]

avg image brightness \(= I(\text{real}) \)

\[ \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{C} \\
  \text{zero} & \rightarrow \text{zero}
\end{align*} \]

[the center of the complex k-space is zero with pure sine or cosine image b/c avg. brightness = 0]

*COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) \]

\[ e^{-ix} \]

\[ \begin{align*}
  \mathbb{R} & \rightarrow \mathbb{C} \\
  \text{flat} & \rightarrow \text{complex}
\end{align*} \]

FT, FT\(^{-1}\)

"Missing" spike results in single spike correlating with \(\cos \) and \(\sin\)

N.B.: this k-space is not Hermitian:

k-space will only have Hermitian symmetry if image is real:

Hermitian symm. when complex conjugate (\(= \text{complex} \) mm w/ sign flipped in image part) is equal to function \(w/\) phase arg:

1D: \( H(k) = H^*(k) \)

2D: \( H(-k_x, k_y) = H^*(k_x, k_y) \)

FT OF COMPLEX IMAGE

spike only on one side of k-space

[N.B. this is exactly what an artifact "spike" does; tho it would have real phase]

[N.B. this is also exactly what a gradient does to image space! ]
GRADIENT COILS

- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the Z-component of the magnetic field \( B_z \).

- For example, the x gradient coil induces a ramp in z-component of the magnetic field when moving in the x-direction:

\[
B_{G,z} = G_x \times
\]

* Since a pure linear gradient of \( B_{G,z} \) in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction (\( B_{G,x} \) and \( B_{G,y} \)).

- The other magnetic field components are usually ignored because they are so small relative to \( B_{G,z} \), since \( B_{G,z} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{G,x} \), \( B_{G,y} \), and \( B_{G,z} \).

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced:

\[
\Delta \phi_G(x) \approx -\frac{x^2 G_x^2 t}{2B_0}
\]

- The Maxwellian terms \( B_{G,x} \) and \( B_0 \) are known and can be included in the reconstruction process.
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim

$$B_z = \frac{\pi}{G_z} \left( B_0 + B_{g,z} \right)$$

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

in practice, Gaussian pulse envelope good too

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq. encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: $90^\circ \rightarrow 52\%$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

Fourier approach details

- Fourier transform approach to slice-selective pulse (linear approx. even though tipping is not linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} df \]

- Time dependent RF stimulation (complex)
- Frequency selection function

Solve with: \( p(f) = \) frequency band:

\[ B_1(t) = A : f_w \ \text{sinc} (\pi f_w t) e^{-i \pi f_c t} \]

- Amplitude controlling flip angle
- Sinc envelope width determined by freq. width \( f_w \) (N.B. wider \( f_w \) is narrower sinc)
- Modulation (complex) at center freq. \( f_c \)

\[ \text{i.e., time-dependent complex (e.g., quadrature) pulse waveform is Fourier transform of frequency spectrum of RF pulse} \]

- Sinc envelope width inversely proportional to \( f_w \)
- Larmor oscills., at center freq.

---

Fourier Transform Pairs, Rules

- Convolution in one domain is multiplication in the other
- Convolution with delta func. impulse moves function to impulse center

---

Fourier Transform Solution to: \( \frac{A}{\varepsilon} \)
SLICE SELECT RF PULSES

Interleaved Acquisition \(\rightarrow\) better S/N &c imperfect slice profile
\(\Rightarrow\) spin history prob if motion

Common RF pulses

- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian

\(\Rightarrow\) pulses need to be "apodized" (have "foot" removed)
\(\Rightarrow\) multiply by function so begin/end of pulse is differentiable

Fat Saturation

- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
\(\Rightarrow\) need high quality (narrow-freq) pulse to avoid saturate water!

HOWTO

1) Fat sat pulse
2) wait T2 so fat signal decays, but no T1 regrowth of fat
3) RF stim for water "protons of interest"

Adding Another Gradient Tilts Slice

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude or RF freq band
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

1) chemical shift change freq \(\rightarrow\) gradient changes freq.
2) stimulate w/ broadband RF \(\rightarrow\) same
3) time-sample FID containing multiple freqs \(\rightarrow\) same
4) FT of FID to get spectrum \(\rightarrow\) FT of FID to get \(\Delta x\) offsets

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)
  - i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>spectroscopy</th>
<th>MRI</th>
</tr>
</thead>
<tbody>
<tr>
<td>samples of oscillations in time-domain</td>
<td>samples of spatial freq. in freq. domain</td>
</tr>
<tr>
<td>FT (\rightarrow) frequency-domain spectrum of shifts</td>
<td>FT (\rightarrow) spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because \(FT \approx FT^{-1}\) (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction

- Different frequency signals are mixed together and recorded as a 1-D signal over time

- A Fourier transform, which can convert back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

- Spatial frequencies get confused/confounded with precession frequencies

**N.B.:** Gradient freq. ramp does not need to be exactly same as recording!!

→ Conceptually wrong!!

→ FT actually converts spatial frequencies to spatial position

→ The spatial frequency increases for each time point in the readout

→ The precession freq. ramp is constant each timestep
FREQUENCY ENCODING (2)

- "Frequency"-encode gradient ($G_x$) turned on during
  during echo causes precession rates to immediately vary with $x$-position

  \[ G_x \rightarrow t \rightarrow \]

  \[ \uparrow B_x \text{ in } x \text{-direction} \]

  \[ G_x \text{ levels} \quad (= \text{slope}) \]

  \[ \text{actually} \quad \downarrow \]

  \[ \uparrow B_x, x \]

- at beginning of gradient on, the phase of signal coming from each $x$-position is the same
  summed phase angle is what we measure

- early after gradient on, phase advances (because of faster precession frequency) arise with greatest phase advance at largest $x$-position

  \[ \text{Single} \quad \text{time point} \quad \text{early} \]

  \[ t \rightarrow \]

  \[ \text{360} \rightarrow \]

  \[ \phi \rightarrow \]

  \[ \text{\text{0} \rightarrow \text{one cycle of spatial frequency \phi} \rightarrow \text{phase angle} \rightarrow \text{0} \rightarrow \text{low spatial freq}} \]

- later during gradient on, phase advances cause multiple wraps around of phase angle across space

  \[ \text{Single} \quad \text{time point} \quad \text{later} \]

  \[ t \rightarrow \]

  \[ \text{360} \rightarrow \]

  \[ \phi \rightarrow \]

  \[ \text{0 \rightarrow multiple cycles of spatial frequency \phi \rightarrow phase angle \rightarrow 0} \rightarrow \text{hi spatial freq}} \]

- in practice, the lowest spatial frequency (\(=0\)) occurs in the middle of the gradient on time because the phase is "Wrapped" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

  \[ \phi \rightarrow \text{max neg} \rightarrow \phi \rightarrow \text{0} \rightarrow \phi \rightarrow \text{max positive} \rightarrow \]

  \[ f = \text{spatial frequency} \]

  \[ G_x \rightarrow t \rightarrow \]

  \[ f = \text{max neg} \rightarrow f = \text{0} \rightarrow f = \text{max positive} \rightarrow \]

  \[ \text{individual RF data samples (after demodulation)} \]
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier Transform: (Temporal freq ↔ time)

\[
H(f) = \int_{t=-\infty}^{t=\infty} h(t) e^{-i 2\pi ft} dt
\]

"k" is often used instead of "f" for the frequency variable

Imaging equation: (Spatial freq ↔ space)

\[
S(f) = \int_{x=-\infty}^{x=\infty} I(x) e^{-i 2\pi fx} dx
\]

Sum across x of object

G_x RF

Get this single readout point by summing signal across x-position (RF coil records sum)

Even though variable is f, it represents one time point during readout

To make image, do inverse Fourier transform of recorded signal S(f)

Increases oscillations come from readout phase wrapping, where f is single spatial freq (e.g., 5) and x goes across object

F: G_x
SE: G_x (t-TE)

f = G_x, that is, spatial freq depend on amount of time gradient was on (this f increases with time!)

don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)
**Alternate Derivation (incl. effects of \( G_x \)) Signal Eq**

- Oscillators at \( \omega = \gamma B \) at each position (just \( x \) for now)

\[
S(t) = m(x) e^{-i \phi(x)} dx
\]

- By definition, freq. \( \omega \) is rate of change of phase, \( \phi \)

\[
\frac{d\phi(x,t)}{dt} = \omega(x,t) = \gamma B(x,t) \quad \text{and} \quad \phi(x,t) = \int_0^t \omega(x,t) dt = \int_0^t \gamma B(x,t) dt
\]

- Assuming phase initially 0, \( B \) affected by gradients

\[
B(x,t) = B_0 + G_x(t) x
\]

So

\[
\phi(x,t) = \gamma \int_0^t B_0 dt + \left[ \gamma \int_0^t G_x(t) dt \right] x
\]

= \( \omega_0 t + 2\pi k_x(t) x \)

- Demodulation removes the \( B_0 \)-caused carrier frequency \( e^{-i\omega_0 t} \) from the first equation

\[
S(t) = \int_x m(x) e^{-i 2\pi k_x(t) x} dx
\]
**PHASE-ENCODE GRADIENT Gy**

- Turn on gradient after excitation but before readout.
- Different levels of Gy:
  - Bz_y
  - Gy
  - Gy
- Higher levels of Gy (slope of Bz in y-direction!)
  \[ \Rightarrow \text{higher spatial freq. (more phase wraps) in } y \text{-direction} \]
- Phase wraps persist after phase-encode gradient off.
- Read-out gradient (Gx) phase wraps then add to phase-encode phase.

