MAGNET HARDWARE

- $B_0$ field from superconducting magnet
- RF transmit/receive
- Gradient coils

$B_0 \rightarrow z$ (longitudinal)
$B_1 \rightarrow x, y$ (transverse)

(1) $B_0$ field

Superconducting coils in liquid helium (no power required after current injected to bring up field using induction)

$1T = 10,000$ Gauss
$\frac{f}{2\pi} = 42.6$ MHz/T

(2) Body gradient coils

Shim coil also embedded in here (not shown)

(3) RF transmit body coil

(4) RF receive-only head coils

$B_1 \rightarrow$ RF transmitter (10 kW)

Circularly polarized $B_1$ field rotating to $B_0$ at Larmor freq (B1 is several orders of magnitude smaller than B0)

Three one million watt amplifiers to add ramps to $B_0$ field

Usu. water cooled
Spin & Precession

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field
- Current loop from spinning charge (right-hand rule)
- N.B.: Classically this would cause EM radiation, spindown
- Stern-Gerlach experiment
  - pass nuclei through strong magnetic field -> split into just 2 beams

Microscopic picture

<table>
<thead>
<tr>
<th>Strong magnetic field, $B_0 = \uparrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk magnetization $= M_0^z$</td>
</tr>
<tr>
<td>Slight excess of &quot;up&quot; (3 ppm)</td>
</tr>
<tr>
<td>x, y components still random</td>
</tr>
<tr>
<td>Precressing vectors are &quot;bunched&quot; at any one moment around circle</td>
</tr>
</tbody>
</table>

Macroscopic picture

<table>
<thead>
<tr>
<th>No strong magnetic field $B_0 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>All vectors same length, random directions</td>
</tr>
<tr>
<td>Bulk magnetization $= 0$</td>
</tr>
</tbody>
</table>

Precession

- Distinguish precession (slow) from spin (fast)
- Treat classically, like spinning top

$$2\pi \frac{B_0}{\gamma} = \omega_0 = \frac{\gamma}{\hbar} B_0$$

- Larmor frequency (eg. 63 MHz)
- Gyromagnetic ratio (eg. 1.5T)

Bulk equilibrium magnetization (parallel to $B_0$)

$$M_0 = |\vec{M}| = \frac{\gamma^2 \hbar}{2} B_0 N_s$$

where $I = \frac{1}{2}$

$$I(\hbar + 1) = \frac{4 K T S}{3}$$

- $\gamma$ = gyromagnetic ratio
- $\hbar$ = Planck's const.
- $B_0 \rightarrow$ i.e., $M_0^z$ proportional to $B_0$ strength
- $N_s \rightarrow$ i.e., $M_0^z$ proportional to number spins
- $K$ = Boltzmann const.
- $T$ = abs. temperature sample
**Bloch Equation**

- Time-dependent behavior of \( \mathbf{M} \) in the presence of an applied magnetic field (excitation & relaxation).

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \mathbf{Y}_B - \frac{\mathbf{M}_x \mathbf{i} + \mathbf{M}_y \mathbf{j}}{T_2} - \frac{\mathbf{M}_z}{T_1} \]

- In the Larmor-rotating coordinate system, a tilt with a phase shift from a standard \( B_1 \) excitation is rotation around \( x'-axis \).

- Longitudinal and transverse relaxations:
  \[
  \frac{dM_z(t)}{dt} = - \frac{M_z(t) - M^0_z}{T_1}
  \]
  \[
  \frac{dM_{x'y'}(t)}{dt} = - \frac{M_{x'y'}(t)}{T_2}
  \]

- Solution to equations above: time-dependent free precession eg's.
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Generalizes to any \( D \)
  \[ \| \vec{c} \| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**
(= inner product)
(= "scaled projection onto")

\[ c = \vec{a} \cdot \vec{b} = [b_x, b_y] [a_x, a_y] = a_x b_x + a_y b_y \]
- Scalar

\[ p = \| \vec{a} \| \cos \theta \]
- \( c = p \| \vec{b} \| \)
- \( \| \vec{b} \| = 1 \)

**Cross Product**
(= outer product)
(can be generalized: see "geometric algebra")

\[ \vec{c} = \vec{a} \times \vec{b} = [b_y \ -b_x, \ a_z \ -b_z, \ a_x \ b_y - a_y b_x] \]
- Vector

Right-hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \): thumb is \( \vec{c} \)
- Unique orthogonal
- Specific to 3D

\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \theta \]
\( \Rightarrow \) max if orthogonal

**Complex Multiply**
(see also quaternions, geometric algebra generalization)

\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x \ -by, \ by \ b_x] [a_x, a_y] = [a_x b_x - a_y b_y, \ a_x b_y + a_y b_x] \]
- Vector

