MAGNET HARDWARE

1. $B\Phi$ field from superconducting magnet
2. Gradient coils
3. Body RF transmit/receive
4. RF receive-only
5. Shim coils

$B\Phi \rightarrow z$ (longitudinal)
$B1 \rightarrow x, y$ (transverse)

RF transmit body coil

RF receive-only head coils

Max gradient:
$[80 \text{ mT/m}]$
$[200 \text{ T/m/sec}]$

1T = 10,000 Gauss
Earth: 0.25 - 0.6 G
25 - 65 mT

$Y/2\pi = 42 \text{ MHz/T}$

Circularly polarized $B1$ field

RF transmitter (30 kW)
RF receiver

$B1$ is several orders of magnitude smaller than $B0$

Non-superconducting water-cooled, external shield

Three 1.5 million watt amplifiers to add ripples to $B\Phi$ field
Spin & Precession

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei with odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field.
- Classical picture:
  - Current loop from spinning charge (right-hand rule).
  - N.B.: Classically this would cause EM radiation, spin-down.

Stern-Gerlach experiment:
- Pass silver atoms through strong magnetic field → split into just 2 beams.

Microscopic picture:
- No strong magnetic field, \( B_0 = 0 \).
- Strong magnetic field, \( B_0 \).
- Strong \( B_0 \) plus oscillating field, \( B_1 \).

Precession:
- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

\( 2\pi f = \frac{\omega}{B_0} \) Larmor frequency.

\( B_0 \) is static field.

\( \text{gyro} \) is static field.

\( \text{gyro} \) is static field.

\( M_2^0 = \frac{\omega}{B_0} \)

Bulk equilibrium magnetization:
- \( M_2^0 = \frac{\gamma h}{4 K T_s} \)
- \( I = /perp \frac{1}{3} \)
- \( N \)

\( \gamma \) is gyromagnetic ratio.

\( h \) is Planck's constant.

\( B_0 \rightarrow M_2^0 \propto \) proportional to \( B_0 \) strength.

\( N \) is proportional to number of spins.

\( K \) is Boltzmann constant.

\( T_s \) is absolute temperature of sample.
BLOCH EQUATION

- Time-dependent behavior of \( \vec{M} \) in the presence of an applied magnetic field (excitation \& relaxation)

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{Y}_B
\]

- In the Larmor-rotating coordinate system, a tilt with a phase shift from a standard \( B_1 \) excitation is rotation around \( x \)-axis.

- Longitudinal and transverse relaxations

\[
\frac{dM_z(t)}{dt} = -\frac{M_z(t) - M_0}{T_1}
\]

\[
\frac{dM_{xy}(t)}{dt} = -\frac{M_{xy}(t)}{T_2}
\]

- Solution to equations above: time-dependent free precession eg's

\[
M_z'(t) = M_z^0 \left(1 - e^{-t/T_1}\right) + M_z'(0) e^{-t/T_1}
\]

\[
M_{xy}'(t) = M_{xy}'(0) e^{-t/T_2}
\]
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (Vector)
  - Applies to complex numbers
  - Generalizes to any D
  \[ \|\vec{c}\| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

  **Dot product**
  (= inner product)
  (= "scaled projection onto")
  \[ c = \vec{a} \cdot \vec{b} = [b_x, b_y, b_z] [a_x; a_y; a_z] = a_x b_x + a_y b_y + a_z b_z \]
  - Scalar
  \[ p = \|\vec{a}\| \cos \theta \]
  \[ c = p \|\vec{b}\| \]
  N.B. equals: \( \vec{a} \cdot \vec{a} \)
  \[ \frac{\vec{a} \cdot \vec{a}}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \]
  \[ \text{length of } \vec{a} \]
  \[ c = \|\vec{a}\| \|\vec{b}\| \cos \theta \]
  \[ \leftrightarrow \text{zero if } \vec{a}, \vec{b} \text{ orthogonal} \]

  **Cross product**
  (= outer product)
  (Can be generalized: see "geometric algebra")
  \[ \vec{c} = \vec{a} \times \vec{b} = [0, -b_z, b_y; b_z, 0, -b_x; -b_y, b_x, 0] [a_x; a_y; a_z] \]
  - Vector
  Right-hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \): thumb is \( \vec{c} \)
  Geometric algebra: bivector plane area
  \[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin \theta \]
  \[ \leftrightarrow \text{max if orthogonal} \]
  Unique orthogonal specific to 3D

  **Complex multiply**
  (See also quaternions, geometric algebra generalization)
  \[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x, -b_y] [a_x; a_y] = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]
  - Vector
  \[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \]
  \[ \leftrightarrow \text{like real nums} \]
  - Angles add
  - Magnitudes multiply
  \[ \text{sum of angles: } \theta_1 + \theta_2 \]
EFFECTS OF $\vec{M}$, $\vec{B}$, and $\theta$ ON PRECESSION FREQ.

Bloch 1st term: $\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}$

Cross prod. properties review:

$$\\begin{align*}
| \frac{d\vec{M}}{dt} | &= | \vec{M} | | \vec{B} | \sin \theta
\end{align*}$$

Starting condition

$\Rightarrow$ now see effects of changing $\vec{M}$, $\vec{B}$, $\theta$

Change $\vec{M}$ length

$\Rightarrow$ $\frac{d\vec{M}}{dt}$ proportionally larger, so canals effect of larger $\vec{M}$$\Rightarrow$ same precession freq. as starting cond.

Change $\theta$ between $\vec{M}$ and $\vec{B}$

$\Rightarrow$ $\frac{d\vec{M}}{dt}$ goes up (then down) as $\sin \theta$
$\Rightarrow$ but circumference also goes up as $\sin \theta$, cancelling again

$\Rightarrow$ same precession freq.

Change $\vec{B}$ length

$\Rightarrow$ $\frac{d\vec{M}}{dt}$ goes up, proportional to $\vec{B}$
$\Rightarrow$ but circumference is same at starting cond.

$\Rightarrow$ increased precession freq. ($\omega = \gamma \vec{B}$)
Simple Matrix Operations

Basic Idea

- A matrix \( \begin{bmatrix} \text{rotates} & \text{scales} \end{bmatrix} \) a vector

\[ \mathbf{b} = M \mathbf{a} \]

3D Example

\[ \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

Add Translate (after rotate/scale)

- Commonly used "hack" for aligning vol

- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} & \text{then} & \text{translates} \end{bmatrix} \) (4th D = 1)

\[ \begin{bmatrix} b_x \\ b_y \\ b_z \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \times 3 \ rot/scale \\ 0 \\ 0 \\ 0 \end{bmatrix} \frac{\partial x}{\partial a_x} \frac{\partial y}{\partial a_y} \frac{\partial z}{\partial a_z} \]

N.B.: Have to keep track of order!!

- Rotate/Scale then Trans \( \neq \) Trans, then Rot/Scale

- Change rot component: untranslate, rot, retranslate

3 Special Cases (3D): Rotate around each major axis without changing length (scale = 1.0)

- Rotate around x-axis:

\[ R_{x}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix} \]

E.g., 90° flip

- Rotate around y-axis:

\[ R_{y}(\alpha) = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix} \]

E.g., 180° flip to avoid add 180° phase after 90° flip on x'

- Rotate around z-axis:

\[ R_{z}(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

E.g., precession with 3β along z'

General Case

- Rotate around general z'-axis:

\[ R_{a'}(\alpha) = R_{z}(\theta) R_{y}(\phi) R_{z}(\alpha) R_{y}(\phi) R_{z}(\theta) \] (quaternions are more efficient)
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

\[ dM_{xy}(t) = - \frac{M_{xy}(t)}{T_2} \]

Solution:

\[ M_{xy}(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

Goal:
1) find eq. whose derivative satisfies diff. eq.
2) also find soln. (one of many) that passes thru init condition

\[ \Rightarrow \text{since own diff. eq. is: derivative of funct. = const. same funct.} \]

\[ \Rightarrow \text{try exponential, since derivative (e^x) = e^x} \]

Den.:

\[ M(t) = \frac{-1}{T_2} \cdot \frac{M(t)}{\text{const.}} \]

N.B. this function is the "unknown" like the x in x + 1 = 3

One soln:

\[ M(t) = e^{-t/T_2} \]

N.B.: same as M(t)

Take deriv. to check

\[ M'(t) = \frac{-1}{T_2} \cdot \frac{M(t)}{\text{same var}} \]

Another soln:

\[ M(t) = \text{const.} \cdot e^{-t/T_2} \]

N.B. again same as M(t)

Take deriv. to check

\[ M'(t) = \frac{-1}{T_2} \cdot \text{const.} \cdot e^{-t/T_2} \]

Initial condition

Information added to soln. (not from diff. eq.!!)

\[ M'(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

Magnetization immed. after RF (B1) ends

\[ M_{xy}(0) \]

\[ t \rightarrow \]
VERIFY SOLUTION TO T1 RECOVERY

- Slightly more complex T1 sol'n compared to T2 sol'n

T2 sol'n verify (as prev)

T1 solution verify

\[
\frac{dM}{dt} = \frac{M_{xy}}{T_2}
\]

original diff eq.

\[
M'(t) = \frac{-1}{T_2} M(t)
\]

make unknown funct M(t) more visible

\[
M(t) = M_{xy}(0) e^{-t/T_2}
\]

proposed solution

\[
M'(t) = \frac{-1}{T_2} M_{xy}(0) e^{t/T_2}
\]

test by take deriv.

\[
M'(t) = \frac{-1}{T_1} (M(t) - M_z^0)
\]

\[
M(t) = \frac{M_z^0}{T_2} \left( 1 - e^{-t/T_2} \right) + M_z(0) e^{-t/T_1}
\]

\[
M'(t) = M_z^0 e^{-t/T_2} + M_z(0) e^{-t/T_1}
\]

\[
M'(t) = 0 + \frac{1}{T_1} M_z^0 e^{-t/T_2} - \frac{1}{T_1} M_z(0) e^{-t/T_1}
\]

\[
M'(t) = \frac{-1}{T_1} \left( -M_z^0 e^{-t/T_2} + M_z(0) e^{-t/T_1} \right)
\]

- derivative in original T1 eq. says M(t) (minus M_z^0)

\[
M'(t) = \frac{-1}{T_1} \left( M(t) - M_z^0 \right)
\]

solution \[
M_z^0 - M_z^0 e^{-th/T_2} + M_z(0) e^{-th/T_1}
\]

- which equals our re-calculated derivative:

\[
M'(t) = \frac{-1}{T_1} \left( -M_z^0 e^{-t/T_2} + M_z(0) e^{-t/T_1} \right)
\]
Bloch Eq. - Matrix Version

\[ \frac{dM}{dt} = \dot{M} \times \vec{B}_0 \]

Differential Eq.:
\[ \frac{d\dot{M}}{dt} = \dot{M} \times \vec{B}_0 \]
\[ \dot{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = R_z(\omega t) \dot{M}(0) \]

Solution:
\[ \dot{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} e^{t/T_2} \cos \omega t e^{t/T_2} \\ -\sin \omega t e^{t/T_2} \\ 0 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

Include Relaxation
\[ \frac{d\dot{M}}{dt} = \dot{M} \times \vec{B}_0 - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1} \]

Differential Eq.:
\[ \frac{d\dot{M}}{dt} = \dot{M} \times \vec{B}_0 \]
\[ \dot{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{T_2} & \frac{1}{T_2} & 0 \\ -\frac{1}{T_2} & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_1} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z^0/T_1 \end{bmatrix} \]

Solution:
\[ \dot{M}(t) = \begin{bmatrix} e^{t/T_2} \cos \omega t e^{t/T_2} \\ -\sin \omega t e^{t/T_2} \\ 0 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0) \end{bmatrix} \]
**Excitation in the Rotating Frame**

- Original Bloch eq. in laboratory frame

\[
\frac{d \vec{M}}{dt} = \vec{M} \times \vec{B}
\]

- Add on-resonance $B_1$ to $\vec{B}_P$

\[
\vec{B} = B_1(t) \left( \cos \omega_0 t \hat{z} - \sin \omega_0 t \hat{j} \right) + B_P \hat{k}
\]

- Matrix version

\[
\frac{d \vec{M}}{dt} = \begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
0 & -\omega_0 & 0 \\
\omega_0 & 0 & w_1(t) \cos \omega_0 t \\
0 & -w_1(t) \sin \omega_0 t & 0
\end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
\]

- Substitution to convert to the rotating frame

\[
\begin{align*}
\vec{M}_{\text{rot}} &= \vec{M} \cdot \hat{z} \left( \omega_0 t \right) \\
\vec{B}_{\text{rot}} &= \vec{B} \cdot \hat{z} \left( \omega_0 t \right)
\end{align*}
\]

- After substitution any off-resonance appears as residual $B_P$ ($B_2$)

(see off-res. notes page)

- Rotating frame < on-resonance

*basic excite, $B_1x$-only

\[
\text{rotating frame < on-resonance} \quad \text{no gradient}
\]

\[
\text{< removes } \omega_0, \cos/\sin
\]

- Rotating frame < off-resonance

*general, $B_1x$-only, incl gradients

\[
\frac{d \vec{M}}{dt} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & w_1(t) \\
0 & -w_1(t) & 0
\end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix}
\]

- Gradient: $\omega(z) = \Gamma_{g,z}$

- Off-res: appears as residual $B_3$, lifting $B_1$ vect.