**2D Imaging Equation**

\[
S(k_x, k_y) = \iint S(k_x, k_y) \cdot e^{-i \omega \cdot (k_x x + k_y y)} \, dx \, dy
\]

- Signal recorded at a single time point (one readout point)
- Complex signal (from phase-sensitive detection)
- Sum across x-y plane
- Image = strength of magnetization at each x-y point
- Done by RF coil
- Scalar (what we try to reconstruct)
- Phase (vector of unit length and particular angle which is function of Gx and Gy)
- Phase angle (of scalar magnetization?) in rotating frame, set by gradients

Ignoring relaxation, spatial frequency \( k_x \) and \( k_y \) have no "inertia"—they stay wherever the gradients last left them.
3-D IMAGING — two phase-encode gradients

- use z-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

\[ S(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, z) e^{-i2\pi(k_x x + k_y y + k_z z)} \, dx \, dy \, dz \]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase; all just add (= 3D rotation of phase stripes)
- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
  -> phase stripes created throughout volume vs. slice

**N.B.** this ignores relaxation effects for now

**one readout point in image space**
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

- Successive read out steps:
  - More rotation
  - Higher spatial freq
  - Small phase encode G_y
  - Large phase encode G_y

N.B.: Stripes have sharp edges from phase wrap (not sinusoid sine of from 2-comp quadrature!)

- 3D phase encode w/ G_y and G_z starts rotated in y-z plane
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[ K = \frac{\chi}{t} \int_0^t G(t) \, dt \]

spatial freq recorded at t = record time
grad strength as function of t

simple form of integral w/ boxcar gradient

\[ K = Gt \]

(k is area under curve)

record data point here

all of the following gradients end up at the same point in k-space:

**Frequency-encode FID**

RF \( 90^\circ \)

Frequency-encode gradient echo

RF \( 90^\circ \)

Frequency-encode spin-echo (plus gradient echo!!)

Phase-encode then frequency encode gradient echo

RF \( 90^\circ \)
**IMAGE RECONSTRUCTION**

\[ S(k_x, k_y) = \int \int I(x, y) e^{i 2\pi (k_x x + k_y y)} \, dx \, dy \]

\[ I(x, y) = \int \int S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} \, dk_x \, dk_y \]

- in practice, finite number of samples, \( N \) and \( M \), are collected
- \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \rightarrow \) discrete sum)

\[ I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i 2\pi n \Delta k_x} \, \Delta k_x \right] e^{i 2\pi m \Delta k_y} \Delta k_y \]
**Sampling**

- Must consider effects of sampling limited points in k-space.
- Limited in range of frequencies sampled ($k_{min} \rightarrow k_{max}$).
- Limited in rate of sampling ($\Delta k$).

N.B., aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling.

- Correct reconstruction
- Infinite frequency range
- Infinately fine sampling

- Correct plus replicas
- Infinite frequency range
- Finite spacing of samples

[As above w/ blurring, ringing]

- Aliasing occurs in spatial domain
- Replicas overlap, causing wraparound

- Finite frequency range
- Too-wide spacing of samples

Thus, finer sampling of same range of spatial freqs increases FOV.
UNDER/OVER SAMPLE

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ \delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

FOV (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is FOV divided by K-space sample count

3 more examples (not incl. less samples to same spatial freq. [bottom last page])

**Basic Image**

**Spatial freq. K-space**

- Same num samp. to 2x spat. freq.
  - (i.e. gradients stronger or time ON longer)
  - N=10
  - \( k_x = 5 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV} = 1 \)
  - \( \delta_x = 0.1 \)

- 2x num samples to same spat. freq.
  - (i.e. gradients weaker or time ON shorter)
  - N=10
  - \( k_x = 10 \)
  - \( \Delta k_x = 2 \)
  - \( \text{FOV} = 2 \)
  - \( \delta_x = 0.05 \)

- 2x number samples to 2x spat. freq.
  - (i.e. gradients stronger or time ON longer)
  - N=20
  - \( k_x = 5 \)
  - \( \Delta k_x = 0.5 \)
  - \( \text{FOV} = 2 \)
  - \( \delta_x = 0.1 \)

- N=25
  - \( k_x = 10 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV} = 1 \)
  - \( \delta_x = 0.05 \)

- Basic image
- Square pix
- \( \text{x-pix half width} \)
- Replicas intrude
  - Scanner makes square image "wrap" occurs
- Square pix
  - Twice x-pix count so FOV = 2x
- This is "phase oversamp"
  - Scanner crops to square replicas move out
- \( \text{x-pix half width} \)
  - Twice x-pix count
  - Same FOV
- This is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Image/brain space
2. Sampled data spatial frequency

- Image convolve

- Multiply

= Equals

- Limit approach to Fourier transform of conv

Fov = \frac{1}{\Delta k}
\Delta k = \frac{1}{\text{Fov}}

Useful FT's

Rect

\text{Rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \cdot \text{sinc} (\pi wk)

Gaussian (special case)

e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2}

Gaussian (adj width)

e^{-a x^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi k^2}{a}}

Comb

\sum_{n=-\infty}^{\infty} \delta \left( x - \frac{n}{\Delta k} \right) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta \left( k - p \Delta k \right)
**Point Spread Function**

\[ \hat{I}(x) = \Delta k \sum_{n: [n,k]} S(n, \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(m, \Delta k) = 1 \]
- Substitute into \( \text{recon} \) to get PSF:
  \[ h(x) = \Delta k \sum_{n: [-N, N]} e^{i 2\pi n \Delta k x} \]
- Simplify
  \[ h(x) = \Delta k \frac{\sin(\pi N \Delta k x)}{\sin(\pi \Delta k x)} \Rightarrow \text{periodic} \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in K-space is an image sinc.

\[ \frac{\sin(Nx)}{x} \]

\[ \text{Image} \]

- How PSF modifies ideal (infinite k) image
  \[ \ast \text{ convolve} \]

\[ \text{FT} \]

- Multiply with acquisition window (truncated hi spati)

\[ \times \text{ multiply} \]

\[ = \text{ringing} \]
**GENERAL LINEAR INVERSE RECON FOR MRI**

\[
S(k_x) = \int \frac{I(x) e^{-i 2\pi k_x x}}{dx} \quad \text{Signal eq. \rightarrow fwd problem}
\]

\[
I(x) = \int_{k_x} S(k_x) e^{i 2\pi k_x x} \frac{dk_x}{dx} \quad \text{Recon eq. \rightarrow inv. problem}
\]

\[
s = F i
\]

\[
\begin{bmatrix}
  s \\
  x, y \\
  k_x \\
  k_y
\end{bmatrix} = \begin{bmatrix}
  F_i \\
  i
\end{bmatrix}
\]

Linear "forward solution"
- Matrix vectors have complex entries
- Can build in any measurable priors

\[
F_{x,y,t} = g(x,y) e^{-i\phi(x,y)} e^{-(\frac{mT \pm m\Delta t + T\gamma t}{2})} e^{-i\gamma B(x,y) x, y \Delta t} e^{-i\left(m\Delta k_x x + m\Delta k_y y\right) \Delta xy}
\]

Cal gain at this location
- Coil phase
- T2 decay
- B0 Error
  - (xy dep.)
  - T2 + phase

**Multi-coil**

\[
\begin{bmatrix}
  s \\
  1 \\
  2 \\
  3 \\
  \vdots \\
  \end{bmatrix} = \begin{bmatrix}
  F_{coil 1} \\
  \vdots \\
  F_{coil n}
\end{bmatrix} i
\]

Naturally incorporates undistorted field map
- Different sensitivity function for each coil!
- Contains additional info about source loc.
- But, need reference scan, lo-res OK
- Need phase corrections for each coil?

\[
i = F^+ s \quad \text{over-determined}
\]

More Powerful Inverse
\[
F^+ = (F^T F)^{-1} F^T \quad \text{Small}
\]

\[
F^+ (F F^T) \quad \text{(x,y, coils)} \rightarrow 16 \times bigger
\]

\[
i = [(F^T F)^{-1} F^T] s \quad \text{Slice-by-slice}
\]

Assume slice, select swamps. 88s
**FAST SPIN ECHO (FSE)**

- One 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.

- Each phase “winder” is “unwound” because leftover phase would be re-focused away by 180° (as EPI where it persists between blips).

**2DFSE**

- The “effective TE” is the TE when center of k-space is collected (largest effect on contrast, largest echo).

- Each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \) \( n = 1, 2, \ldots, M \)

- By arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted.

- Possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train.

- 3DFSE — like 2D except wind/unwind added to thick Slice select (w/ crushers on 180°).

- N.B.: Only one read phase, subsequent 180°s reset from right to left.