\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \]
\( \Rightarrow \) like real num

\[ \text{sum of angles: } \Theta_1 + \Theta_2 \]
**Simple Matrix Operations**

**Basic Idea**
- A matrix \( \begin{bmatrix} \text{rotates} & \text{scales} \end{bmatrix} \) a vector

\[
\mathbf{b} = \mathbf{Ma}
\]

**3D Example**

\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z
\end{bmatrix} =
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{21} & M_{22} & M_{23} \\
    M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
    a_x \\
    a_y \\
    a_z
\end{bmatrix}
\]

**Add Translate (after rotate/scale)**
- Commonly used "hack" for aligning vols
- A 4D matrix \( \begin{bmatrix} \text{rotates/scales} & \text{then} & \text{translates} \end{bmatrix} \) (4th D = 1)

\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z \\
    1
\end{bmatrix} =
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} & 1 \\
    M_{21} & M_{22} & M_{23} & 0 \\
    M_{31} & M_{32} & M_{33} & 0 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    a_x \\
    a_y \\
    a_z \\
    1
\end{bmatrix}
\]

**N.B.:** Have to keep track of order!!
- Rotate/scale then trans ≠ trans, then rot/scale
- Change rot component: untranslate, rot, retranslate

**3 Special Cases (3D):** Rotate around each major axis without changing length

(Scale = 1.0)

- Rotate around X-axis:
  \[
  \mathbf{R}_x(\alpha) =
  \begin{bmatrix}
    1 & 0 & 0 \\
    0 & \cos \alpha & \sin \alpha \\
    0 & -\sin \alpha & \cos \alpha
  \end{bmatrix}
  \]
  E.g., 90° flip

- Rotate around Y-axis:
  \[
  \mathbf{R}_y(\alpha) =
  \begin{bmatrix}
    \cos \alpha & 0 & -\sin \alpha \\
    0 & 1 & 0 \\
    \sin \alpha & 0 & \cos \alpha
  \end{bmatrix}
  \]
  E.g., 180° flip to avoid adding 180° phase after 90° flip on x

- Rotate around Z-axis:
  \[
  \mathbf{R}_z(\alpha) =
  \begin{bmatrix}
    \cos \alpha & \sin \alpha & 0 \\
    -\sin \alpha & \cos \alpha & 0 \\
    0 & 0 & 1
  \end{bmatrix}
  \]
  E.g., precession with B φ along z

**General Case**
- Rotate around general Z'-axis:

\[
\mathbf{R}_{z'}(\phi) = \mathbf{R}_z(-\phi) \mathbf{R}_y(-\phi) \mathbf{R}_z(\alpha) \mathbf{R}_y(\phi) \mathbf{R}_z(\phi)
\]

(Faster axis to Z
Z'-rot.
Quaternions are more efficient)
SOLUTIONS TO SIMPLE DIFFERENTIAL EQUATION

**Diff. Eq.**
\[ dM_{xy}(t) = -\frac{M_{xy}(t)}{T_2} \]
**Solution:**
\[ M_{xy}(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

**Goal:**
1) find \( e^x \) whose derivative satisfies diff. eq.
2) also find soln (one of many) that passes thru init condition

since our diff. eq. is:
\[ \text{derivative of funct.} = \text{const. \ same \ funct.} \]

\( \Rightarrow \) try exponential, since derivative \( (e^x)' = e^x \)

**Deriv. Eq.**
\[ M'(t) = \frac{-1}{T_2} \cdot M(t) \]

**One soln:**
\[ M(t) = e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = \frac{-1}{T_2} \cdot e^{-t/T_2} \]

**OK!**

**Another soln:**
\[ M(t) = \text{const} \cdot e^{-t/T_2} \]

**Take deriv. to check:**
\[ M'(t) = \frac{-1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} \]

**OK!**

**Initial Condition**
\[ M(t) = M_{xy}(0) \cdot e^{-t/T_2} \]

**Magnetization immed. after RF (B1) ends**

**Const:**
\[ M_{xy}(0) \]

**Information added to soln (not from diff eq.)**
**Bloch Eq. - Matrix Version**

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0^+) \\ M_y(0^+) \\ M_z(0^+) \end{bmatrix} = R_z(\omega t) \vec{M}(0^+)
\]

Include Relaxation

Differential Eq.:

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x \hat{c} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1}
\]

Solution:

\[
\vec{M}(t) = \begin{bmatrix} e^{-\frac{t}{T_2}} & 0 & 0 \\ 0 & e^{-\frac{t}{T_2}} & 0 \\ 0 & 0 & e^{-\frac{t}{T_1}} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0^+) \\ M_y(0^+) \\ M_z(0^+) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1-e^{-\frac{t}{T_1}}) \end{bmatrix}
\]
**RF FIELD POLARIZATION**

- Polarization (change of direction) of magnetic field (vs. electric field) generated/recorded by RF coil

- Linearly polarized field
  \[
  \vec{B}_1 = B_1 \cdot \cos \omega t \hat{x} \\
  \text{magn. strength} \cdot \{-1, 1\} \cdot 1
  \]

- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)

- Circularly polarized field (quadrature)
  \[
  \vec{B}_{1\text{circ}} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \\
  = B_1 \cdot e^{-i\omega t}
  \]

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF shim

**Typical 90° flip**
(around x = axis)

**Typical 180° flip**
(around opposite y-axis)

*180° flip reyk
~6× power at 90°*
\[ \Phi(t) = \iiint_{\text{obj}} \mathbf{B} \cdot \mathbf{M} \, d\mathbf{r} \]

\[ V(t) = -\frac{d\Phi(t)}{dt} = -\frac{d}{dt} \iiint_{\text{obj}} \mathbf{B} \cdot \mathbf{M} \, d\mathbf{r} \]

Faraday law of Induction

\[ \text{ignore the change in the } z\text{-component of the magnetization} \]

\[ \text{since it changes so slowly compared to the free precession of } x\text{- and } y\text{-components: } \omega \approx 0 \]

Laboratory frame Bloch solution: $M_L = \text{same}$ $M_T = M_x(0)e^{i\omega t}$

spatially dependent resonant freq in rotating frame - i.e. after subtraction $\omega = \gamma B_0$
**PHASE-SENSITIVE DETECTION**

$V(t) \sim 63 \text{ MHz}$

how we get rotating frame

- method for moving very high frequency Larmor oscillations down to tractable frequency range

Reference signal (63 MHz)

Demodulated signal $\propto$ RF coil signal $\cdot$ reference (transmitter) $\propto \sin (w_0 + \delta w)t \cdot \sin w_0 t$

\[
\sin a \sin b = \frac{1}{2} \left[ \cos(a-b) - \cos(a+b) \right] \\
\sin a \cos b = \frac{1}{2} \left( \sin(a-b) + \sin(a+b) \right)
\]

$\propto \frac{1}{2} \left[ \cos \delta wt - \cos (2w_0 + \delta w)t \right]$ diff between RF input & ref

Filter this one out w/ low pass filter

- two signals are made from a **single** receiving RF coil

- a quadrature coil can be treated the same way (OK to combine after adding $\pi/2$ phase then PSD)

- quadrature coil has better S/N since noise in each part is uncorrelated ($\frac{1}{\sqrt{2}}$ better)

**One freq -- freq domain**

Signal

Reference

Demodulated

After filter

0 6

**Chirp -- time domain**

Chirp

Center

Demodulated

Low freq signal

$S(t)$ complex $= e^{i \delta wt}$ written as

$S(t)$ real $\rightarrow$ PSD $\rightarrow$ $\cos wt$

$S(t)$ imaginary $\rightarrow$ PSD $\rightarrow$ $\sin wt$
**FID - free induction decay, T2**

- FID (free-induction decay) from an RF pulse w/angle $\alpha$

  $$S(t) = \sin \alpha \int_{w=-\infty}^{w=\infty} \rho(w) e^{-t/T_2}(w) e^{-i\omega t} dw$$

  - for a single freq:

  $$S(t) = M_2^0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t}$$

  - Max FID amplitude at $t=0$: $S(0) = M_2^0 \sin \alpha$

- If field inhomogeneous (Lorentzian distribution)

  $$S(t) = \pi M_2^0 \gamma \Delta B_0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t}$$

  - $T_2^*$ = decay rate caused by $\Delta B_0$

  $$\Rightarrow \frac{1}{T_2} = \frac{1}{T_2} + \frac{1}{T_2^*}$$

- Basic FID envelope is proportional to $e^{-t/T_2^*}$

  - NB. actually complex

- Typical time course 10's of ms to sec vs. each precision cycle, which is 10's of nsec

**T2 -** unrecoverable, rapid (intrinsic)

**T2* -** recoverable, static added
ECHOES — spin echo

$T=0$

- Relaxation + Phase dispersion of $f_{10}$ and $f_{1i}$ (both from $B > B_0$)

- Remember: RF just tips vector(s) while retaining length
  Relaxation includes tips and shrinks (and grows for echo)

- $180^\circ$ pulse works too, but echo will be $+\pi$ phase (left side in Figs above)

- Echos generated even if second pulse not $180^\circ$ (see next)

- FID decay (and echo growth/decay) describe by $T_2^*$ from inhomogeneity

- Reduction in height of echo compared to initial described by $T_2$, echo fixes the "star"
**ECHOES — Spin echo**

- \( \alpha_1 - \tau - \alpha_2 \) (both pulses along \( y' \) for simplicity)

**Effect of \( \alpha \)-pulse**

- \( M_{x'} \rightarrow M_{x'} \cos \alpha - M_{z'} \sin \alpha \)
- \( M_{y'} \rightarrow M_{y'} \)
- \( M_{z'} \rightarrow M_{x'} \sin \alpha - M_{z'} \cos \alpha \)