- This means $\vec{M}$ vect. update will contain component that rotates $\vec{M}$ around $z$-axis (in rotating coords = phase)

- Rotating frame < on-resonance

*small tip approx.

\[
\frac{d \vec{M}_{\text{rot}}}{dt} = \begin{bmatrix}
0 & w_1(t) & 0 \\
-w_1(t) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix}
\]

- Small tip easier to solve!
**Bloch Eq. Summary**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_0) \hat{k}}{T_1}
\]

(P lab-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  1. 3 component of \(\frac{d\vec{M}}{dt}\) update vector
  2. Larmor freq. component 7-9 orders magnitude larger than \(T_2, T_1\) decay
  3. \(\vec{B}_1\) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1) **lab frame**
   - Just precession

2) **rotating frame**
   - \(\vec{M}\) stopped
   - That is, \(\vec{B}_0 = 0\)

3) **add \(\vec{B}_1\)**
   - \(\vec{B}_1\) also stopped!
   - But \(\vec{M} \times \vec{B}\) still works!
   - "Precess" around \(\vec{B}_1\) axis

4) **off-resonance**
   - Slow precess, now around tilted \(\vec{B}_{eff}\)
   - Apparent \(B_z\) comp. from residual precess. around \(z\) from off-resonance
**RF Field Polarization**

- Polarization (change of direction) of magnetic field (vs. electric field)

- Linearly polarized field
  \[ \mathbf{B}_1(t) = B_1 \cdot \cos \omega t \hat{\mathbf{x}} \]
  Magnitude \( \{ -1, 1 \} \cdot 1 \)

- N.B.: \( \mathbf{B}_1 \) adds to much larger \( \mathbf{B}_0 \)

- Circularly polarized field (quadrature)
  \[ \mathbf{B}_{1c}(t) = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ \mathbf{B}_0 + \mathbf{B}_1 \]
  \( \mathbf{B}_0 + \mathbf{B}_1 \) wiggles L/R across time

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF field

**Typical 90° flip**
(around x-axis)

**Typical 180° flip**
(around opposite y-axis)
**SIGNAL EQUATION**

\[ \Phi(t) = \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- Magnetic flux through coil
- Scalar integral

\[ V(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int_{\text{obj}} \mathbf{B}(\mathbf{r}) \cdot \mathbf{M}(\mathbf{r}, t) \, d\mathbf{r} \]

- Faraday's law of induction

- Ignore change in z-component of \( \mathbf{M} \) because so slow
- Substitute \( \mathbf{M}(t) \) with lab frame \( M_{xy}(t) = M_{xy}(0) e^{-\frac{t}{T_2}} e^{-i\omega t} \)
- Simplify:
  1. Ignore decay (assume this \( t=0 \))
  2. Assume phase-sensitive detection

**Laboratory frame Bloch solutions:**

- \( M_L \rightarrow \) same
- \( M_T = M_{xy}(0) e^{-\frac{t}{T_2}} e^{-i\omega t} \)

\[ \mathbf{S}(t) = \int_{\text{obj}} B_{xy}(\mathbf{r}) M_{xy}(\mathbf{r}, 0) e^{-i\frac{\omega_0 t}{T_2}} e^{i\frac{\omega t}{T_2}} \, d\mathbf{r} \]

- Demodulate

**Spatially-dependent resonant freq. in rotating frame**

- \( \omega_0 = \gamma B_0 \)

**Standard Signal Expression**

\[ \mathbf{S}(t) = \int_{\text{obj}} M_{xy}(\mathbf{r}, 0) e^{-i\frac{\omega_0 t}{T_2}} \, d\mathbf{r} \]

- Phase angle in rotating frame

\[ \omega t = \frac{\text{radians} \times \text{sec} = \text{radians}}{\text{sec}} (\phi = \omega \Delta t) \]

- Getting difference converts lab to rotating

i.e., at a single time point, RF signal is vector sum across object of local transverse magnetization vectors.
**PHASE-SENSITIVE DETECTION**

- how we get rotating frame

\[ V(t) \xrightarrow{\text{multiply}} \text{Low-Pass Filter} \xrightarrow{\text{PSD}} S(t) \]

- method for moving very high frequency Larmor oscillations down to tractable frequency range

\[ \sim 123 \text{ MHz} \]

- reference signal

\[ \sim 50 \text{ kHz} \]

- digitization

Demodulated signal \( \propto \) RF coil signal \( \times \) reference (transmitter)

\[ \propto \sin[(\omega_0 + \delta \omega) t] \cdot \sin[\omega_c t] \]

\[ \propto \frac{1}{2} \left[ \cos \delta \omega t - \cos(2 \omega_0 + \delta \omega)t \right] \]

- filter this one out
- with low pass filter

**One freq - freq domain**

- Signal
- Reference
- Demodulated
- After filter (rotating frame!!)

**Chirp - time domain**

- Chirp
- Center
- Demodulated
- Low freq signal

- Two signals are made from a single receiving RF coil
- A quadrature coil can be treated the same way (OK to combine after adding \( \pi/2 \) phase, then PSD)
- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \sqrt{2} \) better)

\[ S(t)_{\text{complex}} = M_x e^{-i \delta \omega t} \]
FID - FREE INDUCTION DECAY, T2*

- Signal (FID) resulting from RF pulse w/ angle $\alpha$

$$\bar{S}(t) = \sin \alpha \cdot \left[ \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \cdot dw \right]$$

- An example spectral density ("Lorentzian inhomogeneity")

$$\rho(w) = M_2^0 \cdot \frac{(\Delta \omega_0)^2}{(\Delta \omega_0)^2 + (w - \omega_0)^2}$$

$$\Delta \omega = \gamma B \Delta B \phi$$

$$\rho(w) = \frac{\rho_0}{(c+x)^2}$$

- N.B. center freq, not original integration variable

$$\bar{S}(t) = \frac{\pi \cdot M_2^0 \cdot \gamma \Delta B \phi}{c^2} \cdot \sin \alpha \cdot e^{-t/T_2} \cdot e^{-i\omega_0 t}$$

$$\frac{1}{T_2} = \frac{1}{T_2} + \frac{1}{T_2'(\Delta \omega \phi)}$$

- Overall decay rate including inhomogeneous $B \phi$

$$\rho(w) = \frac{\rho_0}{c^2 + w^2}$$

- $\rho(w)$ for $\phi$ fixed (N.B. not Lorentzian!)

- Suggestive, since 100 million cycles per second

- Only approximated by $e^{-t/T_2}$
ECHOES — spin echo

$90^\circ - \tau - 180^\circ$, $T_2/T_2^*$ & echo

rotating coords

Just after $90^\circ x'$ pulse $f_{lo} + f_{hi}$ have same phase

relaxation + phase dispersion of $f_{lo} + f_{hi}$ (both from $B > B_0$)

Just after $180^\circ y'$ pulse

(y' pulse like x' pulse but RF has $+90^\circ$ phase)

echo caused by re-phasing of $f_{lo} + f_{hi}$ (w/ decay due to $T_2$)

- remember brief RF just tips vectors while retaining length
relaxation includes tips and shrinks ($M_T$) and grows ($M_z$, echo)

- $180^\circ x'$ pulse works, too, but echo will have $+\pi$ phase (left side in figs above)

- echo generated even if second pulse not $180^\circ$ (see next)

FID decay (and echo growth/decay) described by $T_2^*$ from inhomogeneity

reduction in height of echo compared to initial described by $T_2$, echo fixes the 'star'
ECHOES — spin echo

\[ \alpha_1 - T - \alpha_2 - T \] (both pulses along y' for simplicity)

effect of \( \alpha_y \) pulse

\[
\begin{align*}
M_x' &\rightarrow M_x' \cos \alpha - M_z' \sin \alpha \\
M_y' &\rightarrow M_y' \\
M_z' &\rightarrow M_x' \sin \alpha - M_z' \cos \alpha
\end{align*}
\]

(etc. for \( \alpha_x, \alpha_z \))

effect of \( T \) delay

\[
\begin{align*}
M_x' &\rightarrow (M_x' \cos \omega T + M_y' \sin \omega T) e^{-T/\tau} \\
M_y' &\rightarrow (-M_x' \sin \omega T + M_y' \cos \omega T) e^{-T/\tau} \\
M_z' &\rightarrow M_z' (1 - e^{-T/\tau}) + M_z' e^{-T/\tau}
\end{align*}
\]

Immediately after \( \alpha_1 \) pulse

\[
\begin{align*}
M_x'(0,0) &= -M_z' \sin \alpha_1 \\
M_y'(0,0) &= 0 \\
M_z'(0,0) &= M_z' \cos \alpha_1
\end{align*}
\]

for one isochromat of freq. \( \omega \)

After \( T \) delay

\[
\begin{align*}
M_x'(\omega,T) &= -M_z(\omega) \sin \alpha, \cos \omega T e^{-T/\tau} \\
M_y'(\omega,T) &= M_z(\omega) \sin \alpha, \sin \omega T e^{-T/\tau} \\
M_z'(\omega,T) &= M_z(\omega) \left[ 1 - (1 - \cos \alpha) e^{-T/\tau} \right]
\end{align*}
\]

Immediately after \( \alpha_2 \) pulse (no effect on \( M_y' \); rewrite \( y' \) combine \( x \) and \( y \) eqs.)

\[
\begin{align*}
M_{x'y'}'(\omega,T) &= M_z(\omega) \sin \alpha \left( \sin^2 \frac{\alpha_2}{2} e^{i \omega T} - \cos^2 \frac{\alpha_2}{2} e^{-i \omega T} \right) e^{-T/\tau} \\
&- M_z(\omega) \left[ 1 - (1 - \cos \alpha_2) e^{-T/\tau} \right] \sin \alpha_2
\end{align*}
\]

Time dependent

Free precession around \( z' \)

(new write \( M_{x'y'}'(\omega,T) \))

For a large num of freq's:

- For terms 2 & 3 are dephasing → FID of echo
- Term 1: naphasing → naphase at \( t = 2T \)

**Echo Signal from 1**

\[
\begin{align*}
S(t) &= \sin \alpha \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/2} e^{-iw(t-TE)} dw \\
A_E &= \sin \alpha \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-TE/2} dw
\end{align*}
\]

- 90° y, -T - 90° y, \( S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/2} e^{-iw(t-TE)} dw \)
- 90° y, -T - 180° y, \( S_2(t) = \int_{-\infty}^{\infty} \rho(w) e^{-T/2} e^{-iw(t-TE)} dw \)
- 90° y, -T - 180° y, \( S_3(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-T/2} e^{-iw(t-TE)} dw \)

Etc. for \( A_E \)...

**Echo Amplitude, ignoring freq. dependence of \( T \)**

\[
M_2 \sin \alpha \sin^2 \frac{\alpha_2}{2} e^{-TE/2}
\]
**Echo TRAINS** - spin-echo trains.

- It's (too) easy to make echoes...

\[ E_n = \frac{3^{(n-1)} - 1}{2} \]

Echoes after end of nth pulse:
3 RFs \( \rightarrow \) 4 echoes (time)
6 RFs \( \rightarrow \) 121 echoes (11)

Secondary echo: \[ SE_{1,2} \text{ acts like RF pulse} \]
\[ \alpha_3 \text{ makes an echo from it} \]

\[ SE_{1,3}, SE_{2,3} \] - two more conventional two-pulse spin echoes

Stimulated echo: combined effect of 3

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2T spacing.

- Typically, 90° and 180° applied in different axes \((x', y', z', \ldots)\)
  which reduces phase errors due to imperfect 180° pulses
  (since slightly-off rotation around y' affects phase less)
**EXTENDED PHASE GRAPHS**

- Using full Bloch eq. solutions is tedious 😔
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize 90°, 180°)
- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_L$
- Problem #2: Third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

Rule for effect of $6\alpha$

RF pulse on transverse mag

Rule for effect of $8\alpha$

RF pulse on longitudinal mag

→ Echo when phase path crosses zero
3 - Pulse Echo Amplitudes

- Assume $M_0^z = 1$

RF Transmit

RF Receive

Echo  | Time  | Amplitude
------|-------|------------
$SE_{1,2}$  | $t = 2\gamma_1$  | $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\gamma_1 T_2}$

Special cases:
- $\alpha_1 = 90^\circ, \alpha_2 = 180^\circ$  | $e^{-2\gamma_1 T_2}$
- $2\alpha_1 = \alpha_2$  | $\sin^3 \alpha_1 \cdot e^{-2\gamma_1 T_2}$

$Z^0$ ("secondary")

$t = 2T_2$

$t = 2T_1 + 2T_3$

- $\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\gamma_1 T_2}$

STF ("stimulated")

$t = 2T_1 + T_2$

$t = T_1 + T_2$

$\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-T_3 T_{12}} e^{2T_1 T_{12}}$

N.B.: $T_1$

$SE_{2,3}$

$T_1 = 2T_1 + T_2$

$[1 - (1 - \cos \alpha_1) e^{-T_1 T_2}] \sin \alpha_1 \sin^2 \frac{\alpha_3}{2} e^{-(T_1 + 2T_2) T_2}$

N.B.: $T_1$

$SE_{1,3}$

$T_1 = 2(T_1 + T_2)$

$\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(T_1 + T_2) T_2}$

- $T_1$-dependence in $STF$ (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**Hyper Echoes**

1. Solid lines: 3 solid lines
   1 dashed line
   (N.B.: avoid mist... put Z here vs. Block notes)
   [Diagram showing spin echo]

2. \( \alpha_y = 180^\circ - \alpha_y = 180^\circ \)

3. \( \alpha_y = 180^\circ - \alpha_y = 180^\circ \)

(N.B. mirror image coord syst vs. Block notes)

- Hennig & Scheffler (2001)
- Normalize \( \mathbf{M} \) amplitude \( \Rightarrow 1 \)
- Sphere surface defines 2D plane for \( \mathbf{M} \) moved by:
  1) Vector rotation of \( \mathbf{M} \) around tilted axis in transverse \( x-y \) plane by RF with flip, \( \alpha \), and phase, \( \phi : P(\alpha, \phi) \)
  2) Rotation around \( Z \) by phase evolution due to freq offset, \( \omega \) (360 offset) and time, \( t : P(\omega, t) \)

- Three symmetries:
  - Solid lines: phase evolution \( \Rightarrow 180^\circ \) phase evolution or RF
  - Dashed lines: Just \( 180^\circ \) again

**Practical Use**

- Multi-echo example
- Can also use to prepare, then separate read-out
- Practical prob: \( 180^\circ \) pulses deposit a lot of RF (6x 90°)
  - Prob at high fields
- By arranging to get big echo in middle of \( k \)-space
  - Can get by with much less RF power
**Gradient Echoes** - \( T_2^* \), GE chains

- Initial negative gradient dephases spins
- After \( t = T \) of positive gradient, spins rephase
- Does not correct for \( T_2^* \) inhomogeneities, so echo amplitude is:
  \[ A_E = e^{-t/T_2^*} \]
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay
  \[ \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \]
- Key difference between spin-echo (SE) and gradient echo (GE) is that \( B_0 \) inhomogeneities not encoded
  \[ \Rightarrow \text{hence, echoes are } T_2^*\text{-weighted, not } T_2\text{-weighted} \Rightarrow \text{more susceptible to inhomogeneities} \]
- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware \[ \Rightarrow \text{64 echoes} \]
IMAGE CONTRAST

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

\[
\begin{align*}
M_z^0 & \quad \text{Longitudinal magnetization} \\
M_z & \quad \text{Magnetization}
\end{align*}
\]

\[
\begin{align*}
\text{RF} & \quad \begin{array}{c}
1 \quad 90^\circ \quad \text{TR} \\
2 \\
3 \\
4
\end{array} \\
M_z^0 & \quad (1 - e^{-\text{TR/T1}}) + [\text{ignored}]
\end{align*}
\]

- Simple saturation/recovery w/ no echo

\[
M_z \quad \text{before first pulse} = M_z^0 \\
M_z = 0 \quad \text{immed. after first pulse (i.e., 90° pulse)}
\]

- From Bloch eq, \( M_z \) just before second pulse:

\[
M_z^{(2)}(O_-) = M_z^{(0)}(1 - e^{-\text{TR/T1}}) + M_z^{(0)}(0_+) e^{-\text{TR/T1}}
\]

\[
M_z^{(0)} \quad \text{before second pulse} \\
M_z^{(0)} \quad \text{"regrowth-from-zero" term} \\
M_z^{(0)} \quad \text{"left-immed.-after-pulse" term (U.S. decaying)}
\]

- Given

1. 90° pulse
2. no \( M_{xy} \leftarrow \) pure tip: \( M_{xy} = M_z \)

- Tip existing mag

\[
\begin{align*}
M_z^{(n)}(0_-) & = M_z^{(n)}(0_+) = M_z^0 (1 - e^{-\text{TR/T1}})
\end{align*}
\]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[
I(r) = C \rho(r) (1 - e^{-\text{TR/T1}(r)})
\]

Assume this is 0 because we assume that \( M_{xy} \) (transverse) completely decayed so that a 90° pulse doesn’t generate any initial longitudinal.

Assume immediate recoring of signal.