- N.B. all those 180° pulses deposit a lot of RF power: 90° + 180° = 45° × power 30°
MULTI-SLAB 3DFSE, PROBLEMS

- RF in
- Gz
- Gy
- Gx
- RF out

- Echo train e.g., 20
- Etc. to fill 3D k-space
  - Gz is "partition"
  - Gy is "phase encode"
  - Gx readout needs no pre-wind since 180° does it

- TE eff is time from 90° to echo thru center of k-space

- Echoes die out quickly \( \propto e^{-t/T_2} \)
- Since echoes after 90° limited to \( \leq 20 \), can't fill 3-D k-space in a reasonable time

- SAR constraint: \[ \text{SAR} \propto B_0^2 \rho^2 A^2 \] \( \rightarrow 180° \) pulses deposit 4-6x power of 90°

- "Multi-slab" is halfway between slices and single-slab

- Problem at slice boundaries — esp. movement

- Multislab requires slice selective RF pulses \( \rightarrow \) longer than non-selective "hard" pulses

- 4 ms RO
  - hard to get under 8 msec inter-echo spacing

- Limits speed of covering k-space
**SINGLE-SLAB 3D FSE**

- Regular FSE (180° pulse train)
  - Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
  - This "storage" in Z-axis preserves magnetization for longer time
  - Smaller flip angles allow much longer echo trains
  - Enough to collect whole plane of 3-D k-space
  - Different than hyper echoes (not symmetric)

- Contrast must consider STE

\[
\text{SE} = \sin \alpha, \sin^2 \alpha^2 e^{-2\alpha/\tau} \\
\text{STE} = \frac{1}{2} \sin^2 \alpha \sin^2 \alpha^2 e^{-\frac{1}{2} T_2} e^{-2\alpha/\tau}
\]

**Single-slab 3D FSE pulse seq.**

- Variable flip angle (<1 msec)
- Hard (non-selective) pulse not 180°
- Echo num
- Echo trains
- Long TE\text{eff (T2)}
- Short TE\text{eff (T1)}

**NB:** time to scan k-space is \( \propto S \times 5 \)
FAST GRADIENT ECHO (GRASS, FLASH, SPGR, MPRAGE)

- Small tip so TR can be greatly reduced (e.g., 10 msec, less than T2)
- 'Leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

STEADY-STATE COHERENT (GRASS, FISP)

- Unwind phase from phase-encode M\textsubscript{x} before next pulse (there because TR<TE)
- Unwind read gradient, too

STEADY-STATE SPOILED (SPGR, FLASH)

- Spoil with random gradient (but this still allows some \alpha\text{ refocusing})
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T\textsubscript{1}-weighted)

NON-STEADY STATE, MAGNETIZATION-PREP

- Preparation-pulse \rightarrow Strong T\textsubscript{1}-weighting
- Contrast varies in spatial-frequency-dependent way

MP-RAGE

- Longitudinal mag. not affect much by low angle pulses
- Record k\textsubscript{y} = 0 here
Quantitative T1 - Intro, Methods

Motivation

- Image values are arbitrary/relative (diff seg's, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - at 3T, B1 can vary by 25% across the brain
  - this can invert contrast in a fast gradient echo

Pre-scan normalise

- Collect low-res GE image, receive w/body coil (no coil fall-off)
- Set params. to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

T1 divided by T2

- MPRAGE ➔ strong T1-contrast
- SPACE ➔ T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems: noise in regions of low signal

MP2RAGE

- N.B. SSFP-like in partition, phase-encode dir
- Convert to -0.5 to 0.5 image: $S = \text{real}\left[\frac{\hat{S}_{\text{T1}} \cdot \hat{S}_{\text{T1}_2}}{\|\hat{S}_{\text{T1}}\|^2 + \|\hat{S}_{\text{T1}_2}\|^2}\right]$
- Calc. PD & T1 from above (cf. 2 flip angles)
QUANTITATIVE T1 - HELMS 2-FLIP ANGLE METHOD

- Start w/ gradient echo signal e.g., dropping T2-decay: $e^{-TE/2}$

$$ S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \left( 1 - e^{-TR/T1} \right) / \left( 1 - \cos \alpha \cdot e^{-TR/T1} \right) $$

- Simplify/linearize/estimate

  $TR \ll T1$

  Linear approx. of exponentials

  Taylor expansion, simplification of $\sin, \cos$, drop small term

Helms et al. (2008)

$$ S \approx A \cdot \alpha \cdot \frac{TR/T1}{\alpha^2/2 + TR/T1} $$

- Solve for TD and $A$ (proton-density) given signals from 2 diff flip angles

$$ T1_{\text{est}} = 2TR \cdot \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1} $$

$$ A_{\text{est}} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2 \alpha_2 - S_1 \alpha_1} $$

- Tiny error for flip $\leq 15^\circ$

- Problem: FLIP angle varies a lot of 3T (e.g., 25%) from nominal requested (e.g., flip series)

- acq. spin-echo and stimulated echo (EVI): $S = K \cdot \sin^3 \alpha \cdot e^{-TE/T2}$

- solve for $2\alpha'$

  $S_{\text{SE}} = K \cdot \sin^3 \alpha \cdot e^{-TE/T2}$

  $S_{\text{STE}} = K' \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TM/2T1}$

  $\alpha = \cos^{-1} \left( \frac{S_{\text{STE}} \cdot e^{-TM/2T1}}{S_{\text{SE}}} \right)$

- add EPI-like echo train to each FLASH excit.

Jiru & Klöse (2006)
ECHO PLANAR IMAGING (EPI) (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes

- Since there is only one RF pulse per slice, spins never get reset to all the same (= zero freq., center of k-space)

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it

- That explains why successive y phase-encode steps are achieved without changing the size of the Gy "blips"

- Echoes are T2*-weighted (gradient echo)

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes
**SPIN ECHO EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in T2*-weighting

- Deoxygenated hemoglobin is paramagnetic, which reduces signal in a T2*-weighted image due to greater dephasing.

- The excess of oxygenated hemoglobin (probably the result of the need to drive O2 into tissue, which requires more O2 in the blood than is actually used) leads to the positive BOLD effect.

- Spin echo corrects (cancels) static T2* (T2') dephasing, incl. deoxy.

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing.

- Diffusion exposes spins to different fields (reducing gradient echo dephasing).

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels.

- For TE ∼ 100 ms, spins diffuse 10’s of µm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time.

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here).

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells.

- Measure intra/extra v. bipolar pulse which kills signal in faster moving blood in moderate and larger vessels ⇒ over half of SE-BOLD at 1ST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence
- "spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space
- "asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2* weighting (for ky=0 echo)

-- the 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used

- at higher fields, spin echo EPI is more promising
  - signal to noise is higher so we can take spin echo hit
  - contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording
COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs

  but what does this look like in k-space?

- slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space

  (N.B. not addition!)

- to see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space - at all spatial frequencies!!

- simple example w/ "brain" consisting of one spatial freq:

  image domain

  \[ \text{image } \text{domain} \]

  \[ \text{FT} \]

  \[ \text{spatial freq. domain} \]

  \[ \otimes \text{(convolve)} \]

  \[ \text{FT} \]

  \[ \text{acquired image} \]

- N.B. inverse FT of k-space data "smeared" in spatial freq. space is sharp image w/ fall-off (not blurred image)

  "smeared" means normally orthogonal spatial freq's "leak" to adj. freqs.

  [GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center → across all coils]

  [SENSE - general linear inverse approach]

  - N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
- Excite multiple slices at once
- Function of $G_z$ blips is to shift slices in $G_y$ direction
- This occurs because for given slice, a phase pedestal is added to the entire slice
  - This "Fourier Shift Theorem"
  - [N.B.: different than $B_0$ defect-induced incremented phase errors]
- Problem w/all up $G_z$ blips $\Rightarrow$ phase error builds up
  - Trick #1: Start w/2 slices, one at $z=0$, other above
    $\Rightarrow$ if $180^\circ$ (180°) phase shift used, blip up/down same! (no effect at $z=0$)
    $\Rightarrow$ i.e., move top or bottom replica
  - Trick #2: For multiple slices not all at $z=0$, phase no longer same for even/odd
    $\Rightarrow$ but can add phase to equilibrate to k-space before recon.
  - Trick #3: For more than 2 slices:
    $\downarrow$ 1st even odd 1st even odd $\uparrow$ etc.
**MULTI BAND/BLIPPED CHIRP**

- relation between leave-one-out aliasing and nominally fully-sampled SMS

- leave alternate lines out wraps image
- SENSE/GRAFPA to fix block coil view smears K-space data
- nominally, w/SMS we record every line of K-space
- but equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- slice- GRAPPA
- reg GRAPPA → recon missing lines
- single GRAPPA → recon multiple Kspaces
  - For each overlapped slices
  - by training on fully-sampled data
  - at beginning of scan

- interslice "leakage block"
- when training GRAPPA kernel on fully-sampled data,
  - also minimize interslice leakage (split slice-GRAPPA)
- can also do regular GRAPPA on top of this
- reason: for diffusion, loss in S/N from undersample
  - cancelled by shorter TE readout
- gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed
- since entire volume is excited each shot, potentially higher S/N
- must use smaller flip angle to avoid killing $M_r$ since entire volume excited every partition (e.g. every 30 mssec)

- main issue is movement artifact since data assembled from many shots over several secs
- breathing-induced B0 problems in different partitions may cause blur
SPiral imaging

- By using smoothly changing gradients (sinusoids) less gradient power required than w/ trapezoids (less eddy currents)

  Earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/ non-uniform sampling to get constant $\Delta k_x$