**Effect of \( \tau \)-delay**

- \( M_{x'} \rightarrow (M_{x'} \cos \omega \tau + M_{y'} \sin \omega \tau) e^{-\tau/\tau} \)
- \( M_{y'} \rightarrow (-M_{x'} \sin \omega \tau + M_{y'} \cos \omega \tau) e^{-\tau/\tau} \)
- \( M_{z'} \rightarrow M_{z'} (1 - e^{-\tau/\tau}) + M_{x'} e^{-\tau/\tau} \)

**Immediately after \( \alpha_1 \)-pulse**

- \( M_{x'}(w, 0^+; \alpha) = -M_0^z(w) \sin \alpha \)
- \( M_{y'}(w, 0^+; \alpha) = 0 \)
- \( M_{z'}(w, 0^+; \alpha) = M_0^z(w) \cos \alpha \)

**After \( \tau \)-delay**

- \( M_{x'}(w, \tau; \alpha) = -M_0^z(w) \sin \alpha, \cos \omega \tau e^{-\tau/\tau} \)
- \( M_{y'}(w, \tau; \alpha) = M_0^z(w) \sin \alpha, \sin \omega \tau e^{-\tau/\tau} \)
- \( M_{z'}(w, \tau; \alpha) = M_0^z(w) \cos \alpha \left[ 1 - (1 - \cos \alpha) e^{-\tau/\tau} \right] \)

**Immediately after \( \alpha_2 \)-pulse (no effect on \( M_{y'} \); rewrite \( x', \) \( y', \) and \( z' \) eq's)**

- \( M_{x'y'}(w, \tau; \alpha_2) = M_0^z(w) \sin \alpha, \left( \sin^2 \frac{\alpha_z}{2} e^{i \omega \tau} - \cos^2 \frac{\alpha_z}{2} e^{i \omega \tau} \right) e^{-\tau/\tau} \)
- \( M_{z'}(w) \left[ 1 - (1 - \cos \alpha) e^{-\tau/\tau} \right] \sin \alpha \)

**Time dependent (free precession around \( z' \) (rewrite \( M_{x'y'}(w, \tau; \alpha_2) \))**

- \( M_{x'y'}(w, t; \alpha_2) = M_{x'y'}(w, \tau; \alpha_2) e^{-(t - \tau)/\tau e^{-i \omega (t - \tau)} \}

**For a large num of freq's:**

- Terms 1 & 3 are dephasing
- Term 1: zero phase at \( t = 2 \tau \)

**Echo Signal**

- Peak ampl: \( A_E = \sin \alpha, \sin^2 \frac{\alpha_z}{2} \)

**Echo Signal**

- \( S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2 \pm i \omega (t - \tau)} d\omega \)

**Peak Ampl**

- \( A_E = \sin \alpha, \sin^2 \frac{\alpha_z}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2 \pm i \omega (t - \tau)} d\omega \)

**Echo Amplitude, ignoring freq dependence of \( T_2 \)**

- \( A_E = M_0^z \sin \alpha, \sin^2 \frac{\alpha_z}{2} e^{-T_2} \)

**\( 90^\circ - \tau - 90^\circ \)**

- \( S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2 \pm i \omega (t - \tau)} d\omega \)

**\( 90^\circ - \tau - 180^\circ \)**

- \( S_2(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2 \pm i \omega (t - \tau)} d\omega \)

**\( 90^\circ - \tau - 180^\circ \)**

- \( S_2(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2 \pm i \omega (t - \tau)} d\omega \)
Echo TRAINS - spin-echo trains

- it's (too) easy to make echoes...

\[ E_n = \frac{3(n-1)}{2} \]

- echoes after end of nth pulse
- 3 RFs \( \rightarrow \) 4 echoes (here)
- 6 RFs \( \rightarrow \) 121 echoes (!!)

RF transmit

- \( \alpha_1 \)
- \( \alpha_2 \)
- \( \alpha_3 \)

RF receive

- \( \tau_1 \)
- \( \tau_2 \)

Secondary echo: \( \text{SE}_{1,2} \) acts like RF pulse \( \alpha_2 \) makes an echo from it

- \( \text{SE}_{1,2} \) acts like two conventional two-pulse spin echoes
- \( \text{SE}_{1,3} \) - two more conventional two-pulse spin echoes
- \( \tau_2 \)

- Stimulated echo: combined effect of 3
- \( \alpha_1: M_L \rightarrow M_T \)
- \( \alpha_2: \text{leftover } M_T \text{ flipped to } M_L \) (saved)
- \( \alpha_3: \text{flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays (after being held \"in \text{limbo}\" between 180° FID}_2 \text{ and } \text{FID}_3 \); acts like 2-pulse echo

- \( \tau_1 \)
- \( \tau_2 \)
- \( \tau_3 \)

- a useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2\( \tau \) spacing

- \( \tau \)
- \( 2\tau \)