Spectral dens \( p(r) \) & p. density; underlies equil. \( M_z^0 \).
IMAGE CONTRAST

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
- because they are brighter than all the rest
- because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur (e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec
  TR images reached
  after 8 images
IMAGE CONTRAST

IR (still just saturation-recovery — no echo)

- inversion recovery w/ no echo

RF

180° TI 90° TD 180° 90° 180° 90°

TR

“steady state” after here

- 180° deg pulse reverses longitudinal magnetization

\[ M_{2}' = -M_{2} \]

- recovery to end of first TI from long. part of Bloch eq.

\[ M_{2}' = M_{2} \left( 1 - 2e^{-\frac{T_2}{T_1}} \right) \rightarrow \text{flipped into transverse by second pulse (180° 90°)} \]

- longitudinal then regrows from zero

\[ M_{2}' = M_{2} \left( 1 - e^{-\frac{(TR-T1)}{T_1}} \right) \]

- after second 180°, just change sign again

\[ M_{2}' = -M_{2} \left( 1 - e^{-\frac{(TR-T1)}{T_1}} \right) \]

- apply relaxation eq. again

\[ M_{2}' = M_{2} \left( 1 - e^{-\frac{T_2}{T_1}} \right) - M_{2} \left( 1 - e^{-\frac{(TR-T1)}{T_1}} \right) e^{-\frac{T_1}{T_1}} \]

\[ M_{2}' = M_{2} \left( 1 - 2e^{-\frac{T_2}{T_1}} + e^{-\frac{TR}{T_1}} \right) \]

\[ \rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
**IMAGE CONTRAST**

**SE, IR-SE**

- steady state mag (2nd TR) just before 90°
  
  \[ M_z(t_0) = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) + e^{-\frac{TR}{T_2}} \]

- the echo signal \(M_z\) unlike in simple saturation–recovery FID
  
  has an additional delay before it is recorded, so we have to take account of transverse mag relaxation

  \[ A_E = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

- if we assume \(TE\) much less than \(TR\), then we can simplify:

  \[ A_E = M_z^0 \left( 1 - e^{-\frac{TR}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

  *amplitude echo  proton density TR controls T1  TE controls T2* 

- similar equation for SE-IR

  \[ A_E = M_z^0 \left( 1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

  *amplitude echo  proton density TR controls T1  TE controls T2*
**Image Contrast**

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq., again
  - Assume $M_{x'y'}^n(O_+) = 0 \rightarrow$ transverse dephased before next pulse
  
  $M_{z'}^n(O_-) = M_z^0 (1 - e^{-TR/T_1}) + M_{z'}^{n-1}(O_+) e^{-TR/T_1}$

- Assume we have a small tip angle:
  
  $M_z \cos \alpha \rightarrow M_{z'}^n(O_+) = M_z^n(O_-) \cos \alpha$

- Assume we are in dynamic equilibrium:
  
  $M_{z'}^n(O_-) = M_{z'}^{n-1}(O_-) = M_{z'}^{ss}(O_-)$

Pre-pulse steady state longitudinal

Post-pulse transverse magnetization

Gradient echo amplitude

$$A_E = \frac{M_z^0 (1 - e^{-TR/T_1}) \sin \alpha e^{-TE/T_2*}}{1 - \cos \alpha e^{-TR/T_1}}$$

$T_1$ contrast mainly depends on flip angle, not TR $\rightarrow \cos \theta = 1$ $\rightarrow$ eliminates $T_1$ weight since denominator = numerator
**IMAGE CONTRAST**

MDEFT / 3D FLASH

- Saturate, wait for contrast, invert, wait for contrast, FLASH (cont'd out)

A) \( M_z' \) (just after 90\(^\circ\)) = 0 (perfect 90\(^\circ\))

B) \( M_z' \) (after TD) = \( M_z^0 \left(1 - e^{-TD/T2}\right) \) (Blach term #1)

C) \( M_z' \) (just after invert) = \( \cos \phi \ M_z^0 \left(1 - e^{-TD/T2}\right) \)

D) \( M_z' \) (after TI) = \( M_z^0 \left(1 - e^{-TI/T2}\right) + \left[\cos \phi \ M_z^0 \left(1 - e^{-TD/T2}\right)\right] e^{-TI/T2} \)

Special case TI=TD: \( M_z^0 \left[1 - \left[1 - \cos \phi (1 - e^{-TD/T2})\right] e^{-TI/T2}\right] \)

- after the first RF pulse:

E) \( M_z' \) (just after "fista") = \( M_z^0 \left[1 - \left[1 - \cos \phi (1 - e^{-TD/T2})\right] e^{-TI/T2}\right] \sin \alpha \)

\( \rightarrow \) using hard 180\(^\circ\) inversion, con cancel hard alpha B1 inhomogeneities (Thomas et al., '05)
MAGNETIZATION TRANSFER CONTRAST

- Protons in macromolecules & bound to membranes have wide range of resonant freqs ("bound") → T2 = 1 msec
  i.e., signal is not visible w/ usual TE
- Free protons in blood, CSF, water have narrow range of resonant freqs ("free") → T2 = 50 msec

- Mag transfer pulse sequence
  1) Off-center freq pulse to hit "bound" (but don't hit water too hard)
  2) Wait for magnetization transfer from saturated longitudinal M_z of "bound" → M_z of "free"
  3) Result of transfer → attenuation

> N.B. this always happens a little (cf. T1-weighted, T2-weighted)
  Something to keep in mind if hard pulse (wide freq)

- Used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- View w/ MIP: maximum intensity projection along lines

max

→ view as movie
**Signal-to-Noise, Contrast-to-Noise**

- **Signal-to-noise defined as:** 
  \[ \text{SNR} = \frac{\text{avg signal}}{\text{s.d. non-object region}} \]

- **Temporal SNR:** 
  \[ \text{ESNR} = \frac{S_A - S_B}{S_B} \]

- "Contrast" is a difference.

- **Contrast-to-noise ratio:**
  \[ \text{CNR}_{AB} = \frac{S_A - S_B}{\sigma_n} \]

- Spin-echo:
  \[ A_E = M_z^0 \left( 1 - e^{-TR/T1} \right) e^{-TE/T2} \]

**Gradient Echo**

- Long TR
- Short TR

**General rules:** Spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Tissue</th>
<th>T1</th>
<th>T2</th>
<th>PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>950</td>
<td>100</td>
<td>0.80</td>
</tr>
<tr>
<td>WM</td>
<td>600</td>
<td>80</td>
<td>0.65</td>
</tr>
<tr>
<td>CSF</td>
<td>4500</td>
<td>2200</td>
<td>1.00</td>
</tr>
<tr>
<td>Blood</td>
<td>1200</td>
<td>100-200</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
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<th>T1</th>
<th>T2</th>
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</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>760</td>
<td>77</td>
<td>0.69</td>
</tr>
<tr>
<td>WM</td>
<td>510</td>
<td>67</td>
<td>0.61</td>
</tr>
<tr>
<td>CSF</td>
<td>2650</td>
<td>280</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Example:**

\[ T_2^*_{GM} = 50 \]
\[ T_2^{*WM} = 40 \]
SIGNAL-TO-NOISE S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt[4]{N_a z N_x N_y N_z \Delta t}
\]

- Voxel size (to same number)
- Number of repeats (to same size)
- Number of voxels
- Readout timestep

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[\text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}\]

- More voxels (with size of voxels, \(\Delta t\) per readout step constant), \(\sqrt{n}\) effect on S/N
  \[\text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \sqrt{\frac{128 \times 128}{64 \times 64}} = 2 \text{ times better S/N}\]

- # acquisitions \(\sqrt{n}\) better S/N
  \[\text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \sqrt{2} = 1.41 \text{ times better S/N}\]

- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N
  \[\Delta t = \frac{1}{\text{BW}_{\text{read}}}, \text{ digitization timestep during echo acquisition}\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- Must filter out freq's \(> f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**Complex Algebra**

**Real/Imaginary**

add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)

mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

**Angle/Phase**

add: \((A_1, \theta_1) + (A_2, \theta_2) = (A_1 \cos \theta_1 + A_2 \cos \theta_2, A_1 \sin \theta_1 + A_2 \sin \theta_2)\)

multiply (commutative): \((A_1, \theta_1) \times (A_2, \theta_2) = \left( A_1 A_2, \theta_1 + \theta_2 \right)\) → N.B.: 3rd kind of vector multiply, different than dot product and cross product (and G/A. non-commutative pseudoscalar multiply)

divide: \((A_1, \theta_1) \div (A_2, \theta_2) = \left( \frac{A_1}{A_2}, \theta_1 - \theta_2 \right)\)

Complex to Real power: \((A, \theta)^n = (A^n, n\theta)\)

**E^i\theta**

[Expand as series, recognize cos, sin series]

\[e^{i\theta} = \cos \theta + i \sin \theta\]

The real “e” to “purely imaginary” power

\[e^{i\phi} = \cos \phi + i \sin \phi\]

= cos \phi, sin \phi

= Vector on unit circle

\[e^{i\phi^2} = (\cos \phi + i \sin \phi)^n\]

= cos n\phi + i sin n\phi

**Fourier Transform**

\[H(f) = \int h(t) e^{-2\pi i ft} \, dt\]

\[H(\hat{f}) = \int h(t) e^{2\pi i ft} \, dt\]

**Convolution Theorem**

\[F\left[ g(x) \ast h(x) \right] = G(k) \ast H(k) \]

because of FFT, faster if kernel not small

**Convolutions**

\[f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) \, dz\]

Cross-correlation

\[\hat{f}(x) = g(x) \otimes h(x) = \int g(z) \cdot h(x+z) \, dz\]

[the Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function]
Fourier transform (1)

\[
H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-j2\pi ft} \, dt
\]

- How to calculate \( H(f) \) for one \( f = 3 \):
  (real signal: only need 2 correlations)

- \( h(t) \)

- \( e^{-j2\pi ft} \)

- Integrating/summing these multiplies across all \( t \)

- \( \cos (2\pi 3t) \)

- \( \sin (2\pi 3t) \)

- \[ \begin{bmatrix} \cos \sin \end{bmatrix} e^{-j\pi ft} \]

- i.e., two correlations

- real frequency domain

- imaginary frequency domain

- Cartesian (\( A, \phi \))

- Amplitude frequency domain

- Phase frequency domain

- \( \text{like correlating with } \sin \text{ and } \cos \text{ (at each freq.) so we get phase (at each freq.)} \)
Fourier transform (1b)

- $e^{i\phi} = \cos \phi + i \sin \phi$
- $e^{-i\phi} = e^{i(-\phi)}$ (see below)
- $=-\cos \phi - i \sin \phi$

- $e^{i\phi}$ is an "even" function, $\sin$ is an "odd" function

- An orthogonal decomposition

  - $\text{think of discretely sampled } \sin(bx), \cos(bx) \text{ as vectors}$
  - $\text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \text{projection of } \vec{v}_1 \text{ onto } \vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2$

\[
\begin{align*}
\text{Corr}(\cos bx, \sin bx) &= 0 \\
&= \sin \text{ & cos of same frequency are orthogonal} \\
&= \frac{\sin 2x}{\cos 2x} \\
\text{Corr}(\sin bx, \sin 2bx) &= 0 \\
&= \text{different integer freqs of } \sin \text{ & cos are orthogonal} \\
&= \frac{\sin 2x}{\sin 3x} \\
\text{Corr}(\cos bx, \sin 2bx) &= 0 \\
&= \text{as above}
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{x=0}^{x=\pi} \rho(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- Start with spike in image domain
- Take example of spike at $x = 0$
- $[\cos(x), \cos(2x), \cos(kx)]$ all equal 1 there
- All freq's correlate w/ spike at $x = 0$

- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates
- Successively higher freqs

- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-2\pi i k x}$ cos and sin at location of spike
- Real component of FT

- Pos. pair (real) spikes same dist. from origin
- Pos/neg. pair (imaginary) spikes, same dist. orig.
- One spike at distance from origin

$\rightarrow$ This is one way of thinking about what $k$-space means, via correlating it w/ cos's, sin's to get periodic result in image space (Inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

(1) real image ↔ imaginary image
   → y
   x →
   (Zero) → y
   x →

(2) amplitude image ↔ phase image
   → y
   x →
   (Zero) → y
   x →

What you see on screen → view complex vectors directly

(3) complex vectors
   Zero vectors
   → x
   y →

- 3 equivalent representations of image & spatial freq. space
**FOURIER TRANSFORM OF REAL IMAGE (2)**

- what a single k-space point looks like for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

**Image space**
- Offset of stripes is k-space phase
- Brightness of stripes proportional to k-space amplitude

**k-space (spatial freq. space)**
- Distance from center is stripe spacing
- Angle of point perpendicular to angle of stripes
- N.B.: need conjugate point, too

**Inverse Fourier transform**
- (image recon.)

**Phase**
- should be all zero (not same as "stripe phase" above)

**Cartesian dimension of k-space — x- and y- spatial freq**

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin, cos — don't confuse $k_x, k_y$ w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
- 3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)

- Example: cosinusoid in image space, then shifted in x-dir

**REAL IMAGE**

\[ I(x, y) = \cos(x) \]

**FT OF REAL IMAGE**

\[ \mathcal{F}(I(x,y)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y) e^{-i2\pi(xk_x + yk_y)} \, dx \, dy \]

\[ \mathcal{F}(\cos(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(x) e^{-i2\pi(xk_x + yk_y)} \, dx \]

**N.B.:** An example of the "shift theorem" (see below)

\[ I(x, y) = \cos(x - \frac{\pi}{4}) \]

\[ \mathcal{F}(\cos(x - \frac{\pi}{4})) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(x - \frac{\pi}{4}) e^{-i2\pi(xk_x + yk_y)} \, dx \]

**Phase**

\[ 45^\circ \text{ at } k_x = 1, k_y = 0 \]

\[ -45^\circ \text{ at } k_x = -1, k_y = 0 \]
FOURIER TRANSFORM OF IMAGE (S)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

FT OF REAL IMAGE

\[ H(k) = \int h(x) e^{-j2\pi k x} dx \]

Positive center k-space

Avg image brightness \( \leq 1 \) (real)

FT

[the center of k-space is zero w/ pure sin or cos image b/c avg. brightness = 0]

FT, FT\(^{-1}\)

COMPLEX IMAGE

\[ I(x, y) = \cos(x) - j \sin(x) \]

FT OF COMPLEX IMAGE

"missing" spike results in single spike correlating w/ \( \cos \) and \( \sin \)

N.B.: this \( k \)-space is non-Hermitian:
K-space will only have Hermitian symmetry if image is real:

Hermitian symm. when complex conjugate \( (= \text{complex mm w/ sign flipped in imag. part}) \)
Is equal to funct. \( w/ \text{mag arg} \):

1D: \[ H(k) = H^*(k) \]

2D: \[ H(-k_x, k_y) = H^*(k_x, k_y) \]

[Note: this is also exactly what a gradient does to image space! ]
**Gradient Coils**

- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field $B_z$.

- For example, the x gradient coil induces a ramp in z-component of the magnetic field when moving in the x-direction:

$$B_{G,z} = G_x x$$

---

**Since a pure linear gradient of $B_{G,z}$ in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction ($B_{G,x}$ and $B_{G,y}$).**

- The other magnetic field components are usually ignored because they are so small relative to $B_{G,z}$, since $B_{G,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{G,z}$, $B_{G,x}$, and $B_{G,y}$.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms $B_{G,x}$, $B_{G,y}$ are known and can be included in the reconstruction process.

\[ \Delta \phi G_x(x) \approx \frac{-x^2 C_0^2 t}{2 B_0} \]
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim
  
  \[ B_z \]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

  \[ \text{in practice, Gaussian pulse envelope good too} \]

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq. encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: $90^\circ \rightarrow 52^\circ$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho. tipping is non-linear)

\[ B_1(t) \propto \sqrt{\int_{-\infty}^{\infty} p(f) e^{-i2\pi ft} \, df} \]

- Time dependent
- RF stimulation (complex)

**Solve with:** \( p(f) = \text{frequency band} \)

**\[ B_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) e^{-i2\pi f_c t} \]**

- Modulation (complex) at center freq. \( f_c \)
- Sinc envelope width inversely proportional to \( f_w \)
- Larimer oscill. at center freq.