- Sinusoids in both $G_x$ and $G_y$ give spiral $k$-space trajectory

- Constant angular velocity goes too fast at large $k_x$, $k_y$

- Constant linear velocity better but impractical near $k_x=0$, $k_y=0$

- Compromise: start constant angular, end constant linear

**Constant angular velocity**

$$w(t) = \omega_0 t$$

$$k(t) = At e^{i\omega_0 t}$$

$$G(t) = \frac{1}{A} \frac{d}{dt} k(t)$$

$$= A e^{i\omega_0 t} + iAt e^{i\omega_0 t}$$

$$G_x(t) = A \cos \omega_0 t - At \omega_0 \sin \omega_0 t$$

$$G_y(t) = A \sin \omega_0 t + At \omega_0 \cos \omega_0 t$$

**Constant linear velocity**

$$w(t) = \omega_0 T e^{i\omega_0 t}$$

$$k(t) = At e^{i\omega_0 T}$$

$$G(t) = \frac{1}{A} \frac{d}{dt} k(t)$$

$$= \frac{A}{2t} e^{i\omega_0 T} + \frac{A}{2} \omega_0 e^{i\omega_0 T}$$

$$G_x(t) = \frac{A}{2t} \cos \omega_0 T e^{i\omega_0 T} + \frac{A}{2} \omega_0 \cos \omega_0 T$$

$$G_y(t) = \frac{A}{2t} \sin \omega_0 T e^{i\omega_0 T} + \frac{A}{2} \omega_0 \sin \omega_0 T$$

$$G_x$$

$$G_y$$
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- interleaved spirals (like FSE interleaves)
- true IR (vs. MPRAGE)
  
  all echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)

  possible to present sign
  - high, uniform contrast, but lots of waiting (T1), high BW

RF

180° (prep1) TI ~ 700 msec

90°

180°

180°  x 16

180° (prep2)

Gz

black select

Gy

spiral readout unwind

Gx

Sig.

FID

echo1

echo2

Loop order

3D k-space

(“stack of spirals”)

k2

spiral interleaves

k2 echoes

k2

echoes = (after one 90°)
Phase Errors & Echo-Centering Errors

Anything that causes a deviation of the $B_z$ field strength from the expected value $(B_0,z + G_{x,z}x + G_{y,z}y + G_{z,z}z)$ changes precision frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

**Fourier shift theorem**

phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x - x_0) = \int e^{-i2\pi k_x x_0} S(k_x) e^{-i2\pi k_x x} dk_x$$

- Correct w/shimming and $B_0$-mapping/phase unwrapping before reconstruction.

Echo Centering Error

- If realignment of all spins ($k_x = k_y = 0$) doesn’t occur at the middle of read gradient, echo is shifted.
- Since echo is in spatial frequency domain, this causes frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction => magnitude image unchanged.

$$e^{i2\pi k_x x} I(x) = \int e^{i2\pi k_x x_0} S(k_x - k_x_0) e^{-i2\pi k_x x} dk_x$$

- Offset in spatial freq. space.
**FAST SCAN ARTIFACTS**

**EPI vs. Spiral**

**EPI**
- $G_x$ reachach gradient strong \(\rightarrow\) field defects smaller percentage less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weak and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the $x$-direction, for example, $w$ and unmaps phase as a function of $x$-position
- but $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

- for a given $x$-position, the strength of the spurious gradient is constant, so the accumulation of phase error results shift in the $y$-direction ($k_x$-space spin-stripe display)
- the phase error causes a shift in the $y$-direction proportional to $x$-gradient strength (= shear) but no blurring
  (N.B. Shift varies w/x-position) $y \rightarrow y'$

**Spiral**
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error (= more shearing)

- for spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq. $y \rightarrow y'$

- for defects with more complex contents in the $y$-direction (than linear, as above) the vertical shifts (in EPI) will vary with $y$-position, and may result in signals from different $y$-positions being reconstructed on top of each other
**IMAGE-SPACE VIEW OF LOCALIZED B\(\phi\) DEFECT, EFFECT ON RECON**

- Localized B\(\phi\) defects often arise from air pockets embedded in tissue
  - Air in middle/outer ear \(\rightarrow\) indentation in inferior temporal lobe
  - Air under olfactory epithelium \(\rightarrow\) orbitofrontal cto, ant. thal. compression

**Collect one data (k-space) point**

- 4 cycles of phase in y-dir (y-direction)
- Localized \(\Delta B\phi\) defect

**Complex multiply**

- \((=\text{correlate sincos with brain})\)

**Reconstruction from distorted data points**

- Undistorted stripes used by inverse FFT

**- Same defect makes leftward dent in vertical phase stripes**

- Spatial information can be lost when continuous changes in phase are flattened by \(B\phi\) defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

**- Local estimates of \(\Delta B\phi\) needed to correct images**
  1. Fieldmap method: \(<\text{multiple TE's to est. local } B\phi\text{ from } \gamma T E\text{ slope}\>
  2. Point-spread-function: \(<\text{extra phase encode to estimate P.S.F. (should be } \delta\text{-function)}\>

**Close-up of distorted phase stripes (one cycle)**

**N.B.:** Image shift only occurs if shift spatial traces sampled in successive later echoes times (see next page)
LOCALIZED BO DEFECT, EFFECT ON RECON (2)

- when local BO defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, with w/EPI, static BO defect causes more and more local displacement of image phase stripes for each additional ky line

  - that is, later lines have greater spat. freq. offset
  - effectively stretches k-space in ky direction
  - same num. samples to higher spatial freq. shrinks FOV (squishes voxels — see FOV page)

- when image is reconstructed, region with local BO defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:

  1) static local ΔBO defect

  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout

  3) small size of ky phase encode blips relative to BO defect (much less of this effect in freq. encode direction)

- respiration (which affect BO) in 3D FLASH might cause similar effect within k2 partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto Z-component of the B Field -- $B_z$ -- in the x, y, and z directions.

- In practice, gradient coils are non-linear (esp. printed-circuit-like)

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion:
  - A non-linear slice-select gradient will cause a curved slice
  - Non-linear phase and frequency encode gradients will distort in-plane features

- Some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!)

- This can result in errors approaching 1 cm in functional overlays.

- Different coil designs have different directions of distortion (!)

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

These effects do not build up over time in phase-encode direction. Since they only occur when gradients are turned on, these distortions are predictable and can be corrected.

Fourier shift theorem

That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND B0-MAPPING

- Passive iron shims inserted to flatten B0 field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the B0 field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

Linear shim coils impose gradients in x, y, and z
Higher order shims impose higher order sphenoidal harmonic field components (e.g., z^2)

Secondary use is to compensate for inhomogeneities caused by introducing the sample into the B0 field

- Local resonance offsets caused by B0 defects estimated from images
  \[ \text{e.g., sample phase at multiple echo times} \]
- Fit defective field using combination of fields generated by shim coils. Then add these corrections to base shim currents
  \[ \Rightarrow \text{this only corrects spatially gradual field defects} \]
  \[ \Rightarrow \text{local defects due to air in sinuses much higher order than shims} \]

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
NAVI ATO RCHEES

1D navigator

Bz drift problem
- slow up/down drifts in Bz continuously occur
- a pedestal in Bz is pedestal in phase (not gradient)
- which causes spatial shift (Fourier shift theorem)
- in EPI, mainly offsets phase-encode dir b/c small flip angle readout
- result is successive volumes drift in phase encode dir

Grad  ial balance problem
- unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
- causing N/2 (Nyquist) ghosting 2 another phase error

2D navigator: collect 3D sphere in k-space

- rotation of object corresponds to rotation of k-space amplitude pattern
- translation of object corresponds to phase shift of k-space phase (Fourier shift)
- sample at sufficiently high density to pick up high spatial freq features
- N.B.: excite whole volume
- do N/2 hemispheres separately (less T2*, cancel EPI-like error accumulation)

Welch et al. (2002) MRM

\[ z(n) = \frac{2n - N - 1}{N} \]

\[ y(n) = \cos(TN\pi \sin^{-1}z(n)) \sqrt{1 - z^2(n)} \]

\[ x(n) = \sin(TN\pi \sin^{-1}z(n)) \sqrt{1 - z^2(n)} \]

(small poles - slow rate too high)

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR of EPI images (27 ms vs. 2.4 sec)
- may need to smooth rot, trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES $B_v$ inhomogeneities

- Receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way. Variations can be used (cf. GRAPPA, SENSE) and/or corrected.

- Transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH). Potentially worse (why local transmit is still in progress). Usually fixed by using a large transmit coil (e.g. body coil).

- RF penetration at higher fields ($\leq$ higher RF frequencies)
  1) Decreased RF wavelength (closer to size of head) at higher freq.
  2) Increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field.

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain).

- Different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP).