- typically, 90° and 180° applied in different axes \( (x', r \text{en } y', y'... \) which reduces phase errors due to imperfect 180° pulses (since slightly-off rotation around \( y' \) affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. Solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ, 180^\circ$)
- Problem #1: $\alpha$ pulse rotates a position of transverse magnetization into a position that results in rephasing ⇒ reg's QM view
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

Rule for effect of $\alpha$
RF pulse on transverse mag

Rule for effect of $\alpha$
RF pulse on longitudinal mag

→ Echo when phase path crosses zero
**Hyper Echoes**

1. $\alpha_z - 180^\circ - \alpha_2 = 180^\circ$
   - 3 solid lines
   - 1 dashed line

2. $\alpha_y - 180^\circ - \alpha_y = 180^\circ$

3. $\alpha_y - 180^\circ - \alpha_y = 180^\circ$

---

**Practical Use**

- Multi-echo example

- By combining longer sequences observing these symmetries, can generate as strong echo even w/ many inserted $\alpha_z$ pulses

---

- By arranging to get big echo in middle of k-space can get by with much less RF power

---

- Hennig & Scheffler (2001)
  - Ignore amplitude
  - Surface of the sphere then defines a 2D space that you can move around in using:
    
    \[
    \begin{align*}
    \text{flip angle from } z'(\alpha) & \\
    \text{RF phase in } x'y'(\phi) & \\
    \text{"flip/phase" RF shim space} &
    \end{align*}
    \]

- By combining long sequences observing these symmetries, can generate as strong echo even w/ many inserted $\alpha_z$ pulses
**Gradient Echoes** - $T_2^*$, GE Chains

- **Initial negative gradient dephases spins**
- After $t = T$ of positive gradient, spins rephase
- Does not correct for $T_2^*$ inhomogeneities
  - So echo amplitude is
  
  $A_E = e^{-t/T_2^*}$

- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

\[ \frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \Rightarrow A_E = e^{-t/T_2^{**}} \]

**Key difference between spin-echo (SE) and gradient echo (GE)**

- Hence, echoes are $T_2^*$-weighted, not $T_2$-weighted => more susceptible to inhomogeneities

**Echo trains possible w/ gradient echo (CPMG-like)**

- The faster the gradients are switched, the more echoes you get

- EPI hardware => 64 echoes
IMAGE CONTRAST  

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

RF

\[ M_z^0 \to M_z^0 (1 - e^{-TR/T1}) + [\text{ignored}] \]

- Simple saturation/recovery w/ no echo

- Initial conditions: 
  \[ M_z \text{ before first pulse} = M_z^0 \]
  \[ M_z = 0 \text{ imm. after first pulse (i.e., 90° pulse)} \]

- From Bloch eq, \( M_z \) just before second pulse:

\[ M_z^{(n)}(O_-) = M_z^0 (1 - e^{-TR/T1}) + M_z^{(n)}(O_+) e^{-TR/T1} \]

- Given
  1. 90° pulse
  2. no \( M_{xy} \) left

\[ \rightarrow \text{pure tip: } M_{xy} = M_z \]

- Tip existing mag

\[ M_z^{(n)}(O_-) = M_{xy}'(O_+) = M_z^0 (1 - e^{-TR/T1}) \]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[ I(r) = C p(r) \left(1 - e^{-TR/T1(r)}\right) \]

\[ \text{rec. const. spectral dens, p(r)} \]

\[ \text{spectrum dens p(r) & p.d. underlying eqib. } M_z^0 \]
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- Initial fMRI images are usually discarded (why?)
- Because they are brighter than all the rest
- Because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
(e.g. at 3T, flip angle varies almost 25% across brain)

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Diagram showing the process of achieving steady state in an MRI scan.

- At 3T, steady state
  - For typical 1-2 sec TR images reached
  - After 8 images

---

Mathematical expressions and vectors representing the magnetic field components are also present in the diagram.
**IMAGE CONTRAST**

- **IR** (still just saturation-recovery — no echo)

- Inversion recovery w/ no echo

- 180° pulse reverses longitudinal magnetization

  \[ M_z^{(0)} = -M_z^0 \]

- Recovery to end of first TI from long. part of Bloch eq.

  \[ M_z' = M_z^0 \left( 1 - 2e^{-t/2T_1} \right) \]

- Longitudinal then regrows from zero from first Bloch term only

  \[ M_z' = M_z^0 \left( 1 - e^{-\left( TR - TI \right)/T_1} \right) \]

- After second 180°, just change sign again

  \[ M_z' = -M_z^0 \left( 1 - e^{-\left( TR - TI \right)/T_1} \right) \]