Fourier Transform Pair, Rules:
- Convolution in one domain is multiplication in the other
- Convolution with delta function, impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{1}{t} \rightarrow \)

- Frequency \( \longleftrightarrow \) time
- \( \ast \) convolution \( \longleftrightarrow \) multiply
- \uparrow \uparrow \longleftrightarrow \frac{1}{t} \rightarrow
SLICE SELECT RF PULSES

Interleaved Acquisition $\rightarrow$ better S/N b/c imperfect slice profile
      spin history prob if motion

Common RF pulses
  non-selective pulse
    ("hard" pulse)

standard slice select sinc

Gaussian
$\rightarrow$ pulses need to be "apodized" (have "foot" removed)
$\rightarrow$ multiply by function so begin/end of pulse is differentiable

Fat Saturation
- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
- fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq
$\rightarrow$ need high quality (narrow-freq) pulse to avoid saturate water!

How To
  1) fat sat pulse
  2) wait T2 so fat signal decays, but no T1 regrowth if fat
  3) RF stim for water "protons- of interest"

Adding Another Gradient Tilts Slice

- with 3 gradients on, can excite arbitrary angle plane
- translate plane by changing either gradient amplitude
  or RF freq band: $\frac{\gamma}{2\pi}$
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq. $\Rightarrow$ gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF $\Rightarrow$ same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs $\Rightarrow$ same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum $\Rightarrow$ FT of FID to get $\Delta x$ offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican Turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)
- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>Signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq. domain</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because $FT \approx FT^{-1}$ (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient $(G_x)$ causes precession rates to vary linearly in $x$-direction

- Different frequency signals are mixed together and recorded as a 1-D signal over time

- Fourier transform, which can convert back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

- Spatial frequencies get confused/confounded with precession frequencies

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

N.B.: Gradient freq. ramp does not need to be exactly same as recording!!
**FREQUENCY ENCODING (2)**

- "Frequency"-encode gradient ($G_x$) turned on during
  during echo causes precession rates
to immediately vary with x-position

  $G_x \rightarrow t \rightarrow$

- at beginning of gradient on, the phase of
  signal coming from each x-position is the same

  **Summed phase angle is what we measure**

- early after gradient on, phase advances
  (because of faster precession frequency)
  arise with greatest
  phase advance at largest x-position

  \[
  \begin{array}{c}
  \text{Single} \\
  \text{time point}
  \\
  \text{early}
  \\
  \hline
  t
  \\
  \hline
  \end{array}
  \Rightarrow
  \begin{array}{c}
  360 \\
  \hline
  \phi
  \\
  \hline
  \end{array}
  \Rightarrow
  \begin{array}{c}
  \text{one cycle of}
  \\
  \text{spatial frequency}
  \\
  \text{of phase angle}
  \\
  (= \text{low spatial freq})
  \\
  \hline
  \end{array}

- later during gradient on, phase advances cause
  multiple wraps around of phase angle across space

  \[
  \begin{array}{c}
  \text{Single}
  \\
  \text{time point}
  \\
  \text{later}
  \\
  \hline
  t
  \\
  \hline
  \end{array}
  \Rightarrow
  \begin{array}{c}
  360 \\
  \hline
  \phi
  \\
  \hline
  \end{array}
  \Rightarrow
  \begin{array}{c}
  \text{multiple cycles of}
  \\
  \text{spatial frequency}
  \\
  \text{of phase angle}
  \\
  (= \text{hi spatial freq})
  \\
  \hline
  \end{array}

- in practice, the lowest spatial frequency ($=0$)
  occurs in the middle of the gradient on time

  because the phase is "wound" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

  $b=\text{max negative}$

  $b=0$

  $b=\text{max positive}$

  $b$ is spatial frequency

  $G_x \rightarrow t \rightarrow$
**FREQUENCY ENCODING (3)**

why each datapoint is 1 spatial freq

Standard Fourier transform:

\[
H(\nu) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i 2\pi \nu t} dt
\]

"\(\nu\)" is often used instead of "\(f\)" for the frequency variable

Imaging equation:

\[
S(\nu) = \int_{-\infty}^{\infty} I(x) \cdot e^{-i 2\pi \nu x} dx
\]

Sum across \(x\) of object

done by RF coil recording sum

Oscillations come from readout phase wrapping, where \(f\) is single spatial freq (e.g., 5) and \(x\) goes across object

\(f = G_x t\), that is, spatial freq depend on amount of time gradient was on (this \(f\) increases with time!)

don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each \(x\) position)
**Alternate Derivation (incl. effects of $G_x$) Signal Eq**

- Oscillators at $w = \gamma B$ at each position (just $x$ for now)

$$ S(t) = m(x) e^{-i \phi(x)} dx $$

- By definition, freq. $w$ is rate of change of phase, $\phi$

$$ \frac{d\phi(x,t)}{dt} = w(x,t) = \gamma B(x,t) \text{ and } \phi(x,t) = \int_0^t w(x,t) dt = \gamma \int_0^t B(x,t) dt $$

- Assuming phase initially 0, $B$ affected by gradients

$$ B(x,t) = B_0 + G_x(t) x $$

$$ \Rightarrow \phi(x,t) = \gamma \int_0^t B_0 dt + \left[ \gamma \int_0^t G_x(t) dt \right] x $$

$$ = \omega_0 t + 2\pi k_x(t) x $$

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

$$ S(t) = \int_x m(x) e^{-i 2\pi k_x(t) x} dx $$

- Amplitude of each oscillator, gradient-controlled phase
PHASE-ENCODE GRADIENT $G_y$

- Turn on gradient after excitation but before readout

- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
  - Higher spatial freq. (more phase wraps) in y-direction

- Phase wraps persist after phase-encode gradient off

- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase

2D Imaging Equation

$$ S(k_x, k_y) = \iint I(x, y) \cdot e^{-i2\pi(k_x x + k_y y)} \, dx \, dy $$

- Signal recorded at single time point (one readout point)
- Complex signal (from phase-sensitive detection)
- Sum across x-y plane

- Image (strength of magnetization at each x-y point)
- Scalar (what we try to reconstruct)

- Phase (vector of unit length and particular angle which is function of $G_x$ and $G_y$)
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING - two phase-encode gradients

- use $z$-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example
  (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [3PGR]

$$S(k_x, k_y, k_z) = \int \int \int dx dy dz$$

$$I(x, y, z) = e^{-2\pi (k_x x + k_y y + k_z z)}$$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase
  all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
  phase stripes created throughout volume vs. slice:

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

- Since the phase-encode gradient and the freq encode gradient both affect the result is a rotation of phase "stripes" when the two add

- successive read out steps:

  - More rotation higher spatial freq

  - Small phase encode G_y

  - Large phase encode G_y

  - Read out G_x

  - TE

  - t_1, t_2, t_3

- 3D phase encode w/ G_y and G_z starts rotated in y-z plane

  - Large phase encode G_z

  - Small phase encode G_y

  - Y

  - TE

  - X

  - t_1, t_2, t_3
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ k = \frac{1}{t} \int_{0}^{t} G(t) \, dt \]

spatial freq recorded at t = record time

gradient strength as function of t

simple form up integral w/ boxcar gradient

record data point here

all of the following gradients end up at the same point in k-space:

**Frequency-encode FID**

\[ RF \xrightarrow{90^\circ} G_x \]

samples

**Frequency-encode gradient echo**

\[ RF \xrightarrow{90^\circ} G_x \]

**Frequency-encode spin-echo (plus gradient echo!!)**

\[ RF \xrightarrow{90^\circ} \tau \bar{B}_0 \tau \]

\[ G_x \]

**Phase-encode then frequency encode gradient echo**

\[ RF \xrightarrow{90^\circ} Gy \]

\[ G_x \]

\[ G_x + Gy \]
**IMAGE RECONSTRUCTION**

\[
S(k_x, k_y) = \iint \sqrt{I(x, y)} e^{i 2\pi (k_x x + k_y y)} \, dx \, dy
\]

**Image (real)**

**Gradient - caused phase maps (complex)**

\[
I(x, y) = \iint S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} \, dk_x \, dk_y
\]

**Signal (complex)**

**RF coil sums**

- Ideally, image is real
- In practice, complex
- Use amplitude image:

\[
\frac{A}{2} \quad \text{i} \theta
\]

Adding exponents

- Same as multiplying two \( e^{i 2\pi \cdot \text{exponent}} \)

\[
= \iint S(k_x, k_y) e^{i 2\pi x k_x} e^{i 2\pi y k_y} \, dk_x \, dk_y
\]

Same as two sequential 1D FFTs (actual code)

\[
= \int_{k_y} \left[ \int_{k_x} S(k_x, k_y) e^{i 2\pi x k_x} \, dk_x \right] e^{i 2\pi y k_y} \, dk_y
\]

- In practice, finite number of samples, \( N \) and \( M \), are collected
- \( k_x \) and \( k_y \) directions of k-space (integral \( \rightarrow \) discrete sum)

\[
I(x, y) = \sum_{m=-N/2}^{N/2-1} \left[ \sum_{n=-M/2}^{M/2-1} S(n, m) e^{i 2\pi n \Delta k_x} e^{i 2\pi m \Delta k_y} \right] e^{i 2\pi m \Delta k_y \Delta k_x}
\]

Sampling interval in k-space
**Sampling**

Aliasing, FOV

- must consider effects of sampling
  - limited points in k-space
  - limited in range of frequencies sampled \((k_{\text{min}} \rightarrow k_{\text{max}})\)
  - limited in rate of sampling \((\Delta k)\)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

---

**Space**

**Spatial Frequency**

---

- correct reconstruction
- infinite frequency range
- infinitely fine sampling
- infinite frequency range
- finite spacing of samples
- as above w/ blurring, ringing
- as above w/ blurring, ringing
- finite frequency range
- finite spacing of samples
- aliasing occurs in spatial domain
- replicas overlap, causing wraparound
- finite frequency range
- too-wide spacing of samples
- thus, finer sampling of same range of spatial freqs increases FOV
UNDER/OVER SAMPLE

$FOV_x = \frac{1}{\Delta k_x}$

$\delta_x = \frac{FOV_x}{N} = \frac{1}{N \Delta k_x}$

$FOV$ (distance to repeat) is reciprocal of spatial frequency sampling interval

Pixel size is $FOV$ divided by $K$-space sample count

3 more examples (not incl. less samples to same spat. freq. (bottom last page))

Basic Image

Same num samp. to $2 \times$ spat. freq.
(i.e. gradients stronger or time ON longer)

$N=10$
$K_x=5$
$\Delta k_x=1$
$FOV=1$
$\delta_x=0.1$

$2 \times$ num. samples to same spat. freq.
(i.e. gradients weaker or time ON shorter)

$N=10$
$K_x=10$
$\Delta k_x=2$
$FOV=2$
$\delta_x=0.05$

$2 \times$ number samples to $2 \times$ spat. freq.
(i.e. gradients stronger or time ON longer)

$N=20$
$K_x=5$
$\Delta k_x=0.5$
$FOV=2$
$\delta_x=0.1$

$N=25$
$K_x=10$
$\Delta k_x=1$
$FOV=1$
$\delta_x=0.05$

- Basic Image
- Square pix
- X-pix half width
- Replicas intrude
  [Scanner makes square image "wrap" occurs]
- Square pix
- Twice x-pix count so $FOV=2\times$
- This is "phase oversamp" [Scanner crops to square replicas move out]
- X-pix half width
- Twice x-pix count
- Same $FOV$
- This is decrease pixel size w/o change $FOV$
Fourier Transform Solution to Replicas

1. Image/brain space

2. Sampled data spatial frequency

- Convolve

\[ \mathbf{f}(x) \ast \mathbf{g}(x) \]

= equals

\[ \mathbf{F}(\omega) \ast \mathbf{G}(\omega) \]

Useful FTs

- Rect
  \[ \text{Rect} \left( \frac{x}{w} \right) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi W x) \]

- Gaussian (special case)
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi K^2} \]

- Gaussian (adj width)
  \[ e^{-ax^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{a}} e^{-\frac{\pi K^2}{a}} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n \Delta k) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(K - p \Delta k) \]

limit approach to Fourier transform of comb

\[ \text{Fov} = \frac{1}{\Delta k} \]
\[ \Delta k = \frac{1}{\text{Fov}} \]
POINTER SPREAD FUNCTION

\[ \hat{I}(x) = \Delta k \sum_{n} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:

\[ S(m \Delta k) = 1 \]

- Substitute into recon to get PSF:

\[ h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x} \]

- Simplify

\[ h(x) = \Delta k \frac{\sin (\pi N \Delta k x)}{\sin (\pi \Delta k x)} \Rightarrow \text{periodic} \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc

\[ \text{image} \quad \overset{\text{convolve}}{\rightarrow} \quad \text{FT} \quad \overset{\times \text{multiply}}{\rightarrow} \quad \text{frequency} \]

How PSF modifies ideal (infinite \( k \)) image

\( \times \) ringing

acquisition window (truncates high spatial frequency, \( f \))
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int_I(x) e^{-i2\pi k_x x} \, dx \]

\[ I(x) = \int_{k_x} S(k_x) e^{+i2\pi k_x x} \, dk_x \]

Signal eq. \( \rightarrow \) fwd. problem

Recon eq. \( \rightarrow \) inv. problem

\[ s = \mathbf{F}\mathbf{i} \]

\[ \mathbf{s} = \begin{bmatrix} s_{x,1} \\ s_{x,2} \\ \vdots \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & \cdots \\ F_{21} & F_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \end{bmatrix} \]

Linear "forward solution" matrix vectors have complex entries can build in any measurable priors

\[ \mathbf{F}_{x,y,t} = g(x,y) e^{-i\phi(x,y)} e^{-i(\Delta T + TE)/2} e^{-i\gamma B(x,y) \Delta T \text{AMD}} e^{-i\left(m \Delta k_x x + n \Delta k_y y\right)} \]

cal gain at this location coil phase T2 decay B0 Error (x,y dep.) freq + phase

multi-coil

\[ \mathbf{s} = \begin{bmatrix} s_{1,1} \\ s_{1,2} \\ \vdots \\ s_{2,1} \\ \vdots \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} F_{11} & F_{12} & \cdots \\ F_{21} & F_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \end{bmatrix} \]

naturally incorporates undistorted field map different sensitivity function for each coil!
different weighting function for each coil!
contains additional info about source loc.
but, need reference scan, lo-res is ok
(need phase corrections for each coil?)

\[ \mathbf{i} = \mathbf{F}^+ \mathbf{s} \quad \text{over-determined} \]

More common inverse \( \mathbf{F}^+ = (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \)

(x,y) \( \rightarrow \) small

\[ \mathbf{F}^T (\mathbf{F}^T)^{-1} \mathbf{F}^T \text{ (x,y,c0il)} \rightarrow 6 \times \text{ bigger for 4 coils} \]

\[ \mathbf{i} = [(\mathbf{F}^T)^{-1} \mathbf{F}^T] \mathbf{s} \]

slice-by-slice assume slice select swamps others
**FAST SPIN ECHO (FSE)**

- one 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient
- each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips)

**2D FSE**

- RF
  - G₀: 90°, 180°, 180°, 180°, 180°, 180°, ...
  - G₁: slice select, slice select
  - G₂: phase encode (w/ wind)
  - G₃: unwind 1, unwind 2, [none]
  - G₄: read, response, read
  - G₅: read, read

**Signal**

- effective TE

- the "effective TE" is the TE when center of k-space is collected (largest effect on contrast, largest echo)
- each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \) \( n = 1, 2, ..., M \)

- by arranging to collect \( ky = 0 \) early, PD-weighted instead of T2-weighted
- possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train

- 3D FSE — like 2D except
  - wind/unwind added to thick slice select (w/ erasers on 180°)

**N.B.** only one read/rephase, subsequent 180° resets from right to left

**N.B.** all those 180° pulses deposit a lot of RF power:

- \( 90° + 180° = 45° \times \) power 30°
MULTI-SLAB 3DFSE PROBLEMS

- echoes die out quickly $\propto e^{-t/T_2}$
- since echoes after $90^\circ$ limited to $< 30$, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 \theta_2 A_b$ affects 180° pulses deposit 4-6x power of 90°
- "multi-slab" is halfway between slices and single-slab
- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\rightarrow$ longer than non-selective 'hard' pulses

- Ref. RO: hard to get under 18 msec, inter-echo spacing limits speed of covering k-space
SINGLE-SLAB 3D FSE

Flip angle series

- Regular FSE (180° pulse train)
- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
- This "storage" in z-axis preserves magnetization for longer time
- Smaller flip angles allow much longer echo trains
- Enough to collect whole plane of 3-D k-space
- Different than hyper echoes (not symmetric)

SE = sinα, sin²α/2 e⁻²π/T2
STE = ½ sinα sinα sinα e⁻T/12 e⁻²π/12

Single-slab 3DFSE pulse seq.