  Normalization ("pre-scan normalize")

  - Record low-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s).
  - Divide small coil body coil at each voxel to determine receive field.
  - Use receive field to normalize main image(s)

[see also: qT1, MP2RAGE, T1/T2]
**Diffusion-Weighted Imaging**

Simple diffusion weighting:
- RF: $90^\circ$ pulse
- $G_z$: select
- $G_y$: phase encode
- $G_x$: readout

- "Apparent diffusion coefficient" map
- To get large $b$, need $G \uparrow$, $ST \uparrow$, (need big $G$'s)
- Long $ST$ gives spurious $T_2$-weighting
- Can use stimulated echoes:
  - $90^\circ$ RF $\rightarrow$ $ST_1$, $90^\circ$ RF $\rightarrow$ $ST_2$ $\downarrow$
  - Transverse diffusion (lobe 1) $\rightarrow$ back to transverse diffusion (lobe 2)

1) Anisotropic Diffusion (Gaussian)
- Measure $D$ along multiple axes
- Have to measure tensor, not scalar
- Even for determining one primary direction

\[ D = \begin{bmatrix} D_x & D_y & D_z \\ D_y & D_x & D_z \\ D_z & D_y & D_x \end{bmatrix} \]

D = $u^T \cdot D \cdot u$

Diffusion Surface (Non-Gaussian)
- Need to measure diffusion in many directions ($\geq 6$) to properly characterize even 2 main directions

2) Non-Gaussian length scale by multiple $b$-values
- Fit line to semi-log signal as function of $b$ (Mulkern, 1999)
- If not straight line: multi-exponential, e.g.,
  - $S = A_1 e^{-bD_1} + A_2 e^{-bD_2}$
  - $\log$ signal $\uparrow$ (noise)$\rightarrow$ $600$
  - $b$ scale $\downarrow$
  - Data both $b$-hi, $b$-lo $\rightarrow$ diff sig $\uparrow$

Tract Tracing
- 1) Markov process
- 2) Crossing fibers
- 3) "Free way ramp" prob
- 4) Sharp turns (into gyri)

Classical diffusion coefficient:
- Spins acquire phase during first $ST$
- If spins diffuse (=move) along gradient by time $T$, signal is lost because negative $ST$ doesn't re-phase
- Attenuation: $A(D) = \frac{S_0}{S_p} = e^{-bD}$
  - where $b = \frac{Y^2 G^2 ST^2}{T - ST/(3T)}$

Calculate $D$ image from $b=0$
- $b$ large (add $b$'s better)

Long $T$ allows
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)

\[ \text{[expanded in time for clarity]} \]

- eddy-currents are long time-constant currents in metal of scanner that distort B field \( \rightarrow \) spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/particles, time constants)

\[ \text{Nagy et al., 2014 MRM} \]

\[ Y_{\text{TRSE}} = 0 = Y_1 - Y_2 - Y_3 + Y_4 \]

\[ \text{twice-refocused Spin-echo for center k-space} \]
**PERFUSION - ARTERIAL SPIN LABEL**

- **Basic idea:** Collect control & tagged image assume directional input flow.
  - Tag is 180° pulse tag is 180° pulse
  - 180° tag is sign not problem when delay long enough (see below).

  - Continuous ASL (CASL): continuously tag a plane.
  - Pseudo-continuous ASL (pCASL): greatest on, blood gets adiabatically inverted as it passes through location w/Corresponding resonant tag.

  - Pulsed ASL (PASL): e.g., EPSTAR, FAIR, PICORE, QUIPSS II
  - Tag block of tissue below slice(s).

- Small diffs between control and tag (~1%)
  - Reverses accurate balancing of control & tag images, control mag, transfer.
- Contrast problems: transit delays biggest confounding factor.
  - Relaxation rate differential venous clearance (vs. microvessels, which get stuck.)

- QUIPSS II - Quantitative Perfusion
  1. Pre-saturate spins in target slices.
  2. Tag - 180° pulse below slices.
  3. Control - 180° pulse above slices (to control off-resonance)
  4. Saturate tagged block to end tag (TI).
  
  - Both tag and control.
  
  - Can use train of thin slices pulsed at top of tag band.
  - Can image most distal slice first to cancel delays.

- EPI or spiral images of target slices (T2).
  - Image most distal slice first to cancel delays.

- Fast between slice so imaging excitations don't get interpreted as flow.

\[
\Delta M \approx \text{flow} \times \left(2M_0 \text{ TI, } e^{-T1_2/T1A}\right)
\]

- Can extract flow and BOLD. Adjacent substrates minimize movement artifact.

- 1) Alternate tag and control,GRE TE = 30 ms
  - Both BOLD-weighed
  - Control + tag \rightarrow BOLD

- 2) Dual Echo Spiral
  - K = 0 early \(\Rightarrow\) high S/N flow
  - TE = 30 ms \(\Rightarrow\) BOLD
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires
  RF on continuously to adiabatically invert blood flowing
  through one plane
  - can only image one slice (due to dephasing from gradient)
  - hard to keep RF on continuously on modern scanners (esp. 3T)
  - can use special purpose RF transmit (separate xmit channel)

A) Original CASL

[RF] [Gz] [image formation module ("readout") → [multiple possibilities]


[RF] [Gz] [begin readout]

- Problem: multiple pulsers create aliased slice planes

\[ RF(t) = \frac{1}{\Delta t} \text{comb}(\frac{t}{\Delta t}) \otimes \text{rect}(\frac{t}{\delta}) \]  \\

- [use: convolution of 2 funct equals multiplying their FTs]

\[ F[RF(t)] = \text{comb}(6\delta t) \cdot \delta \text{ sinc}(\pi \delta b) \]

- Aliased labeling planes at: \( b = n/\Delta t \) in frequency space, modulated by broad sinc()

- Use Hamming or hyperbolic secant to reduce replicas

C) pCASL w/ shaped gradients

- Tag pulses have phase offset respecting gradient
- Control identical except every other has \( +\pi \) phase

Diagram:

- RF,
- 0.8 msec, tag,
- 0.2 msec pulse,
- FLASH,
- EPI,
- SMS,
- \( 3D \),
- no net flip
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g. 1H) by surrounding electron orbitals

- e.g., acetic acid:
  - Oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:
  - Larmor oscillations are multiplied (PSD) by center freq to obtain $\Delta f$ (not MHz, high freq)
  - data before FT is a series of time-domain samples of the mix of shifted-freq offsets
  - FT turns data into "shift spectrum"

- N.B.: opposite "direction" of FTs!

<table>
<thead>
<tr>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMR</td>
<td>time domain oscillation $\rightarrow$ freq spectrum $\rightarrow$ shift spectrum</td>
</tr>
<tr>
<td>MRI</td>
<td>spatial $\rightarrow$ spatial object (like time domain signal)</td>
</tr>
</tbody>
</table>

Pulse Sequence

- since we are already using phase (freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal $\rightarrow$ and FT, it like chemists do!
PRESS, MEGA-PRESS

Usu. single voxel by using 3 orthogonal slice-selects (tho can add PEG gradients & more excitations to get multiple vox)

PRESS - 3 orthogonal slice-select

MEGA-PRESS - add "editing" RFs to suppress solvent (water)

G_1, G_2 - asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

G_3 - G_1 symmetric around 2π/800

FT to get shift spectrum
PHASE-ENCODED STIMULUS & ANALYSIS

- periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

- calculate significance
  - ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
  - ignore harmonics, low freq (= movement)

- smooth
  - vector average of complex significance (A, φ) with that at nearest neighbor surface points

- display
  - plot phase using hue and saturation to indicate significance

- delay correction
  - record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
  - vector average after reversing angle of one
  - penalizes inconsistent more than just avg of angles

- remove constant (avg) and linear trend

- FFT, convert to A, φ

- freq = total TR's / 2

- bright vs t

- strongly periodically activated single voxel time course
**CONVOLUTION**

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{\infty} f(z) \cdot g(x-z) \, dz \]

- definition of convolution \((f \ast g)(x)\)
- commutative

**linear time/space invariant system**

- how to calculate one term
- sum across all \(z\) to get the value of the convolution at point \(x\)
- move kernel to calculate next \(x\)

**Why reverse makes sense**

**blc commutative, this is thinking like:**

\[ \int_{-\infty}^{\infty} g(x) \cdot f(x-z) \, dz \]

**impulse response function (HDR)**

impulses (expt. design)

- impulse occurred a while ago
  - small effect
- impulse occurred recently
  - larger effect

**how to calculate convolution output for this time point (only 3 terms in sum, all other zero)**

**NB. cross-corr same as convolution except no reversal**

\[ g(x+z) \text{ instead of } g(x-z) \]

**NB. auto-corr same, except no-reversal and use same-funct for both } f, g **
**GENERAL LINEAR MODEL**

\[ \tilde{y} = \mathbf{X}\hat{\mathbf{h}} + \mathbf{S}\hat{\mathbf{b}} + \tilde{\mathbf{n}} \]

- goal is to solve for the hemodynamic response functions, \( \hat{\mathbf{h}} \)

\[ \mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{S}\mathbf{b} + \mathbf{n} \]

\( \mathbf{y} \) - data

\( \mathbf{X} \) - design

\( \mathbf{S} \) - drifts

\( \mathbf{b} \) - weights

\( \mathbf{n} \) - noise

\( \mathbf{h} \) - unknowns

\( \mathbf{t}_{\text{exp}} \) - temporal

\( \mathbf{t}_{\text{hemo}} \) - hemodynamic

\( \mathbf{h}_{\text{on}} \) - experimental

\( \mathbf{h}_{\text{off}} \) - design

\( \mathbf{S}_{\text{on}} \) - poly

\( \mathbf{S}_{\text{off}} \) - poly

\( \mathbf{b} \) - poly

\( \mathbf{c} \) - fixed

\( \mathbf{S} \) - assume white noise

1) Conv. \( \mathbf{X} \) w/ fixed \( \hat{\mathbf{h}} \)

2) solve for \( \hat{\mathbf{b}} \)

3) \( \hat{\mathbf{y}} = \mathbf{X}\hat{\mathbf{b}} + \tilde{\mathbf{n}} \)

\( \mathbf{x} \) - scalar

\[ \begin{bmatrix} \mathbf{y} \\ \mathbf{\tilde{n}} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{h} \\ \mathbf{b} \end{bmatrix} + \begin{bmatrix} \mathbf{n} \end{bmatrix} \]

\( \hat{\mathbf{h}} \) - maximum likelihood estimate

1) Assume white noise, solve for \( \hat{\mathbf{h}} \)

2) \( \hat{\mathbf{h}} = (\mathbf{X}^T\mathbf{P}_S\mathbf{X})^{-1}\mathbf{X}^T\mathbf{P}_S\mathbf{y} \) \quad where \( \mathbf{P}_S = \mathbf{I} - \mathbf{S}(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T \)

3) Significance (how to construct F-ratio)
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions $s$, we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

$$\hat{y} = X\hat{h} + \hat{e}$$  
$$\hat{y}$$ = data estimate w/ error  
$$X\hat{h}$$ = signal  
$$\hat{e}$$ = noise

Projection matrix $P_x$ operates on $\hat{y}$ to give projection of data into experiment space $X$.