- Apply relaxation eq. again

  \[ M_z' = M_z^0 \left( 1 - e^{-TI/T_1} \right) - M_z^0 \left( 1 - e^{-\left( TR - TI \right)/T_1} \right) e^{-T_1/T_1} \]

  \[ M_z' = M_z^0 \left( 1 - 2e^{-TI/T_1} + e^{-TR/T_1} \right) \]

\[ \Rightarrow \text{this is magnetization flipped to transverse, made recordable} \]
**IMAGE CONTRAST**

- **SE, IR-SE**

- **RF**
  - $90^\circ$ signal $180^\circ$ causes echo
  - slice select

- **G_z**
  - refocus phase encode
  - center of RF

- **G_y**
  - center of read out

- **G_x**
  - echo

- **RF_{in}**
  - TE
  - TR

- **Steady state mag** (2nd TR) just before $90^\circ$  
\[ M_z'(0) = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) + e^{-\frac{TR}{T_2}} \]

- The echo signal ($M_z$) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation.

\[ A_E = M_z^0 \left( 1 - 2e^{-\frac{(TR-TE/2)}{T_1}} \right) + e^{-\frac{TR}{T_2}} \]

- if we assume $TE$ much less than $TR$, then we can simplify:

\[ A_E = M_z^0 \left( 1 - e^{-\frac{TR}{T_1}} \right) e^{-\frac{TE}{T_2}} \]

- Similar equation for SE-IR

\[ A_E = M_z^0 \left( 1 - 2e^{-\frac{TE}{T_1}} + e^{-\frac{TR}{T_2}} \right) e^{-\frac{TE}{T_2}} \]
- use basic longitudinal relaxation from Bloch eq. again

\[ M_z'^{(n)}(0_\pm) = M_z^0 (1 - e^{-TR/T1}) + M_z'^{(n-1)}(0_\pm) e^{-TR/T1} \]

- assume we have a small tip angle: \[ M_z \cos \theta \Rightarrow M_z'^{(n)}(0_\pm) = M_z'^{(n)}(0_-) \cos \alpha \]

\[ M_z'^{(n)}(0_-) = M_z^0 (1 - e^{-TR/T1}) + M_z'^{(n-1)}(0_-) \cos \alpha e^{-TR/T1} \]

- assume we are in dynamic equilibrium: \[ M_z'^{(n)}(0_-) = M_z'^{(n-1)}(0_-) = M_z^{ss}(0_-) \]

\[ M_z^{ss}(0_-) = \frac{M_z^0 (1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}} \]

\[ M_z^{ss}(t) = \frac{M_z^0 (1 - e^{-TR/T1}) \cdot \sin \alpha e^{-T2}}{1 - \cos \alpha e^{-TR/T1}} \]

\[ \text{gradient echo amplitude} \]

\[ A_E = \frac{M_z^0 (1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2}}{1 - \cos \alpha e^{-TR/T1}} \]

TI contrast mainly depends on flip angle, not TR \( \Rightarrow \cos \phi = 1 \) \( \Rightarrow \) eliminates TI weight since denominator = numerator.
- Saturate, wait for contrast A, invert, wait for contrast B, FLASH (continued)

A) $M_z'(\text{just after } 90^\circ) = 0$ (perfect $90^\circ$)

B) $M_z'(\text{after } TD) = M_z^0 (1 - e^{-TD/T_2})$ (Bloch term #1)

C) $M_z'(\text{just after invert}) = \cos \phi M_z^0 (1 - e^{-TD/T_2})$

D) $M_z'(\text{after } TI) = M_z^0 (1 - e^{-TI/T_1}) + [\cos \phi M_z^0 (1 - e^{-TD/T_1})] e^{-TI/T_1}$

Special Case $TI = TD$:

$M_z' = M_z^0 [1 - e^{-TI/T_1}]^2$

- After the first $\alpha$ pulse:

E) $M_z'(\text{just after } 180^\circ) = M_z^0 [1 - [1 - \cos \phi (1 - e^{-TD/T_1})] e^{-TI/T_1}] \sin \alpha$
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}} \)
- Temporal SNR: \( \text{tSNR} = \frac{S_{\text{obj}} - S_{\text{bg}}}{S_{\text{obj}} + S_{\text{bg}}} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[
\text{CNR}_{\text{AB}} = \frac{S_A - S_B}{\text{s.d. non-object region}} = \text{SNR}_A - \text{SNR}_B
\]

- Spin-echo: \( A_e = M_z (1 - e^{-TR/T1}) e^{-TE/T2} \)

**Haemorrhage**

<table>
<thead>
<tr>
<th>Tissue</th>
<th>( T1 )</th>
<th>( T2 )</th>
<th>PD</th>
</tr>
</thead>
<tbody>
<tr>
<td>GM</td>
<td>950</td>
<td>100</td>
<td>0.85</td>
</tr>
<tr>
<td>WM</td>
<td>600</td>
<td>80</td>
<td>0.65</td>
</tr>
<tr>
<td>CSF</td>
<td>4500</td>
<td>2200</td>
<td>1.50</td>
</tr>
<tr>
<td>Blood</td>
<td>1200</td>
<td>100-200</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Gradient Echo**

\[
A_e = \frac{M_z (1 - e^{-TR/T1})}{1 - \cos \alpha} e^{-TE/T2*}
\]