Variable flip angle (<1 msec)
Hard (non-selective) pulse not 180°

RFm

Echo trains

G_y

G_z

G_x

FID spin echo FID

Spin echo plus stimulated echo

Long TE eff (T2)

Short TE eff (T1)

K

Echo train

Train

NB: time to scan k-space is ≈ 5X apparent contrast time b/c of "storage" (e.g. TE eff = 585 ms looks like FSE TE = 140 ms)
FAST GRADIENT ECHO (GRASS, FLASH, MPRAGE)

- Small tip so TR can be greatly reduced (e.g., 10 msec, less than T2)
- 'Leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

**RF**
- $\alpha$: select (balanced)
- $G_z$: phase
- $G_y$: make GE, read
- $G_x$: read

**Signal**
- TR - as small as 10 msec

**STEADY-STATE COHERENT (GRASS, FISP)**
- Unwind phase from phase-encode $M_T$ before next pulse (there because TR < TE)
- Unwind read gradient, too

**Non-STeady State, magnetization-prep**

- (Shown as 3D sequence - possible with ones above, too)

**STEADY-STATE SPOILED (SPGR, FLASH)**
- Spoil with random gradient (but this still allows some $\alpha$ refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (TI-weighted)

**NON-STeady State, magnetization-prep**

- Preparation pulse → strong TI-weighting
- Contrast varies in spatial-frequency-dependent way

- Longitudinal mag. not affect much by low angle pulses

- Record $k_y = 0$, here
QUANTITATIVE T1 — INTRO, METHODS

Motivation

- Image values are arbitrary/relative (dif sects, manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local B1 field can cause contrast variation
  - At 3T, B1 can vary by 25% across the brain sig
  - This can invert contrast in a fast gradient echo
    \( \alpha \approx 25\% \) error!

Pre-scan normalise

- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set params. to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

\[ \text{T1 divided by T2} \]

- MPRAGE \( \Rightarrow \) strong T1-Contrast
- SPACE \( \Rightarrow \) T2-weighted (no T1 weighting)
- T1/T2 removes coil fall-off
- Problems < noise in regions of low signal

MP2RAGE

1st volume \( \rightarrow \) PD-weighted

2nd copy of volume \( \rightarrow \) max T1-weighted

- N.B. SSFP-like in partition, phase-encode dir
- Convert to \(-0.5 \rightarrow 0.5\) image:\n  \( S = \text{real} \sum \frac{S_{T1}^{*} \cdot S_{T1}}{|\sum S_{T1}|^2 + |S_{T1}|^2} \)
- Calc. PD & T1 from above
  \( \text{cf. 2 flip angles} \)
QUANTITATIVE T1 - HELMS 2-FIP ANGLE METHOD

- Start with gradient echo signal e.g., dropping T2-decay: \( e^{-\text{TE}/2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-\text{TR}/T1}}{1 - \cos \alpha \cdot e^{-\text{TR}/T1}}
\]

- Simplify/linearize/estimate

\( \text{TR} \ll T1 \)

Linear approx. of exponentials

Taylor expansion simplification of \( \sin, \cos \), drop small term

\[
S \approx A \cdot \frac{\text{TR}/T1}{\alpha^{2}/2 + \text{TR}/T1}
\]

- Solve for \( T1 \) and \( A \) (proton-density) given signals from 2 diff flip angles

\( \max: \alpha^{2}/2 = \text{TR}/T1 \)

\[
T1_{\text{est}} = 2 \text{TR} \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1}
\]

\[
A_{\text{est}} = \frac{S_1 S_2 (\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2 \alpha_2 - S_1 \alpha_1}
\]

- Tiny error for flip \( \leq 15 \text{deg} \).

Estimating at flip 15 deg.

- Problem: flip angle varies a lot at 3T (e.g., 25%) from nominal requested (e.g., flip series)

- Acq.: spin-echo and stimulated echo (EVI)

\[
S = K \cdot \sin^3 \alpha \cdot e^{-\text{TE}/T2}
\]

\[
S_{\text{TE}} = \frac{K}{2} \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-\text{TE}/T2} \cdot \frac{e^{-\text{TM}/T1}}{S_{\text{SE}}}
\]

\[
\alpha = \cos^{-1}\left(\frac{S_{\text{TE}} \cdot e^{-\text{TM}/T1}}{S_{\text{SE}}}\right)
\]

- Add EPI-like echo train to each FLASH excitation.
ECHO PLANAR IMAGING, EPI

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space).

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
SPIN ECHO EPI

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxyhemoglobin (probably the result of the need to drive O$_2$ into tissue, which requires more O$_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo cancels (cancels) static $T_2^*$ ($T_2'$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For $TE \approx 100$ ms, spins diffuse 10's of $\mu$m, which is larger than diameter of Small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at cancelling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels
**SPIN ECHO EPI**

- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.

- "Asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2* weighting (for ky=0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially replaced asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising because signal to noise is higher so we can take spin echo hit.

- Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording.
COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- Coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff fall-offs
  - but what does this look like in k-space?
- Slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space
  - (N.B., not addition!)
- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space— at all spatial frequencies!!
- Simple example w/ "brain" consisting of one spatial freq:
  
  \[
  \begin{align*}
  \text{Image domain} & \quad \text{Spatial freq. domain} \\
  \text{Image ("brain")} & \quad \text{FT} \\
  \text{Coil fall-off function} & \quad \text{(multiply)} \\
  \text{Acquired image} & \quad \text{FT} \\
  \text{FT} & \quad \text{(convolve)} \\
  \text{FT} & \quad \text{FT} \\
  \text{K-space} & \quad \text{K-space}
  \end{align*}
  \]

- N.B. inverse FT of k-space data "smeared" in spatial freq. space is sharp image w/ fall-off (not blurred image)
- "Smeared" means normally orthogonal spatial freqs. "leak" to adj. freqs.

[GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center across multi coils]

[SENSE - general linear inverse approach]

- N.B.: neither would work unless normally orthogonal spatial freqs. blurred!
Pulse sequence(SMS/MULTIBAND/BLIPPED CARRI)

\[ V_2 \]
\[ A_f \]
\[ RF_{in} \]
\[ RF_{out} \]

- Excite multiple slices at once
- Function of \( G_2 \) blips is to shift slices in \( G_y \) direction

- This occurs because for given slice, a phase pedestal is added to the entire slice
  \[ \Rightarrow \text{``Fourier Shift Theorem''} \]
  \[ \text{``N.B.: different than } B_0 \text{ defecnt-induced incremented phase errors''} \]
- Problem w/ all up \( G_2 \) blips \( \Rightarrow \) phase error builds up

**Trick #1**
- Start w/ 2 slices, one at \( z=0 \), other above
  \[ \Rightarrow \text{if } \pi (180^\circ) \text{ phase shift used, blip up/down same! (no effect at } z=0) \]
  \[ \Rightarrow \text{i.e., move top or bottom replica} \]

**Trick #2**
- For multiple slices not all at \( z=0 \), phase no longer same for even/odd
  \[ \Rightarrow \text{but can add phase to equilibrate to } k \text{-space before recov.} \]

**Trick #3**
- For more than 2 slices:
  
  \[ 1 \text{st even odd even odd etc} \]
MULTI BAND/BLIPPED CHIRP (cont.)

Relation between leave-one-out aliasing and nominally fully-sampled SMS

- Leave alternate lines out, wraps image
- SENSE/GRAPPA to fix block coil view swears K-space data
- Nominally, w/ SMS we record every line of K-space
- But equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- Slices - GRAPPA
  - reg GRAPPA → recon missing lines
  - slice GRAPPA → recon multiple K-spaces
    For each overlapped slices by training on fully-sampled data at beginning of scan
    i.e., not SMS

- Inter-slice "leakage block"
  - When training GRAPPA kernel on fully-sampled data, also minimize inter-slice leakage (split-slice GRAPPA)
  - Can also do regular GRAPPA on top of this
    Reason: for diffusion, loss in S/N from undersample cancelled by shorter TE readout
    Gain from reduced image distortion from shorter readout
ECHO-VOLUME IMAGING EVI

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

- entire k-space must be filled before 3D image is reconstructed

- since entire volume is excited each shot, potentially higher S/N

- must use smaller flip angle to avoid killing $M_L$ since entire volume excited every partition (e.g. every 80 msec)

- main issue is movement artifact since data assembled from many shots over several secs

- breathing-induced BP problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant Δk_x

- sinusoids in both G_x and G_y give spiral k-space trajectory

RF

G_z

G_y

G_x

Sig

constant angular velocity goes too fast at large k_x, k_y

constant linear velocity better but impractical near k_x=0, k_y=0

compromise: start constant angular, end constant linear

**Constant angular velocity**

\[w(t) = \omega_0 t\]

\[k(t) = A t e^{i \omega_0 t}\]

\[G(t) = \frac{1}{\Delta t} \frac{d}{dt} k(t)\]

\[= A e^{i \omega_0 t} + i A t \omega_0 e^{i \omega_0 t}\]

\[G_x(t) = A \cos \omega_0 t - A t \omega_0 \sin \omega_0 t\]

\[G_y(t) = A \sin \omega_0 t + A t \omega_0 \cos \omega_0 t\]

**Constant linear velocity**

\[w(t) = \omega_0 T\]

\[k(t) = A T e^{i \omega_0 T}\]

\[G(t) = \frac{1}{\Delta t} \frac{d}{dt} k(t)\]

\[= \frac{A}{2t} e^{i \omega_0 T} + \frac{A}{2} \omega_0 e^{i \omega_0 T}\]

\[G_x(t) = \frac{A}{2t} \cos \omega_0 T + \frac{A}{2} \omega_0 \cos \omega_0 T\]

\[G_y(t) = \frac{A}{2t} \sin \omega_0 T + \frac{A}{2} \omega_0 \sin \omega_0 T\]
SPIRAL 3D IR FSE  (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)
  - All echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- Possible to present sign
- High uniform contrast, but lots of waiting (T1), high BW

RF

180° (prep1)  T1 ~ 700 msec

90°  180°  180°  x 16  180° (prep2)

G_x

G_y

G_z

Sig.

FIID  echo_1  echo_2

Loop order

3D k-space
("stack of spirals")

Spiral interleaves
k_z interleaves
k_z echoes

echoes ~ (after one 90°)
Phase Errors & Echo-Centering Errors

Anything that causes a deviation of the $B_z$ field strength from the expected value $(B_{0,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z)$ changes precession frequency and therefore, expected phase angle. Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

- Correct with shimming and $B_0$-mapping/phase unwrapping before reconstruction.

Echo Centering Error

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.
- Since echo is in spatial frequency domain, this is frequency shift.
- Spatial frequency shift results in wrapping in phase image after reconstruction. Magnitude image unchanged.

\[
I(x-x_0) = \int e^{-i 2\pi k_x x} S(k_x) e^{i 2\pi k_x x} \, dk_x
\]

N.B.: This is a pedestal of phase, not a gradient.

- Fourier freq. shift theorem: freq. shift in freq. domain causes phase shift in spatial domain.
FAST SCAN ARTIFACTS

brain-induced field defects lead to phase errors

EPI
- G_x readout gradient strong → field defects smaller percentage
  less deformation of k_x (vertical stripe components)
- G_y "blips" are weak and total G_y readout time
  much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the x-direction, for example, maps
  and unmaps phase as a function of x-position
- but G_x big, so effect on freq.-encode direction is much
  less than on phase-encode direction, where error accumulates

Spiral
- with centric-out spirals phase errors accumulate
  in a radial direction
- thus, higher spatial frequencies have more error (= more shearing)
- for spurious x-direction gradient as above, there is
  a radial blurring, rather than a vertical shift
  because higher frequency phase stripes misaligned
  relative to low spatial freq.

- for defects with more complex contours in the y-direction
  (than linear, as above) the vertical shifts (in EPI) will
  vary with y-position, and may result in signals from different
  y-positions being reconstructed on top of each other
**IMAGE-SPACE VIEW OF LOCALIZED $B\phi$ DEFECT, EFFECT ON RECON**

- Localized $B\phi$ defects often arise from air pockets embedded in tissue.
  - Air in middle/outer ear $\rightarrow$ indentation in inferior temporal lobe.
  - Air under olfactory epithelium $\rightarrow$ orbitofrontal cvx, ant, thal. compression.

**Collect one data (k-space) point**

- $4$ cycles of $B\phi$ in y-dir (y-position) + localized $B\phi$ defect
- Complex multiply ($=\text{correlate sin/cos with brain}$)
- Brain structure sampled with distorted stripes
  - Local upward displacement
  - Image phase (phase encode dir)
  - N.B.: Image shift only occurs if shift spatially spans sampled w/ successively later echotimes (see next page)

**Reconstruction from distorted data points**

- $... + \left[ \begin{array}{c} \text{amplitude and phase} \\ \text{of this component} \end{array} \right] \times \left[ \begin{array}{c} \text{undistorted} \\ \text{stripes used by inverse FFT} \end{array} \right] + \left[ \begin{array}{c} \text{Same for} \\ 5 \text{ cycles} \end{array} \right] + ... = \text{brain}

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by $B\phi$ defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of $B\phi$ needed to correct images
  1) Fieldmap method: Use multiple TEs to estimate local $B\phi$ from $\phi/TE$ slope
  2) Point-spread-function: Extra phase encode to estimate PSF (should be $\delta$-function)
     - Deconvolve distorted image in phase-encode direction
LOCALIZED $B\phi$ DEFECT, EFFECT ON RECON

- when local $B\phi$ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (ct. echo cont. error)

- however, with w/EPI, static $B\phi$ defect causes more and more local displacement of image phase stripes for each additional ky line

[---that is, later lines have greater spat. freq. offset
- effectively stretches k-space in ky direction
- same num samples to higher spatial freq.
  shrinks FOV (squishes voxels -- see FOV page)]

- when image is reconstructed, region with local $B\phi$ defect shifted oppositely

- thus, local shift effect due to combination of 3 things:

  1) static local $\Delta B\phi$ defect

  2) successive increases in phase error for successive spat. freq. measurements during long EPI readout

  3) small size of ky phase encode blips relative to $B\phi$ defect (much less of this effect in freq. encode direction)

- respiration (which affect $B\phi$) in 3D FLASH might cause similar effect within k2 partition (if successive spat. freqs.)
\[ \text{GRADIENT NON-LINEARITIES} \]

- Ideally, the \( G_x, G_y, \) and \( G_z \) gradient coils attempt to impress a linear variation onto the \( Z \)-component of the \( B \) field — \( B_z \) — in the \( x, y, \) and \( z \)-directions.