- when nuisance functions $s$, are considered, problem: $S$ may not be orthogonal to $X$.

For example: linear trend not orthogonal to std. block design.

Remember: "orthogonal" means dot prod. = 0.

Geometric Picture (Liu et al., 2001, Neuroimage)

Orthogonal projection

$$\hat{y} = P_x \hat{y}$$

Calc. dot prod

Orthogonal projection onto nuisance ($P_s y$)

Data orthogonal to nuisance

Error ($e$) not explained by reference and nuisance ($F$ denom.)

Same as projection onto reference only in special case where $S \perp X$.
1) MNI auto-Talairach → generates 4x4 matrix
- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on xcorr
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut-off
  - but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
- histogram of pixel values in 10 mm thick HQR slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  - interpolated scaling factor 1 to HQR
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D
- find points in 5x5x5 within 10% of WM, set new scale for them
- build Voronoi to interpolate scales unset above
- soap-bubble-smooth Voronoi boundaries (3 iterations)
- re-scale each voxel

3) Skull Stripping (output: "brain")
- "shrink-wrap" algorithm
- start with ellipsoidal template
- minimize brain penetration and curvature
  - curvature: spring force
    - from center-to-neighbor vect sum
  - brain penetration
    - apply force along surface normal that prevents surface from entering gray matter

Talairach, Normalize, Strip Skull
- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something')
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

\[ \mathbf{F}_{\text{smooth}} = \lambda \tan \sum_{\text{neigh}} (I - \mathbf{n}_{\text{center}} \mathbf{n}^T_{\text{center}}) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

\[ \mathbf{F}_{\text{MRI}} = \lambda \mathbf{n}_{\text{center}} \left[ \frac{1}{d} \right] \max \left[ 0, \tanh \left[ 5 \left( I(\mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}}) - I_{\text{threshold}} \right) \right] \right] \]

Snapshot of surface and "core sample" from one vertex
4) Non-isotropic filtering (output: "win") — "floss" and "speckle"
   - preliminary hard thresholds: output
   - find ambiguous/boundary voxels
     \( \rightarrow \) 20\% or more of 26 immediate neighbors different
   - find plane of least variance
     for each direction (from icosahedral super-tessellation)
     consider 5x5x5 volume around 1 voxel
     find plane of least variance in this hemisphere
     medium filter with hysteresis
     \( \rightarrow \) if 60\% of within-slab differ, reverse classification
     \( \rightarrow \) "flosses" sulci without blurring

5) Find cutting planes
   midbrain
   callosum, to separate hemispheres (SAG)
   midbrain, to avoid fill into cerebellum (T1W)
   Talairach to start:
   fill WM in SAG or T1W till min area

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out — for each hemisphere
   - up/down cycles within each plane
   - plane-by-plane
   - "wormhole filter" (3x3x3 = center + 26)
     \( \rightarrow \) fill (unfilled) voxel if 66\% neighbors differ — eliminates structures within, 1-D structure
7) Surface Tessellation (output: rh.orig, lh.orig)

- variable num neighbors possible!
- quads to triangles

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  \( \Rightarrow \) so cross-products oriented properly

- long list of values associated with each numbered vertex
  e.g. position (orig, morphed)
  area (orig, morphed)
  curvature (intrinsic, Gaussian)
  "Sulcussness" (summed L movement during unfolding)
  cortical thickness
  fMRI data \( \leq \)
  EEG/MEG dipole strength

- separate fMRI data set must be aligned, sampled
  fMRI voxels larger
  Sample at each surface vertex
  nearest-neighbor "soap bubble" smoothing
  to interpolate data onto hi-res mesh

- some quantities only well-defined on surface
  \( \Rightarrow \) gradient of magnitude of cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON

- smoothing/inflation/WM, pial done as derivative of energy functional

\[
J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}}
\]

- scalar tangential error (fixed by redistributing vertices)
- scalar normal error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[
J_{\text{normal}} = \frac{1}{2\text{ #vert}} \sum_{\text{center}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2
\]

- across all vertices
- \(\frac{1}{2}\) so no coefficient on derivative
- tangential across all vertices
- normal neighbors of one vertex
- vertex unit normal
- vector from current center to one neighbor (position vector diff)

\[
J_{\text{tangential}} = \frac{1}{2\text{ #vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}_x^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 + \left[ \mathbf{t}_y^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2
\]

- "squishing & wash"
- \(x\)-direction in tangent plane
- project vector to neighbor onto \(x \& y\)
- \(y\)-direction in tangent plane

\[
J_{\text{image}} = \frac{1}{2\text{ #vert}} \sum_{\text{centers}} \left[ I_{\text{target}} - I(\mathbf{r}_{\text{center}}) \right]^2
\]

- target for WM: mean of voxels labeled WM in 5 mm neighborhood
- target for pia: global — small num for CSF-like

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[- \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{target}} - I(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

- go the opposite direction (vector) of largest scalar error for one vertex
- \(\cos\) gone b/c const
- \(\sum\) over neighbors
- normal \(\mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \)
- \(\mathbf{n}_{\text{center}}\) scaled by unit normal vector

\[- \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{tangential}} \left[ \mathbf{t}_x^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right] \mathbf{t}_x^{\text{center}} + \left[ \mathbf{t}_y^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right] \mathbf{t}_y^{\text{center}}\]
SULCUS-BASED CROSS-SUB. ALIGN

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: \((S_{\text{out}} - S_{\text{targ}})^2\)
- Bootstrap morph to one brain make avg target remorph to avg targ

<table>
<thead>
<tr>
<th>Smooth WM</th>
<th>Inflated</th>
<th>Sphere</th>
<th>Registered Sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub_1</td>
<td>Inflate</td>
<td>Sphere</td>
<td>Morph</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub_2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sub_n</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (fsaverage "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- Can use sampled-to-icos individual subj coords to draw
  icosahedral surface in shape of an indiv. brain
- N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. sub.)
SOURCE OF EEG/MEG

PSPs

- anisotropic cables
- aligned spatially
- coherent/biased stim

GM
WM
- no distant signal from axon spike
- too close

isotropic
"closed field" (invisible at distance)

N.B.: spikes only detected by 15um microelectrode in gray matter!

Head

1) - Local dipole
2) - EEG through skull, skin
3) - Sweating because skull 1/80 conductivity of brain

MEG

- radial dipoles lost
- tangential dipole generates Gabor-like scalp distr.
  & B field
INTRACORTICAL CIRCUITS & ORIGIN OF EEG

Cell Types
- Excitatory (spiny)
  - Pyramidal
  - Spiny stellate (e.g. V1 layer 4c)
- Inhibitory (smooth)
  - Basket
  - Double bouquet
  - Chandelier
  - Clutch

Circuits
- Huge complexity
- First principal components: input → layer 4
- Micro-electrode recording (e.g. 10 μm tip):
  - High pass → spikes
  - Low pass → local field potentials
- Spikes only recordable in gray matter
- White matter spikes only recordable with pipette wi very fine tip b/c inward & outward currents are spatially close in axon/spike (>1 μm)

Intra/inter cortical connections cartoon

"Lower" (e.g. V1)
1. Feedback
2/3 feed-forward
4 input
5 motor output
6 feedback
Ascending input (e.g. dLGN) → output to sup. collic. → feedback avoids layer 4

"Higher" (e.g. V2)
2/3
4
5
6
Motor striatum

Layer 2/3 → feedforward
Layer 5/6 → output feedback
Spike has opposite polarity here!
**GRADIENT, DIVERGENCE, CURL**

**Gradient** \( \nabla \) (generalized derivative)

\[
\nabla s(\mathbf{r}) = \frac{\partial s(\mathbf{r})}{\partial x} \mathbf{i} + \frac{\partial s(\mathbf{r})}{\partial y} \mathbf{j} + \frac{\partial s(\mathbf{r})}{\partial z} \mathbf{k}
\]

- Turns scalar function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of \( s \) in \( x \)-direction at point \( \mathbf{r} \)
- Unit vector in \( x \)-dir