**General rules: Spin-echo, long TR GE**

<table>
<thead>
<tr>
<th>proton-density weighted</th>
<th>( \text{TR} \approx T1 ) (big T1 diffs)</th>
<th>( \text{TE} \approx T2 ) (big T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>( \text{TR} \approx T1 ) (big T1 diffs)</td>
<td>( \text{TE} \approx T2 ) (big T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>( \text{TR} \approx T1 ) (big T1 diffs)</td>
<td>( \text{TE} \approx T2 ) (big T2 diffs)</td>
</tr>
</tbody>
</table>
**SIGNAL-TO-NOISE S/N**

- Generalized dependence of SNR on 3D imaging parameters

\[
\frac{\text{SNR/voxel}}{\Delta x \Delta y \Delta z} = \sqrt{N_a^2 N_x N_y N_z \Delta t}
\]

- Size (volume) of voxels (with the number of voxels held constant)
  - Linear effect on S/N
  - \( \ell \) e.g., \( 3 \times 3 \times 3 \text{mm} \rightarrow 4 \times 4 \times 4 \text{mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N} \)

- More voxels (with size of voxels, \( \Delta t \) per read step constant)
  - \( \ell \) e.g., \( 64 \times 64 \rightarrow 128 \times 128 \rightarrow \sqrt{128 \times 128} \) \( / \sqrt{64 \times 64} \) = 2 times better S/N

- Number of acquisitions \( \sqrt{\ell} \) better S/N
  - \( \ell \) e.g., 1 acq \( \rightarrow \) 2 acq \( \rightarrow \) \( \frac{\sqrt{2}}{\sqrt{1}} = 1.41 \) times better S/N

- Larger timestep during readout, \( \sqrt{\Delta t} \) better S/N

\[ \Delta t = \frac{1}{\text{BW}_{\text{read}}} \text{, digitization timestep during echo acquisition} \]

- BW$_{\text{read}}$ determined by cutoff freq analog lowpass filter
- \( \Delta t \) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \( \Delta t \)
- Must filter out freqs > \( f_{\text{max}} = \frac{1}{2\Delta t} \) because they alias

![Diagram of signal-to-noise ratio with waveform and frequency components]
**Complex Algebra**

- **Real/Imaginary**
  - Add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - Mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

- **Angle/Phase**
  - Add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - Mult: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\) → N.B.: 3rd kind of vector mult. different than dot product and cross product

- **Real to Complex Power**
  \[ e^{i\phi} = \begin{cases} \cos \phi + i \sin \phi \\
\text{recognize \cos, \sin series} \\
\text{as \exp of series} \end{cases} \]
  \[ e^{i\phi} = (\cos \phi + i \sin \phi)^n \]
  \[ \cos n\phi + i \sin n\phi \]

- **Fourier Transform**
  \[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-i2\pi ft} dt \]
  \[ \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(\theta) e^{-i2\pi \theta t} d\theta \]

- **Convolution Theorem**
  \[ \mathcal{F}[g(x) * h(x)] = G(k) * H(k) \]
  \[ \text{because of FFT, faster if kernel not small} \]

- **Convoluton**
  \[ f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) dz \]
  \[ f(x) = g(x) \otimes h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x+z) dz \]

- **Convolution**
  \[ f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) dz \]
  \[ \text{Cross-covlem} \]
  \[ f(x) = g(x) * h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x+z) dz \]

- **Fourier Transform of Two Functions Multiplied by Each Other**
  \[ \text{equals the convolution of} \]
  \[ \text{the Fourier transform of each function} \]
Fourier transform (1)

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-i \frac{2\pi ft}{6}} dt \]

- How to calculate \( H(f) \) for one \( f (=3) \):

- For one \( \theta \)

\[ \theta = \pi t \]

\[ e = e^{i \theta} \]

\[ \cos (3t) \]

\[ \sin (3t) \]

i.e., two correlations

Real frequency domain

Imaginary frequency domain

Like correlating with sin and cos (at each freq) so we get phase (at each freq)
Fourier transform \( (1b) \)

\[
e^{i\phi} = \cos \phi + i \sin \phi
\]

\[
e^{-i\phi} = e^{i(-\phi)}
\]

\[
= \cos(-\phi) + i \sin(-\phi)
\]

\[
= \cos \phi - i \sin \phi
\]

\( \cos \) is an "even" function, \( \sin \) is an "odd" function

\[
\text{even}
\]

\[
\cos(x)
\]

\[
\downarrow \text{equals}
\]

\[
\cos(-x)
\]

\[
\downarrow \text{flips}
\]

\[
-\cos(x)
\]

\[
\text{odd}
\]

\[
\sin(x)
\]

\[
\downarrow \text{flips}
\]

\[
\sin(-x)
\]

\[
\downarrow \text{equals}
\]

\[
-\sin(x)
\]