- In practice, gradient coils are non-linear (esp. printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice:
  
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently:
  
  - For 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching \( 1 \text{ cm} \) in function-structure overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the \( B_x \) and \( B_y \) components are known.

- These effects do not build up over time in phase-encoded directions since they only occur when gradients are turned on.

- Fourier shift theorem:

- \( \text{these distortions are predictable and can be corrected} \)

\( \text{that is the assumption that gradients cause no field in the } B_x \text{ and } B_y \text{ direction} \)
SHIMMING AND $E_0$-MAPPING

- Passive iron shims inserted to flatten $E_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $E_0$ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc. (= several hundred ppm)

[Linear shim coils impose gradients in $x, y, and z$
Higher order shims impose higher order spherical harmonic field components (e.g. $z^2$)]

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $E_0$ field

- Local resonance offsets caused by $E_0$ defects estimated from images
  - e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents
  - This only corrects spatially gradual field defects
  - Local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to un-warp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq. encode gradient is strong relative to defects)
- 1D navigator
  - slow up/down drift in $\Phi$ continuously occur
  - a slowly moving $\Phi$ is nearly in phase (not gradient)
  - when causes spatial shift (Fourier shift theorem)
  - in EPI, mainly affects phase-encode dir b/c small blip traj readout
  - result is successive volumes drift in phase encode dir

  Gradient balance problem
  - unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
  - causing $N/2$ (Nyquist) ghosting $\geq$ another phase error

- 3D navigator: collect 3D sphere in k-space
  - rotation of object $\rightarrow$ rotation of k-space amplitude pattern
  - translation of object $\rightarrow$ phase shift of k-space phase (Fourier shift)
  - sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N,S hemispheres separately (less $T_2^*$, cancel EPI-like error accumulation)
  - Welch et al. (2002) MRM

  Welch's formula:
  \[ z(n) = \frac{2n - N - 1}{N} \]

  \[ y(n) = \cos \left( \frac{\pi n}{N} \sin^{-1} z(n) \right) \]

  \[ x(n) = \sin \left( \frac{\pi n}{N} \sin^{-1} z(n) \right) \]

  (skip poles - slew rate too high)

  RF
  - can be used for prospective motion correction (rotate, translate w/ gradients)
  - better estimate, because of speed, than Full TR & EPI images (27 ms vs 2.4 sec)
  - may need to smooth rot/translation estimates across time (e.g. Kalman filter)
**RF Field Inhomogeneities**

- **Receive coil inhomogeneities** alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way. Variations can be used (e.g., GRAPPA, SENSE) and/or corrected.

- **Transmit coil inhomogeneities** affect the flip angle in a spatially varying way (can affect contrast: FLASH). Potentially worse (why local transmit is still in progress) usu. fixed by using a large transmit coil (e.g., body coil).

- **RF penetration at higher fields** (\(\leq\) higher RF frequencies)
  - is less uniform:
    1) decreased RF wavelength (closer to size of head) at higher freq.
    2) increased permittivity \((\epsilon)\) and conductivity \((\sigma)\) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain).

- Different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

  **Normalization** ("pre-scan normalize")
  - Record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)
  - Divide small coil body coil at each voxel to determine receive field
  - Use receive field to normalize main image(s)

  [see also: \(\text{T\,1, MP2RAGE, T\,1/T\,2}\)]
DIFFUSION - WEIGHTED IMAGING

Simple diffusion weighting

- spins acquire phase during first $\Delta T$
- if spins diffuse (= move) along gradient by time $T$, signal is lost because negative $\Delta t$ doesn't re-phase
- attenuation: $A(D) = \frac{S_0}{S} = e^{-bD}$
  where $b = G^2 G^2 \Delta t^2 (T - \Delta t/3)$

"Apparent diffusion coefficient" map
- to get large $b$, need $G \uparrow \Delta T \uparrow$(need big $G$'s)
- Long $\Delta t$ gives spurious $T_2*$-weighting
  can use stimulated echoes: $90^\circ$ RF $\rightarrow \Delta t$, $90^\circ$ RF $\rightarrow 90^\circ$ RF $\rightarrow \Delta t$

1) Anisotropic Diffusion (Gaussian)
- measure $D$ along multiple axes
- have to measure tensor, not scalar
  $D$ even for determining one primary direction

$D = \begin{bmatrix} U_x & U_y & U_z \\ D_{xx} & D_{xy} & D_{xz} \\ & D_{yy} & D_{yz} \\ & & D_{zz} \end{bmatrix}$
$\mathbf{D} = \mathbf{u}^T \cdot \mathbf{D} \cdot \mathbf{u}$
Scalar diffusion $\rightarrow$ diffusion tensor measurement direction

Diffusion Surface (Non-Gaussian)
- need to measure diffusion in many directions ($\geq 6$) to properly characterize even 2 main directions

2) Length Scale by multiple $b$-values
- fit line to semi-log signal as $f(b)$
- fit $\log$ log as $\log$ $S = A_1 e^{-bD_1} + A_2 e^{-bD_2}$

Tract Tracing
1) Markov process
2) Crossing fibers
3) "Free way ramp" prob
4) Sharp turns (into gyri

Voxels
Axial
Coronal
Sagittal
Diffusion basis set
Data both $b$-fract
Data both $D$-fract
(Mulkern, 1999)
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)

- RF_

- \( G_z \)

- \( T \)

- other diff.

- RF_

- induced by onset and offset of gradients

- eddy-currents are long time-constant currents in metal of scanner that distort B field \( \rightarrow \) spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)

- (also, keep crushers orthogonal to diffusion-encoding gradients)

- Nagy et al. (2014) MRM

- RF_

- \( G_z \)

- \( T \)

- other diff. diffusion

- RF_

- navigators

\( \gamma_{TRSE} = 0 = \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4 \)

- twice refocused Spin-echo (for center k-space)
**PERFUSION - ARTERIAL SPIN LABEL**

- Basic idea: tag blood below area of interest, collect control & tagged image, assume directional input flow.

  - Continuous ASL (CASL) — continuously tag a plane, greatest on, blood gets adiabatically inverted as it passes through location w/correct resonant tag.
  - Pseudo-continuous ASL (pCASL) — see next.
  - Pulsed ASL (PASL) — e.g., EPISTAR, FAIR, PICORE, QUIPSS II.

- Small diff between control and tag (~1%).
  - Requires accurate balancing of control & tag images, control mag. transfer.

- Contrast problems:
  - Transit delay = biggest confounding factor.
  - Relaxation rate diff vs. venous clearance.

- QUIPSS II — Quantitative Perfusion

  1. Pre-saturate spins in target slices.
  2. Tag - 180° pulse below slices.
  3. Control - 180° pulse above slices (to control off-resonance).
  4. Saturate tagged block to end tag (TI).

- Solutions for quantitative:
  - Use Ti longer than TR.
  - Omit QUIPSS tag ending.

Turbo ASL

- Use Ti longer than TR.
- Omit QUIPSS tag ending.

- Tag pulse control
- Tag image control pulse
- Tag image tag pulse
- Control image

- 2X faster but limited slice*

\[ \Delta M \approx \text{flow} \times \left[ 2M_0 \cdot \frac{1}{T1a} e^{-T1b/T1a} \right] \]

- Can extract flow and BOLD adjacent subregions minimize movement artifact.
- Alternate tag and control, GRE TE = 30 ms.

- Dual Echo Spiral
  - K = 0 early => high S/N.
  - TE = 30 ms => BOLD.
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires
  RF on continuously to adiabatically invert bloomed flowing
  through one plane
  - can only image one slice (due to dephasing from gradient)
  - hard to keep RF on continuously on modern scanner (esp. BOLD)
  - can use special purpose RF transmit (separate xmt channel)

A) Original CASL

\[
\begin{array}{c}
\text{RF} \\
G_z
\end{array} \quad \text{image formation module ('readout') \rightarrow [multiple possibilities]}
\]

B) pCASL - pseudo continuous arterial spin labeling  Dai, Alsop (2008)

\[
\begin{array}{c}
\text{RF} \\
G_z
\end{array} \quad \text{begin readout}
\]

- Problem: multiple pulsers create aliased slice planes

\[
RF(t) = \frac{1}{\Delta t} \text{comb}(t) \ast \text{rect}(t/\Delta t)
\]

\[
F[RF(t)] = \text{comb}(6\delta t) \ast \delta \text{sinc}(\pi \delta t)
\]

\[
\text{aliased labeling planes at: } \delta = n/\Delta t \text{ in frequency space, modulated by broad sinc()}
\]

- Use Hamming or hyperbolic secant to reduce replicas

C) pCASL with shaped gradients

\[
\text{RF} \quad G_z
\]

- Tag pulses have phase offset respecting gradient
- Control identical except every other has \(\pi\) phase
- No net flip

\[
\text{FLASH} \quad \text{EPI} \quad \text{SMS} \quad \text{Stack spirals 3D}
\]
**OFF RESONANCE EXCITATION**

- **Main idea**: examine evolution of $\vec{M}$ vector in rotating coordinate system set to "off-resonance" $\vec{B}_1$ field freq ($\omega_{rf}$), not Larmor freq of $\vec{M}$ ($\omega_0$)

- Normally, if rotating coord syst freq set to Larmor freq ($\omega_{rf} = \omega_0$), an actually precessing $\vec{M}$ will be stationary (ignoring decay) $\implies$ implies effective $B_2 \approx 0$ in rotating

- Now, move $\vec{M}$ to rotating coord syst at $\vec{B}_1$ freq lower than $\omega_0$ (assume $\vec{B}_1 = 0$ = RF): existing $\vec{M}$ will now appear to precess around z-axis:

  \[
  \Delta \omega_0 = \omega_0 - \omega_{rf}
  \]

  - Thus, viewing $\vec{M}$ vector in off-resonance rotating coord syst makes it look like additional $\vec{B}_2$ field is causing "extra" precession

  - "Extra" $\vec{B}_2$ component is proportional to $\Delta \omega_0$ offset $\implies$ can be pos or neg: rotation freq too low $\implies$ pos $\vec{B}_2$ rotation freq too high $\implies$ neg $\vec{B}_2$

- Extra $\vec{B}_2$ adds to $\vec{B}_1$, resulting in slow precession around tipped axis: $\vec{B}_{eff}$ (effective)

- Extra z-gradient can have same effect on $\Delta \omega_0$ (changes $\omega_0$ instead of changing $\omega_{rf}$)

  \[
  \vec{B}_{eff} = \left(\frac{\Delta \omega_0}{\gamma}\right) \hat{k} + B_2 \hat{k} + B_1 \hat{i}
  \]

  - Effective $\vec{B}$ in rotating frame set to $\vec{B}_1$ freq
  - Apparent "extm" $B_2$ from Larmor-$\vec{B}_1$ freq mismatch
  - Optional z-gradient (pos or neg)
  - Transverse RF stim (here, around x-axis)

**adiabatic RF pulse** $\approx$ Flow-driven CASL tag

**On-Resonance**

- "precession" (-=flip) exactly around $\vec{B}_1$
- $\vec{B}_{eff} = \vec{B}_1$
- (RF subtracted out)

**Off-Res.**

- "Precession" $\implies$ $\vec{M}$ precesses slowly around $\vec{B}_{eff}$
- $\vec{B}_{eff}$ tilted (for $\vec{B}_1$)
- $\vec{B}_1$ from RF coil

**Slightly Off-Res.**

- Almost back to on-res. at top diagram

**Off-Res., opposite way**

- $\vec{B}_1$ tipped
- Can cause (RF) flip

**Adiabatic RF pulse** $\Rightarrow$ Flow-driven CASL tag

- RF: sweep freq
- $W_0$: constant (for given slice)
- RF: const freq
- $W_0$: sweeps because spins flow along gradient direction
- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g., $^1H$) by surrounding electron orbitals

- e.g., acetic acid: 
  - oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:
  - Larmor oscillations are multiplied (PSD) by center freq to obtain $\Delta f$ (not MHz, high freq)
  - data before FT is a series of time-domain samples of the mix of shifted-freq offsets
  - FT turns data into "shift spectrum"

Pulse Sequence

- since we are already using phase (freq) encoding for space, we need an "extra dimension" with all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal $\Delta f$ and FT-it like chemists do!
PRESS, MEGA-PRESS

- usu. single voxel by using 3 orthog. slice selects (tho can add PG gradients & more excitations to get multiple vox)

PRESS - 3 orthog. slice select

MEGA-PRESS - add "editing" RF's to suppress solvent (water)

MEGA 180° pulses set to freq. of solvent

G2, G3 - asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses

FT to get shift spectrum
**Phase-encoded Stimulus & Analysis**

**Calculate Significance**
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, lo freq (= movement)

**Smooth**
- Vector average of complex significance \((A, \phi)\) with that at nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay Correction**
- Record responses to opposite directions of stimulus (ccw/lcw, in/out, up/down)
- Vector average after reversing angle of one, penalizes inconsistent more than just avg of angles

**Typically 0.5-5% amplitude**

**Strongly periodically activated single voxel time course**

**Remove constant (avg) and linear trend**

**Reversed CCW**
- Vector average
- CW significance

**FFT, convert to \(A, \phi\)**

**Real**
\[ t \rightarrow \]

**Imaginary**
\[ t \rightarrow \]

\[ t = \text{total TRs} \]

\[ 0 \]

\[ \frac{\text{total TRs}}{2} \]
**CONVOLUTION**

\[ h(x) = f(x) \ast g(x) = \sum_{z=-\infty}^{z=+\infty} f(z) \cdot g(x-z) \, dz \]

- Definition of convolution \((f \ast g)(x)\)
- Commutative

- **Linear time/space invariant system**
  - How to calculate one term
  - Sum across all \(z\) to get the value of the convolution at point \(x\)
  - Move kernel to calculate next \(x\)

**Why reverse makes sense**

- Bc commutative, this is thinking like: \(\int g(x) \cdot f(x-z) \, dz\)

**Impulse response function (HDR)**

- Impulses (expt. design)
  - Impulse occurred a while ago
  - Small effect
  - Impulse occurred recently
  - Larger effect

**Intuitive non-reversed view of convolution output**

- Start here!