**Divergence** \( \nabla \cdot \mathbf{V} \) (div. "dot prod")

\[
\nabla \cdot \mathbf{V}(\mathbf{r}) = \frac{\partial V_x(\mathbf{r})}{\partial x} + \frac{\partial V_y(\mathbf{r})}{\partial y} + \frac{\partial V_z(\mathbf{r})}{\partial z}
\]

- Turns vector field into scalar field
- Vector function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of \( \mathbf{V} \) in \( x \)-direction at point \( \mathbf{r} \)

**Curl** \( \nabla \times \mathbf{V} \) (div. "cross product")

\[
\nabla \times \mathbf{V}(\mathbf{r}) = \left( \frac{\partial V_z(\mathbf{r})}{\partial y} - \frac{\partial V_y(\mathbf{r})}{\partial z} \right) \mathbf{i} + \left( \frac{\partial V_x(\mathbf{r})}{\partial z} - \frac{\partial V_z(\mathbf{r})}{\partial x} \right) \mathbf{j} + \left( \frac{\partial V_y(\mathbf{r})}{\partial x} - \frac{\partial V_x(\mathbf{r})}{\partial y} \right) \mathbf{k}
\]

- Turns vector field into vector field
- Vector function defined at each \( x, y, z \) point \( \mathbf{r} \)
- Change of just \( z \)-component of \( \mathbf{V} \) in \( y \)-direction at point \( \mathbf{r} \)

**Vector identities**

\[
\nabla \times \nabla s = \mathbf{0} \quad \text{curl of the gradient of any scalar field is zero}
\]

\[
\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad \text{divergence of the curl of any vector field is zero}
\]

\[
\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}
\]
POTENTIAL (\( \Phi \)), ELECTRIC FIELD (\( \nabla \Phi \)) \& CSD (\( \nabla \cdot ( - \nabla \Phi ) = \nabla^2 \Phi \))

- low-frequency field approximation
  - electric fields uncoupled from magnetic (vs. electromagnetic radiation)
  - pre-Maxwellian approx. (EEG freqs << 1 MHz)
  - calculate electric fields as if magnetic fields don't exist
  - calculate magnetic fields strictly from distribution of currents
  - ignore capacitive effects, too

Scalar potential, \( \Phi \) (what we measure with electrode)

\[
\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}
\]

1. \( \vec{E} \) defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)
2. Current density, \( \vec{J} \) (not curr. source den.s! ) is proportional to \( \vec{E} \) (still a vector!)
3. \( \vec{J} = \sigma \vec{E} \) (Ohm's law, \( \sigma \) is conductivity)

CSD is Laplacian of \( \Phi \) (= div \( \vec{E} \))

\[
\nabla \cdot ( - \nabla \Phi ) = \text{scalar field} = - \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] = -\nabla^2 \Phi
\]

3D CSD gold standard (rat BAER paper)

- \( \Phi \) data \( \rightarrow \) \( \nabla \Phi \) \( \rightarrow \) \( \nabla \cdot ( - \nabla \Phi ) \)
- scalar field source/sink movie as function of \( z \)


**1D CSD**

- Raw, event-related signal relative to ground, \(-\frac{d}{h}\) (eg skull)
- High pass vs low pass
- Spikes (upside down field potential)
- LFP (local field potential)
- Both types of data can be recorded from same electrode

**Rationale:** CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet

**1D CSD**

- \(\nabla^2 \phi\) means find spatial (ie, 1D depth) curvature of potential
- Discrete approx: center = \(\frac{\text{above+below}}{2}\)
- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink

**2D CSD**

- 2D array of electrodes on pial surface or on scalp

**Rationale:** all electrodes record along same surface so assume depth profiles are constant

**For scalp recordings, sources and sinks are at the scalp**
- (not a depth loc method unless done in 3D)
- Cannot tell curvature from sign of potential!
  - Convex \(\Rightarrow\) source, \(\text{pos}\) concave \(\Rightarrow\) sink, \(\text{neg}\)

**Note:**
- Conf use calc of calc w/ always a second deriv.
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

CSD - C

Layer 4

Source
Sink 4

50 msec

Source
Sink

50 msec

P1

N1

P2

- phase-locked CSD p
  Gamma shifts with each cycle
\[ \nabla \cdot (\sigma \nabla \Phi) = \nabla \cdot \vec{J} \]

**Divergence**

- Conductivity constant (a tensor constant if inhomogeneous in different directions)
- Gradient of scalar potential (what we measure in practice at each point)
- Impressed currents
- Currents due to ionic flow that "appear out of nowhere" (Von Neumann)

\[ \nabla \times \vec{B} = \mu_0 (\vec{J} - \sigma \nabla \Phi) \]

**Curl**

- Magnetic field
- Permeability
- Impressed currents
- Conductivity
- Gradient of scalar potential (= \(-E\))

---

- Propagation of potentials, magnetic fields instantaneous (no capacitance)
- Simultaneous Eqs to solve: \( \vec{J} \) are sources, \( \Phi, \vec{B} \) are data
- Linear

\[ \nabla \times \vec{B} = \mu_0 (\vec{J} - \sigma \nabla \Phi) \]

Potential (\( \Phi \)) and magnetic fields (\( \vec{B} \)) produced by a weighted sum of two current source distributions are equal to weighted sum of fields produced by each current source distribution by itself.
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t} \]

(from Nunez, 1981)

\[ \vec{B} = \nabla \times \vec{A} \]

"vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

magnetic field

currents (in given medium)

induced

\[ \vec{H} = \frac{1}{\mu} \vec{B} \]

permeability \( \mu \)

conductivity \( \sigma \)

permittivity \( \varepsilon \)

characteristic substance linear in all three

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \frac{\vec{J} + \partial \vec{D}}{\partial t} \right) \]

if linear in conductivity and dielectric, too, and fields periodic w/\( \tau \)

\[ \nabla \times \nabla \times \vec{E} = -2\pi \mu_0 (\sigma + 2\pi f\varepsilon) \vec{E} \]

to neglect:

\[ \left| \nabla \times \nabla \times \vec{E} \right| \approx \frac{|\vec{E}|}{L^2} \]

1) \( |\nabla \times \nabla \times \vec{E}| \approx |\vec{E}|/L^2 \) when \( L \) is short over which \( \vec{E} \) varies significantly

2) \( \mu \) of tissue similar to empty space

3) Assume conservative (large) \( \varepsilon \), dielectric unit, and EEG freq.

\( \mu \) number is about \( 10^{-6} \) \( \Rightarrow \) small
\[ \Phi_1 = \frac{S}{4\pi \sigma r} \]  
for potential recorded for source monopole.

\[ \Phi_2 = \frac{S}{4\pi \sigma} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) \]  
for potential recorded for source-sink pair.

\[ \Phi_2 \approx \frac{S}{4\pi \sigma} \left( \frac{1}{r^2} \right) \]  
for scalar approximation for "far enough away" measurements.

\[ \mathbf{B}_2 \approx \frac{\mu_0}{4\pi} \frac{S \mathbf{d} \times \mathbf{r}}{r^3}, \quad r \gg d \]  
for vector approximation.

\[ \Phi_i(t) = e_i s_i(t) \]  
for electrode gain.

\[ \mathbf{b}_i(t) = m_i s_i(t) \]  
for magnetic component of \( \mathbf{B} \).

\[ \Phi(t) = \sum_j e_j s_j(t) \]  
for all sources.

\[ \mathbf{b}(t) = \sum_j m_j s_j(t) \]  
for all sources.

\[ \mathbf{x}(t) = \sum_j b_j(t) s_j(t) \]  
for all sources.

\[ \mathbf{X}(t) = \mathbf{G} \mathbf{s}(t) \]  
for linear superposition with fixed electrodes and sensors.
Forward Solution

- well-posed (one answer)
- linear: \[ b(A) + b(B) = b(A+O) \]
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic

- skull / brain conductivity
- "smearing" (cf. cable theory)
- remember, we only need to be able to calc. weight for each dipole / electrode pair independently

- 3-shell boundary element

- arbitrary shape
- homogeneous conductivity
- solution = infinite homogeneous + vertex
- for magnetic, only need one shell b/c currents thru skin/skull too small to make sig. \( B \)
- matrix of correction factors

- finite element
- most general computational intensive w/ small grid
- many unknown parameters to estimate
Forward Solution (2)

\[ V_i = \sum_j E_{ij} S_j + n_i \]

Position \((x, y, z)\) \\
Angle \((\Theta, \phi)\) in brain

Matrix form

\[
\begin{bmatrix}
V
\end{bmatrix} = \\
\begin{bmatrix}
E
\end{bmatrix} \begin{bmatrix}
S
\end{bmatrix} + \\
\begin{bmatrix}
n
\end{bmatrix}
\]

\[ V, S, n \text{ vary} \]

Lower case bold \(\Rightarrow\) vector
Upper case bold \(\Rightarrow\) matrix

Electric recordings
Magnetic recordings

\[
\begin{bmatrix}
V
\end{bmatrix} = \\
\begin{bmatrix}
E
\end{bmatrix} \begin{bmatrix}
S
\end{bmatrix} + \\
\begin{bmatrix}
n
\end{bmatrix}
\]

Current Source Dipole Amplitudes (same length as above)

Note: only one current source for each column in the \(E+B\) matrix!
WHY LOCALIZE?

- most of ERP literature based (instead) on temporal "components"
  but: 1) *underlying local cortical generators* (from microelectrode LFP)
  - extended in time (400 msec)
  - multiphasic in every cortical area
  - temporally non-static depending on *stimulus*
    e.g. simple contrast brightness cliffs can modulate retinal delay by 50 msec!

2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels

3) *stimulus manipulations will change temporal overlap* may cause "component" peak to disappear without changing cortical areas being activated

4) verified by *intra-cortical LFP/CSD* (Schroeder et al, 1998)

---

macaque monkey
intra-cortical data

these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey CT

LFPs from approx. layer 4 in cortex ("input layer")

Psychologists now identify a few temporal "component" peaks...

But each one comes from every one of these cortical areas!!

---

- by contrast, the spatial signature of the signal from one cortical area is static — a better area-based component
- origin of "components" — easier to record many temporal points (EEG started w/ few electrodes, many time points)
  - easier to "paste" high level psychological functions onto a few waveform deflections
Derivation of Ill-posed Inverse

(from Dale & Sereno, 1993)

\[ x = As + n \]

\[ A = \text{forward solution matrix (E+I)} \]

\[ s = \text{source vector} \]

\[ n = \text{sensor noise vector} \]

\[ E[r_w] = \langle \| Wx - s \|^2 \rangle \]

\[ \text{expectation: } E[r_w] = \sum_k P_k k \]

Assume \( n, s \) normal, zero-mean \( W \) corresponding covar. matrics \( C, R \)

\[ E[r_w] = \langle \| W(As+n) - s \|^2 \rangle \]

\[ = \langle \| (WA-I)s + Wn \|^2 \rangle \]

\[ = \langle \| Ms + Wn \|^2 \rangle \]

where \( M = WA-I \)

\[ = \langle \| Ms \|^2 + \| Wn \|^2 \rangle \]

\[ = \text{diag is noise variance (already squared) tr(MRMT) + tr(WCWT)} \]

\[ \text{trace is sum of diag elements} \]

\[ \text{[re-expand]} = \text{tr (WARA}^TW - RA}^TW - \text{WAR} + R) \]

\[ + \text{tr (WCWT)} \]

Explicitly minimize by taking derivative wrt \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARA}^T - 2RA}^T + 2WC \]

\[ W = RA}^T \]

\[ W(A(AA}^T + C) = RA}^T \]

\[ W = RA}^T(AA}^T + C)^{-1} \]

\( W \) is inverse solution operator:

\[ \text{W is inverse solution operator} : \begin{bmatrix} \text{Sensors} \\ W \end{bmatrix} \]

\[ \text{equivalent to minimum norm and Tikhonov regularized inverse if } C, R \text{ proportional to identity matrix (i.e., sensor noise & sources independent and equal variance)} \]

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\[ W = R A^T (A R A^T + C)^{-1} \]

- "Minimum norm" solution
  (find \( \hat{\theta} \) with smallest norm \( \| \hat{\theta} \| \))

- the minimum norm solution appropriately downplays deeper (=weaker scalp signal) sources since these are more likely to fall into the noise floor

- "Problems" of minimum norm:
  - deeper sources get displaced to the surface

- Small superficial sources "win" because of approx inverse square form of true solution,
  \( \Rightarrow \) smaller norm of distributed superficial soln

- Can't fix by increasing priors of deep sources!!
  \( \Rightarrow \) that will give deep sources given noise as input!!
**Inverse Solutions to Ill-Posed Compared**

\[ s = Wx \]

- **Sources**
- **Sensors**
- **Sensor Data**

- How to use the inverse solution, \( W \)
- Same \( W \) for all time points

**Minimum Norm Solution**

- i.e., norm \( ||x|| \) of solution
- is smallest of infinitely many alternate solutions

**Linear Inverse Operator**

\[ W = RA^T(ARA^T + C)^{-1} \]

- From error minimization derivation

\[ W = (A^Tc^{-1}A + R^{-1})^{-1}A^TC \]

- Alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses)

\[ \begin{bmatrix} W \end{bmatrix} = \left( \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \right) + \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} C \end{bmatrix} \]

- Both square in # of sources (large)
- Hard inverses
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on surface often have different orientations, surface normal constraint can help (since fwd soln A, B very different)
- but, since point spread functions typically extends across sulci, artificial sign reversals occur

- Solutions

1) Ignore sign \( \rightarrow \) saves useful orientation info!

2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole
   \( \rightarrow \) more appropriate when averaging across subjects, since detailed configurations vary a lot
   \( \rightarrow \) also, fills in bottom of sulci (else unsigned stripes)
- insert FMRI values for Rii's
- but still allow other sites to have non-zero Rii's
- pathologies occur if solution restricted completely to FMRI points by setting non-FMRI Rii's to zero → set to small number instead!

- this allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispensed to small distributed values at a large number of vertices

visible in both EEG/MEG and FMRI

visible only in EEG/MEG and not FMRI → distributed at small amplitude across many vertices
NOISE SENSITIVITY NORMALIZATION

\[ W = \text{RA}^T (\text{AR} \text{A}^T + \mathbf{C})^{-1} \]

- Multiply inverse operator by noise sensitivity matrix, \( \mathbf{D} \) (diagonal)

\[ D_{ii} = \text{diag} : \sqrt{\mathbf{WCW}^T} \]

\[ \mathbf{W}^{\text{ns-norm}} = \mathbf{DW} \]

normalize each row of inverse operator by noise sensitivity at that location

\[ \mathbf{s}_i^{\text{ns-norm}} = (\mathbf{W}^{\text{ns-norm}} \mathbf{x})_i = (\mathbf{DWx})_i = \frac{\mathbf{W}_i \mathbf{x}}{\sqrt{(\mathbf{WCW}^T)_i}} \]

if assume Gaussian white noise,

\( \text{noise covariance } \mathbf{C}_i \) is multiple of \( \mathbf{I} \), so

\[ \mathbf{W}_i^{\text{us-norm}} = \frac{\mathbf{W}_i^{\text{orig}}}{|| \mathbf{W}_i^{\text{orig}} ||} \]

i.e., scale each row of \( \mathbf{W} \) by single value — the norm of that row

row of \( \mathbf{W} \) is:

\[ \frac{\mathbf{W}_i^{\text{orig}} \mathbf{x}}{|| \mathbf{W}_i^{\text{orig}} ||} \]

\[ \frac{\mathbf{W}_i}{|| \mathbf{W}_i ||} \]

\[ \text{inverse solution coefficients for one source} \]

\[ \text{scale by norm of this row} \]

i.e., if inverse solution for source is big (e.g., deep source), noise normalized inverse for that source reduced by scaling
**Noise Sensitivity Normalization (2)**

Shallow source (unit strength)  

- fund big  
- inv small

Deep source (unit strength)  

- fund small  
- inv smaller because of minimum norm

\[
S_i = \frac{\tilde{W}_{i, \text{orig}} \cdot \tilde{X}}{\| \tilde{W}_{i, \text{orig}} \|}
\]

- effect on inverse solution: more like significance vs actual power
- effect on point-spread function is to equalize shallow & deep

Point spread functions:

- Shallow spread out more than min norm
- Deep shrink to same as shallow

Point spread functions are noise normalized.
**Conclusions**

- More EEG or more MEG better.
- EEG better than MEG (cf. radial) (EEG far less currently less accurate).
- Biggest gain from adding small # EEG (n MEG) (e.g. 20) to many MEG (n EEG) (e.g. 150).
- Easier to add many MEG, so: optimal < 30 EEG, < 300 MEG.
- EEG/MEG forward-solution-scaling-factor error causes more cross-talk.
**Music (1)**

(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

- Using **sensor covariance**

\[
D = \langle x x^T \rangle = \sigma_x^2 I + \sum \sum \sigma_{ij} C_{ij}(i,j) A_i A_j^T
\]

\[
\text{sensor noise} \quad \text{source condition} \quad \text{times function}
\]

\[
D = \mathbf{U} \Lambda \mathbf{U}^T = \begin{bmatrix} U_1 & U_2 & \ldots & U_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \\ \vdots \\ U_n^T \end{bmatrix}
\]

\[
\text{eigenvector} \quad \text{corresponding eigenvalue}
\]

= find most significant **spatial patterns**

in sensors over time

---

**Project forward solns onto these spatial patterns**

(“**project**” = dot prod = similarity) for each point in brain

\[
\mathbf{\xi}_i = A_i \mathbf{U} \Lambda \mathbf{U}^T A_i^T
\]

\[
\text{one num for each source} \quad \text{all eigenvectors of sensor spatial patterns} \quad \text{find for one source}
\]

\[
\Rightarrow \text{big single number if forward soln looks like } \mathbf{U} \text{'s}
\]
(2) \[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

\[ R = \begin{bmatrix} R_1 & \cdots & R_n \end{bmatrix} \]

\[ W = R A^T (A R A^T + C)^{-1} \]

- like parallel resistance
  \[ R_{\text{parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \cdots} \]

- any high resistance (\( R_i \)) decreases overall resistance (small \( R_{ii} \))

- i.e., if forward soldi
  has appearance like any low eigenvalue spatial pattern, it gets devalued
- How it fixes min norm problem

- vertical point

- multiple points

- N.B., if two sources (i.e., diff. highly correlated)

- Note: dual music; lacks possible

- "dual music" marks possible

- NB: if two sources (i.e., diff. highly correlated)

- Note: dual music; lacks possible