**NB**

- Cross-correlation same as convolution except no reversal
  - \(g(x+z)\) instead of \(g(x-z)\)

- Auto-correlation same, except no reversal and use same function for both \(f, g\)

How to calculate convolution output for this time point (only 3 terms in sum, all other zero)
**GENERAL LINEAR MODEL**

\[ \hat{y} = \mathbf{X}\hat{h} + \mathbf{S}\hat{b} + \hat{n} \]

Data = design + HDR + drifts + weights + noise

- **Goal:** Solve for the hemodynamic response functions, \( \hat{h} \)

\[ \mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{n} \]

\( \mathbf{y} \) \( \mathbf{X} \) \( \mathbf{h} \) \( \mathbf{n} \)

- **Temporal**
  - Experimental design
  - Off

\( \tau \)

\( \mathbf{X} \)

\( \mathbf{h} \)

\( \mathbf{n} \)

\( \mathbf{y} \)

\( \mathbf{S} \)

\( \hat{h} \)

\( \hat{b} \)

\( \hat{n} \)

\( \mathbf{X} \) (temporal)

\( \mathbf{h} \) (hemo)

\( \mathbf{n} \) (noise)

\( \mathbf{S} \) (unknowns)

\[ \mathbf{y} = \mathbf{X}\mathbf{h} + \mathbf{n} \]

\( \mathbf{y} \) \( \mathbf{X} \) \( \mathbf{h} \) \( \mathbf{n} \)

1. **Convolution**: Assume white noise, solve for \( \hat{h} \)
2. **Convolution**: Solve for \( \hat{h} \)
3. **Convolution**: Assume white noise, solve for \( \hat{h} \)

\[ \hat{h} = (\mathbf{X}^T\mathbf{P}_S^\perp \mathbf{X})^{-1}\mathbf{X}^T\mathbf{P}_S^\perp \mathbf{y} \]

where \( \mathbf{P}_S^\perp = \mathbf{I} - \mathbf{S}(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T \)

\( \mathbf{P}_S^\perp \) (projection matrix)

\( \mathbf{S} \) (stimulus pattern)

\( \mathbf{h} \) (hemo)

\( \mathbf{n} \) (noise)

\( \mathbf{y} \) (data)

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\( \mathbf{h} \) (hemo)

\( \mathbf{n} \) (noise)

\( \mathbf{y} \) (data)
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions ($s$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[ \hat{y} = X\hat{h} + \hat{e} \]
\[ \hat{y} = X\hat{h} + \hat{e} \]

Projection matrix, $P_x$, operates on $y$ to give projection of data into experiment space, $X$.

- When nuisance functions, $s$, are considered, problem: $s$ may not be orthogonal to $x$.

For example: linear trend not orthogonal to std. block design.

Remember: "orthogonal" means dot prod. = 0, corr = 0.

Orthogonal projection:

- Project onto $X$ with $P_x y$.
- Project onto $S$ with $E_x y$.

Oblique projection:

- Project onto $X$ with $P_x y$.
- Project onto $S$ with $E_x y$.

Data orthogonality:

- $X$: Space of data modeled by all reference and nuisance.
- $S$: Orthogonal projection onto nuisance ($P_s y$).
- $E_x y$: Oblique projection onto nuisance ($E_x y$).
- $P_x y$: Orthogonal projection onto reference ($P_x y$).
- Error ($e$) not explained by reference and nuisance ($F$ denoted).

Geometric Picture:

(Liu et al., 2001, Neuroimage)

Error ($e$) not explained by reference and nuisance ($F$ denoted).

Orthogonal projection onto reference ($P_x y$)

Same as projection onto reference only in special case where $S \perp X$.
1) MNI auto-Talairach \rightarrow generates 4x4 matrix
- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on xcorr
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut off, only 2 points near center of brain!
  \rightarrow but much better than standard! \sim fit to bounding box

2) Intensity Normalization (output: "T1")
- histogram of pixel values in 10 mm thick T1R slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  \rightarrow interpolated scaling factor 1 to T1R
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D
  \rightarrow find points in 5x5x5 within 10% of WM, get new scale for them
  \sim build Voronoi to interpolate scales \sim set above
  \sim smooth Voronoi boundaries (3 iterations)
  \rightarrow re-scale each voxel

3) Skull Stripping (output: "brain")
- "shrink-wrap" algorithm
- start with ellipsoidal template \rightarrow sub-tessellated icosahedron
- minimize brain penetration and curvature
- curvature: spring force
  \sim (from center-to-neighbor vect sum)
- brain penetration
  \sim apply force along surface normal that prevents surface from entering gray matter
SEGMENTATION & SURFACE RECON

- Implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something").
- More formally, we would define cost function, then take its derivative (gradient) to minimize it.

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t) \]

\[ \text{rule for each vertex, } \mathbf{r}_{\text{center}} \]

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (\mathbf{I} - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

\[ \text{vector to neighbor vertex} \]

\[ \text{stronger than normal} \quad (0.5) \]

\[ \text{expands by distribution to neighbor vertex} \]

\[ \text{minus projection of neighbor onto normal = tangential} \]

\[ \text{vector substract of} \]

\[ \lambdabar_{\text{normal}} \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\text{#vertices}} \sum \sum (\mathbf{n}_v \mathbf{n}_v^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v) \]

\[ \text{average normal component} \]

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \frac{30}{d} \max \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{thresh}} \right] \right] \]

\[ \text{d sample points into brain along the direction of normal} \]

\[ \text{outside (dark)} \]

\[ \text{outside (dark)} \]

\[ \text{skin (light)} \]

\[ \text{skull (dark-light-dark)} \]

\[ \text{WM} \]

\[ \text{GM} \]

\[ \text{ideal skull strip} \]

\[ \text{Surface moving outward} \]

\[ \text{etc} \]

\[ \text{Snap shot of surface and "core sample" from one vertex} \]
4) Non-isotropic filtering (output: "win") - "floss" and "speckle"
  - preliminary hard threshold: output C$^\text{fM}$ NN
  - find ambiguous/boundary voxels
    - 20% or more of 26 immediate neighbors different
  - find plane of least variance
    - for each direction (from icosahedral super-tessellation)
      - consider 5x5x5 volume around 1 voxel
    - find plane of least variance in this hemisphere
      - medium filter w/ hysteresis
    - if 60% of within-slab differ, reverse classification
    - "flosses" sulci without blurring

5) Find cutting planes
   - callosum, to separate hemispheres (SAG)
   - midbrain, to avoid fill into cerebellum (COR)
   - Talairach to start;
     fill WM in SAG or COR till min area

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out - for each hemisphere
   - up/down cycles within each plane
   - plane-by-plane
   - "wormhole filter." (3x3x3 = center + 26)
     - fill (unfilled) voxel if 66% neighbors differ
     - eliminates structures w/ thin, 1-D structure
SEGMENTATION & SURFACE RECON: 3D → 2D

7) Surface Tessellation (output: rh.orig, lh.orig)

- Variable num neighbors possible!
- Quads to triangles

3D data set → I(x, y, z)

2D surface

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  → so cross-products oriented properly

- Long list of values associated with each numbered vertex
  e.g., position (orig, morphed), area (orig, morphed), curvature (intrinsic, Gaussian), "sulcussness" (summed 1 movement during unfolding), cortical thickness

- Separate fMRI data set must be aligned, sampled
  - fMRI voxels larger
  - Sample at each surface vertex
  - Nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh
  - Some quantities only well-defined on surface
    - Gradient of magnitude of cortical map measure (e.g., eccentricity)

Face list (vertex nums)

Vertex list (coords)

FMRI → surface

Initial sample

Same value here

Interpolate on surface
SEGMENTATION & SURFACE RECON
Smooth, inflate, final surfaces

-smoothing/inflation/WM, pial done as derivative of energy functional

\[ J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}} \]

\[ J_{\text{normal}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 \]

\[ J_{\text{tangential}} = \frac{1}{2} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right]^2 \]

\[ J_{\text{image}} = \frac{1}{2} \sum_{\text{centers}} \left[ I_{\text{center}} - I(r_{\text{center}}) \right]^2 \]

-take directional derivative of energy functional (to find steepest uphill)

\[ -\frac{\partial J}{\partial r_{\text{center}}} = \lambda_{\text{image}} [I_{\text{targ}} - I(r_{\text{center}})] \nabla I(r_{\text{center}}) \]

\[ + \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}}) \right] n_{\text{center}} \]

\[ + \sum_{\text{neighbors}} [t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}})] t^x_{\text{center}} + [t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neigh}})] t^y_{\text{center}} \]
**SULCUS-BASED CROSS-SUB. ALIGN**

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"

- Add term to energy function: "sulcus-ness" error: \((S_{\text{ref}} - S_{\text{trag}})^2\)

- Bootstrap morph to one brain make avg target remorph to avg target

---

**Smooth WM**

![Diagram of brain registration process]

- Each sub's native surf has diff # vertices

- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- Average sphere made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (f average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")

- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\(\Rightarrow\) N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)
**SOURCE OF EEG/MEG**

**PSPs**
- Anisotropic cables + aligned spatially + coherent/biased stim one end
- No distant signal from axon spike

**Isotropic**
- "Closed field" (invisible at distance)
- N.B.: spikes only detected by 15 µm microelectrode in gray matter!

**Head**
1) - Local dipole
2) - EEG through skull, skin
3) - Swearing because skull 1/80 conductivity of brain

**MEG**
- Radial dipoles lost
- Tangential dipole generates Gabor-like scalp distrib of B field
**INTRACORTICAL CIRCUITS & ORIGIN OF EEG**

**Cell Types**
- **Excitatory (spiny)**
  - Pyramidal
  - Spiny stellate (e.g., V1 layer 4c)
- **Inhibitory (smooth)**
  - Basket
  - Double bouquet
  - Chandelier
  - Clutch

**Circuits**
- **Huge complexity**
- **First principal components**: input → layer 4 → layer 2/3 → feedforward → layer 5/6 → output feedback
- **Microelectrode recording** (e.g., 10 μm tip)
  - High pass → spikes
  - Low pass → local field potentials
- **Spikes only recordable in gray matter**
- **White matter spikes only recordable with pipette w/ very fine tip b/c inward & outward currents so spatially close in axon/spike (>1 μm)**

**Intra/inter cortical connections cartoon**
- "Lower" (e.g., V1)
  - Layer 2/3 feedforward
  - Layer 4 input
  - Layer 5 motor output
  - Layer 6 feedback
  - Ascending input (e.g., dLGN)
  - Output to sup. collic.
- "Higher" (e.g., V2)
  - Layer 2/3 feedforward
  - Layer 4 input
  - Layer 5 motor output
  - Layer 6 feedback
  - Motor striatum
  - Feedback avoids layer 4
  - To higher areas
**GRADIENT, DIVERGENCE, CURL**

**Gradient** ($\nabla$) (generalized derivative)

$$\nabla s(\mathbf{r}) = \frac{\partial s(\mathbf{r})}{\partial x} \mathbf{i} + \frac{\partial s(\mathbf{r})}{\partial y} \mathbf{j} + \frac{\partial s(\mathbf{r})}{\partial z} \mathbf{k}$$

- Turns scalar field into vector field at each $x, y, z$ point, $\mathbf{r}$
- Scalar function defined at each $x, y, z$ point, $\mathbf{r}$
- Change of $s$ in $x$ direction at point $\mathbf{r}$
- Unit vector in $x$ dir

**Divergence** ($\nabla \cdot$) (deriu dot prod)

$$\nabla \cdot \mathbf{v}(\mathbf{r}) = \frac{\partial v_x(\mathbf{r})}{\partial x} + \frac{\partial v_y(\mathbf{r})}{\partial y} + \frac{\partial v_z(\mathbf{r})}{\partial z}$$

- Turns vector field into scalar field at each $x, y, z$ point, $\mathbf{r}$
- Vector function defined at each $x, y, z$ point, $\mathbf{r}$
- Change of just $x$ component of $\mathbf{v}$ in $x$-direction at point $\mathbf{r}$

**Curl** ($\nabla \times$) (deriu cross product)

$$\nabla \times \mathbf{v}(\mathbf{r}) = \left( \frac{\partial v_z(\mathbf{r})}{\partial y} - \frac{\partial v_y(\mathbf{r})}{\partial z} \right) \mathbf{i} + \left( \frac{\partial v_x(\mathbf{r})}{\partial z} - \frac{\partial v_z(\mathbf{r})}{\partial x} \right) \mathbf{j} + \left( \frac{\partial v_y(\mathbf{r})}{\partial x} - \frac{\partial v_x(\mathbf{r})}{\partial y} \right) \mathbf{k}$$

- Turns vector field into contra vector field at each $x, y, z$ point, $\mathbf{r}$
- Vector function defined at each $x, y, z$ point, $\mathbf{r}$
- Change of just $z$ component of $\mathbf{v}$ in $y$-direction at point $\mathbf{r}$

**Vector identities**

$$\nabla \times \nabla s = 0$$  curl of the gradient of any scalar field is zero

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$  divergence of the curl of any vector field is zero

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$$
Potential ($\Phi$), Electric Field ($\nabla \Phi$) \(\text{CSD.} \quad (\nabla \cdot (-\nabla \Phi) = \nabla^2 \Phi)$

**Low-frequency field approximation**

- Electric fields uncoupled from magnetic (vs. electromagnetic radiation)
- Pre-Maxwellian approx. (EEG freqs \(\ll 1\) MHz)
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitive effects, too

**Scalar potential, $\Phi$** (what we measure with electrode)

\[
\bar{E} = -\nabla \Phi - \frac{\partial \hat{A}}{\partial t}
\]

- **Electric field vector**
- Turns scalar field ($\Phi$) into vector field ($\bar{E}$)
- Ignore couplings; vector potential \(\Rightarrow\) because EEG \(\ll 1\) MHz

**CSD is Laplacian of $\Phi$** ($= \text{div} \bar{E}$)

\[
\nabla \cdot (-\nabla \Phi) = \text{scalar field} = - \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] = -\nabla^2 \Phi
\]

- **Divergence**
- Measures rate of change of vector field across space
- Makes scalar!
- \(\nabla \cdot \) turns vector field into scalar field

**3D CSD gold standard** (rat BAER paper)

- $\Phi$ data \(\nabla \rightarrow \nabla \Phi \rightarrow \nabla \cdot (-\nabla \Phi)$
- Scalar field source/sink movie as function of $t$. 

- **Note:** High counts, $\vec{E}$ large. 
- **Note:** Vector in $y$ dir, not vector along circle.
**1D CSD**

- Raw, event-related signal relative to ground, $\xi$ (e.g., skull)

- High pass
- Low pass

- Spikes (upside down local field potential)

- LFP (local field potential)

- Both types of data can be recorded from same electrode

Rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet

$\Rightarrow$ assume approx constant ($\approx 0$) parallel to cortex

**2D CSD**

- 2D array of electrodes on pial surface or on scalp

Rationale: all electrodes record along same surface, so assume depth profiles are constant

$\nabla^2$ means find spatial (i.e., 1D depth) curvature of potential

- Discrete approx: center $= \frac{\text{above} + \text{below}}{2}$

- N.B. in example above, even though all 3 potentials are positive, smaller value of center point implies sink!

**1D CSD**

- For scalp recordings, sources and sinks are at the scalp

- Can't tell curvature from sign of potential!

- Concave $\Rightarrow$ sink, convex $\Rightarrow$ source

- For depth loc tech method unless done in 3D

- Don't confuse dim of calc with always a second deriv.
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

P1

N1

P2

- phase-locked CSD \rho
  gamma shifts with each cycle

- surface electrode is
  typical ref
**Maxwell Equations**

- **Electrostatics, Magnetostatics**
- **Low freq limit**
- **Like CSE but times \( \sigma \), conductivity**
- \( \varepsilon = \sigma V = I, V = IR \)
- **N.B. These are all defined at a (every) point in space**

**Ohm's law**:
\[ \nabla \cdot \left( \sigma \nabla \Phi \right) = \nabla \cdot \mathbf{J} \]

**Conductivity**
- constant (a tensor constant if inhomogeneous in different directions)
- gradient of scalar potential
- what we measure (as implicitly)
at each point
- divergent

**Impressed currents**
- currents due to ionic flow
- that appear out of nowhere
- (neural batteries)

\[ \nabla \cdot \mathbf{B} = 0 \]

**Incidentally, this Maxwell equation**
- violated by a linear gradient
- in \( \mathbf{B} \) in \( x \)
- \( \Phi \) or \( \Phi \cdot \mathbf{d}n \)

\[ \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J} - \sigma \nabla \Phi \right) \]

**Curl**
- magnetic field
- permeability
- impressed currents
- conductivity
- gradient of scalar potential

---

- Propagation of potentials, magnetic fields instantaneous (no capacitance)
- Simultaneous eqs to solve if \( \mathbf{J} \) are sources, \( \Phi, \mathbf{B} \) are data
- Linear

\[ \mathbf{J}_t + \mathbf{J}_p = \Phi \mathbf{B} \]
\[ \mathbf{J}_p \rightarrow \Phi, \mathbf{B} \]
\[ \mathbf{J}_t \rightarrow \Phi, \mathbf{B} \]
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \]

(field component due to charge distribution)

\[ \mathbf{B} = \nabla \times \mathbf{A} \]

(\text{"vector potential"})

\[ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \]

(magnetic field in given medium induced)

\[ \mathbf{D} = \varepsilon \mathbf{E} \]

(permittivity \(\varepsilon\))

\[ \mathbf{H} = \frac{1}{\mu} \mathbf{B} \]

(permeability \(\mu\))

permeability \(\mu\), conductivity \(\sigma\), and permittivity \(\varepsilon\) characterize substance linear in all three.

\[ \nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \]

If linear in conductivity and dielectric, too, and fields periodic w/\(\lambda\)

\[ \nabla \times \nabla \times \mathbf{E} = -2\pi f^2 \mu (\sigma + 2\pi f^2 \varepsilon) \mathbf{E} \]

\[ \text{to neglect: } \frac{2\pi f^2 \mu (\sigma + 2\pi f^2 \varepsilon) |\mathbf{E}|}{|\nabla \times \nabla \times \mathbf{E}|} \ll 1 \]

1) \[|\nabla \times \nabla \times \mathbf{E}| \propto |\mathbf{E}|/L^2\] where \(L\) is dist over which \(\mathbf{E}\) varies significantly
2) \(\mu\) of tissue similar to empty space
3) Assume conservative large \(\sigma\), dielectric unit, and EEG freq.

\[ \text{order of number is about } 10^{-6} \rightarrow \text{small} \]
**MONPOLE, DIPOLE FORWARD SOL’N**

\[ \Phi_1 = \frac{s}{4\pi \sigma r} \]  

Potential recorded for source monopole

\[ \Phi_2 = \frac{s}{4\pi \sigma} \left( \frac{1}{\sqrt{r_1}} - \frac{1}{\sqrt{r_2}} \right) \]  

Potential recorded for source-sink pair

\[ \vec{\Phi} \approx \left( \frac{1}{4\pi \sigma} \right) \frac{s}{r^3} \hat{r} \]  

Measuring point vector (center of dipole to point)

\[ \vec{B}_2 \approx \left( \frac{\mu_0}{4\pi} \right) \frac{s}{r^3} \hat{r} \]  

N.B.: Both assume inside infinite isotropic conductor

**1. Linear superposition with fixed electrodes and sensors**

\[ \Phi_i(t) = \text{gain} \times \text{source strength} \]

\[ \bar{b}_i(t) = \bar{m}_i \times s(t) \rightarrow \text{Squid measures component of } \vec{B} \]

**2. Approximations for "far enough away" measurements (subtracting two 1/r’s gives inverse square)**

\[ \vec{J} = \nabla \Phi \]

\[ \vec{E} = \nabla \vec{J} \]

\[ \vec{J} = \sigma \vec{E} \]  

***Since \( \vec{F} \) also in numerator, this now inverse square***

\[ \Phi_i(t) = \sum_j \text{gain} \times \text{source strength} \]

\[ \bar{b}_i(t) = \bar{m}_i \sum_j s_j(t) \]

\[ \bar{X}(t) = \sum_j \text{gain} \times \text{source strength} \]  

**3. Steepest up hill direction**

\[ \vec{E} = -\nabla \Phi \]

\[ \vec{J} = \sigma \vec{E} \]  

**4. Current sources in homogeneous medium (e.g., salt water tank)**

\[ \vec{J} \]  

8. Electric field (also current lines)

9. Isopotential lines (\( \Phi \))

10. Source e.g.

11. Sink e.g.
Forward Solution

- well-posed (one answer)
- linear: \( b(A) + b(B) = b(A+B) \)
- approximations due to unknown electrical properties of head

- 3-shell spherical analytic
  - skull / brain conductivity
  - "smearing" (cf. cable theory)
  - remember, we only need to be able to calc. weight for each dipole/electrode pair independently

- 3-shell boundary element
  - arbitrary shape
  - homogeneous conductivity
  - solution = infinite homogeneous + matrix of correction factors

- finite element
  - most general
  - computational intensive w/ small grid
  - many unknown parameters to estimate
**Forward Solution**

\[ V_i = \sum_j E_{ij} S_j + \eta_i \]

Matrix form:

\[
\begin{bmatrix}
V \\
\end{bmatrix} =
\begin{bmatrix}
E \\
\end{bmatrix}
\begin{bmatrix}
S \\
\end{bmatrix} +
\begin{bmatrix}
\eta \\
\end{bmatrix}
\]

\[ v = E s + n \]

Lower case bold \( \rightarrow \) vector
Upper case bold \( \rightarrow \) matrix

Electric recordings

Magnetic recordings

\[
\begin{bmatrix}
V \\
m \\
\end{bmatrix} =
\begin{bmatrix}
E \\
B \\
\end{bmatrix}
\begin{bmatrix}
S \\
\end{bmatrix} +
\begin{bmatrix}
\eta \\
\end{bmatrix}
\]

\[ \hat{x} = \mathbf{A} \hat{z} + \hat{n} \]

Note: only one current source for each column in the \( E + B \) matrix!
WHY LOCALIZE?

- most of ERP literature based (instead) on temporal "components"
  but: 1) underly local cortical generators (from microelectrode LFP)
    - extended in time (400 msec)
    - multiphasic in every cortical area
    - temporally non-static depending on stimulus
      e.g. simple contrast brightness shifts can modulate retinal delay by 50 msec!
  2) thus, any "component" consists of sum of activity from multiple cortical areas at different hierarchical levels
  3) stimulus manipulations will change temporal overlap
    - may cause "component" peak to disappear without changing cortical areas being activated
  4) verified by intra cortical LFP/CSD (Schroeder et al, 1998)

macaque monkey
intra cortical data

- by contrast, the spatial signature of the signal from one cortical area is static — a better area-based component
- origin of "components":
  - easier to record many temporal points (EEG started w/ few electrodes, many time points)
  - easier to "paste" high level psychological functions onto a few waveform deflections

these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey CT
Derivation of Ill-posed Inverse
(from Dale & Sereno, 1993)

\[ x = As + n \]
\[ A = \text{forward solution matrix} \ (E + B) \]
\[ s = \text{source vector} \]
\[ n = \text{sensor noise vector} \]

\[ \text{solve for inverse operator} \]
\[ \text{expectation: } \sum_k p_k \text{ variance} \]

\[ \text{assume } n, s \text{ normal, zero-mean with corresponding covar. matrices } C, R \]

\[ \text{Err}_w = \left\langle \| W(x + n) - s \|^2 \right\rangle \]

\[ = \left\langle \| (WA-I)s + Wn \|^2 \right\rangle \]

\[ = \left\langle \| Ms + Wn \|^2 \right\rangle \quad \text{where } M = WA-I \]

\[ = \left\langle \| Ms \|^2 \right\rangle + \left\langle \| Wn \|^2 \right\rangle \quad \text{diag is noise variance (already squared)} \]

\[ = \text{tr}(MRMT) + \text{tr}(WCW^T) \quad \text{trace is sum of diag elements} \]

(re-expand) \[ = \text{tr}(WARA^TW^T - RATA^TW^T - WAR + R) + \text{tr}(WCW^T) \]

Explicitly minimize by taking derivative w.r.t. \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARA^T - 2RATA^T + 2WC \]

\[ WARA^T + WC = RATA^T \]

\[ W(AATA^T + C) = RATA^T \]

\[ W = RATA^T(AATA^T + C)^{-1} \]

\[ W \text{ is inverse solution operator: } \begin{bmatrix} \text{sensors} \\ W \end{bmatrix} \]

equivalent to minimum norm and Tikhonov regularized inverse if \( C, R \) are proportional to identity matrix (i.e., sensor noise & source independent and equal variance)
Inverse Solution

\[ W = RA^T(ARA^T + C)^{-1} \]

- \( R \) sources uncorrelated
- \( A \) noise uncorrelated

\( W \) is the "minimum norm" solution

- (find \( \tilde{W} \) with smallest norm = 1131)

- Minimum norm solution appropriately downplays deeper (= weaker scalp signal) sources since these are more likely to fall into the noise floor.

- "Problems" of minimum norm:
  - Deeper sources get displaced to the surface

- Small superficial sources "win" because of approx. inverse square form of field solution.
  - Smaller norm of distributed superficial solution.

- Can't fix by increasing priors of deep sources!!
  - That will give deep sources given noise as input!!
**Inverse Solutions to Ill-Posed Compared**

1. Sources:
   \[ s = Wx \]
   - Sensor data
   - How to use the inverse solution, \( W \)
   - Same \( W \) for all time points

2. "Minimum norm" solution:
   - \( ||W|| \) of solution is smallest of infinitely many alternate solutions

3. Linear inverse operator:
   \[ W = RA^T (ARA^T + C)^{-1} \]
   - From error minimization derivation
   - Easier inverse
   - \( ARA^T \) \( \Rightarrow \) Square in # of sensors (small)

4. Alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!):
   \[ W = (A^TC^{-1}A + R^{-1})^{-1}A^TC \]
   \[ \left[ \begin{array}{c} W \\ C \\ A^T \\ R \end{array} \right] = \left[ \begin{array}{c} A^T \\ C^{-1}A \\ A \\ R \end{array} \right] \]
   - Both square in # of sources (large)
   - Hard inverses
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on surface often have different orientation, surface normal constraint can help (since fwd sol'n A,B very different)
- But, since point spread functions typically extends across sulci, artificial sign reversals occur

- Solutions

1) Ignore sign → saves useful orientation info!
2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

→ more appropriate when averaging across subjects, since detailed variations vary a lot
→ also, fills in bottom of sulci (else unsigned stripes)
FMRI Constrained Inverse

- Use inverse-2

- Insert FMRI values for Ri's

- But still allow other sites to have non-zero Ri's

- Pathologies occur if solution restricted completely to FMRI points by setting non-FMRI Ri's to zero → set to small number instead!

- This allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: Sources that are only visible in EEG/MEG will be dispersed to small distributed values at a large number of vertices visible in both EEG/MEG and FMRI

visible only in EEG/MEG and not FMRI → distributed at small amplitude across many vertices
Noise Sensitivity Normalization

(1)

Forward: \( x = Ax \) well posed

Inverse: \( s = W x \) ill posed

Solve: \( x = As + n \) for \( s \)

\[
W = R A^T (AR A^T + C)^{-1}
\]

- multiply inverse operator by noise sensitivity matrix, \( D \) (diagonal)

\[
D_{ii} = \text{diag} \left\{ \sqrt{W C W^T} \right\}_{ii}
\]

\[
w^{\text{norm}} = D w
\]

\[
s^{\text{norm}}_i = (w^{\text{norm}} x)_i = (D W x)_i = \frac{W_i x}{\sqrt{(WCW^T)_i}} = \sqrt{\frac{(W x x^T W^T)_i}{(WCW^T)_i}}
\]

If assume Gaussian white noise, noise covariance, \( C \), is multiple of \( I \), so

\[
w^{\text{norm}}_i = \frac{W_i^{\text{orig}}}{\| W_i^{\text{orig}} \|}
\]

i.e., scale each row of \( W \) by single value = the norm of that row

row of \( W \) is:

\[
\frac{W_i^{\text{orig}}}{\| W_i^{\text{orig}} \|}
\]

Invert solution coefficients for one source

Scale by norm of this row

i.e., if inverse soln for source is big (e.g., deep source), noise norm inverse for that source reduced by scaling
Noise Sensitivity Normalization (2)

Shallow source (unit strength)

- Low big
- Inv small

Deep source (unit strength)

- Low small
- Inv smaller because of minimum norm

\[ S_i = \frac{\hat{W}_i \cdot \hat{X}}{||\hat{W}_i||} \]

- Therefore, fixed estimate increased relative to shallow

- Effect on inverse solution: more like significance than actual power

- Effect on point-spread function is to equalize shallow and deep

Point-spread functions:

- Noise normed
- Point spread data normed
**Crosstalk** (rows of resolution matrix, WA)

\[ \xi_{ij} = \frac{(W\tilde{A})_{ij}}{(W\tilde{A})_{ii}} = \frac{|W_i \tilde{a}_i|^2}{|W_i \tilde{a}_i|^2} \]

**Point spread** (columns of resolution matrix, WA)

\[ \rho_{ij} = \frac{|(W\tilde{A})_{ji}|^2}{|(W\tilde{A})_{ii}|^2} = \frac{|W_j \tilde{a}_j|^2}{|W_i \tilde{a}_i|^2} \]

**Avg crosstalk**

\[ ACM_i = \frac{\sum_j \xi_{ij}^2}{j} \]

**Avg point spread**

\[ APSF_i = \frac{\sum_j \rho_{ij}^2}{j} \]

- PSF & crosstalk maps identical for standard inverse (WA, the "resolution matrix" is symmetric)
- PSF affected by noise-normalized; crosstalk same (DWA not symmetric)

**Conclusions**

- More EEG or more MEG better

- EEG better than MEG (cf. radial) (EEG fdr better currently less accurate)

- Biggest gain from adding small # EEG (n MEG) (e.g., 30) to many MEG (n EEG) (e.g., 150)

- Easier to add many MEG, so: optimal < 30 EEG

- EEG/MEG forward-solution-scaling-factor error

  causes \( \rightarrow \) more crosstalk
**MUSIC** (1)

(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

- using sensor covariance

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum_i \sum_j \sigma_{ij} C_{ii} A_i A_i^T
\]

\[
\sim \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right] \left[ \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right]^T
\]

\[
D = U \Lambda U^T = \left[ \begin{array}{c|c} \Lambda & 0 \\ \hline 0 & \Lambda_n \end{array} \right]
\]

\[
= \text{find most significant spatial patterns in sensors over time}
\]

Project forward solutions onto these spatial patterns

("project" = dot prod = similarity) for each point in brain

\[
\mathbf{z}_i = \mathbf{A}_i^T U \Lambda U^T A_i
\]

\[
\Rightarrow \text{big single number if forward soln looks like } U_i's
\]
W = RA \cdot (ARA^T + C)^{-1} \cdot A^T A

\text{Minimum normal inverse}

\text{How to weight the parallel resistance}

\begin{align*}
R_{\text{parallel}} &= \frac{1}{\frac{1}{R_i} + \frac{1}{R_s}}
\end{align*}
- How it works

one time point

\[ \sqrt{3} \]

multiple time points

- How it fixes min norm problem

\[ \frac{\text{equivalent to each other}}{\uparrow} \]

N.B., if two widely separated sources (i.e., different solns) are highly correlated, MUSIC will eliminate both. Since no single furth soln will look like that 2-separated dipole pattern (e.g., L/R A-I)

In various "dual MUSIC" hacks possible