An orthogonal decomposition

- think of discretely sampled \( \sin(bx) \), \( \cos(bx) \) as vectors
- \( \text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \) projection of \( \vec{v}_1 \) onto \( \vec{v}_2 \)

\[
\text{Corr}(\cos(b_1 x, \sin(b_1 x)) = 0
\]

\[
= \sin \& \cos \text{ of same frequency are orthogonal}
\]

\[
\text{Corr}(\sin(b_2 x, \sin(b_2 x)) = 0
\]

\[
= \text{different integer freqs of } \sin(\& \cos \text{ are orthogonal}
\]

\[
\text{Corr}(\cos(b_1 x, \sin(b_2 x)) = 0
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{-\infty}^{\infty} g(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- start with spike in image domain
- take example of spike at \( x = 0 \)

\[
\begin{bmatrix}
\cos(x), \cos(2x), \cos(kx)
\end{bmatrix}
\text{all freqs correlate w/ spike at } x = 0
\]

- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the \( e^{-2\pi ikx} \) cos and sin at location of spike

\[
\text{successively higher freqs}
\]

- \( \cos \) vs. \( \sin \):
  - \( \cos \): even
  - \( \sin \): odd

\[
\text{positive spikes same dist from origin: } \rightarrow \text{pick cos's}
\]
\[
\text{positive & negative spikes, same dist: } \rightarrow \text{pick sin's}
\]

- this is one way of thinking about what one point in k-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

1. Real image \( \rightarrow \) Imaginary image

\[
\begin{array}{c}
\text{real image} \\
x \rightarrow \\
\text{imaginary image} \\
x \rightarrow \\
(zero)
\end{array}
\]

Fourier Transform

2. Amplitude image \( \rightarrow \) Phase image

\[
\begin{array}{c}
\text{amplitude image} \\
x \rightarrow \\
\text{phase image} \\
x \rightarrow \\
(zero)
\end{array}
\]

Inverse Fourier Transform

What you see on screen \( \downarrow \) View complex vectors directly

3. Complex vectors

- 3 equivalent representations of image & spatial frequency space
FOURIER TRANSFORM OF AN IMAGE (3)

- what a single k-space point looks like in image space (polar coordinates A, φ instead of r, i)

image space

k-space (spatial freq. space)

offset of stripes is k-space phase

brightness of stripes is k-space amplitude

distance from center is stripe spacing

angle of point is angle of stripes

value from 0 to 360°

amplitude

inverse Fourier transform

(image recon.)

(should be all zero not same as "stripe phase" above)

phase

(should be all zero not same as "stripe phase" above)

phase

(should be all zero not same as "stripe phase" above)

Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ Sin t/cos — don't confuse kx, ky w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
FOURIER TRANSFORM OF IMAGE (4)

- 3 equivalent representations of complex numbers in image space and spatial-freq. space \( (k\)-space)  
- example: cosinusoid in image space, then shifted in x-dir

REAL IMAGE

\[ I(x,y) = \cos(x) \]

FT OF REAL IMAGE

\[ \text{FT of } I(x,y) \]

\[ k_x = 1, \quad k_y = 0 \]
\[ k_x = -1, \quad k_y = 0 \]

\[ \text{max pos, max neg} \]
\[ \text{max pos, max neg} \]

\[ \text{complex} \]

\[ I(x,y) = \cos(x - \frac{\pi}{4}) \]

\[ \text{FT of } \cos(x - \frac{\pi}{4}) \]

\[ \text{FT of } \sin(x - \frac{\pi}{4}) \]

\[ \text{FT of } \cos(x - \frac{\pi}{4}) \]

\[ \text{FT of } \sin(x - \frac{\pi}{4}) \]

\[ \text{FT of } \cos(x - \frac{\pi}{4}) \]

\[ \text{FT of } \sin(x - \frac{\pi}{4}) \]

\[ \text{same as above} \]

\[ \text{complex} \]

\[ \text{real component less than above because rot:} \]

\[ \text{phase now } 45^\circ \text{ at } \]
\[ k_x = 1, \quad k_y = 0 \]
\[ (-45^\circ \text{ at } \]
\[ k_x = -1, \quad k_y = 0) \]

U.B.: an example of the "Fourier Shift Theorem" (see below)
FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

REAL IMAGE

\[ I(x,y) = 1 + \cos(x) \]

![Image of real image with center and zero areas marked]

\[ \text{center of k-space:} \]

\[ H(k) = \int x \cdot e^{-i\pi kx} \]

avg image brightness \( \leq \) 1 (real)

FT OF REAL IMAGE

![Diagram of FT of real image with positive center k-space]

![Diagram of inverse FT with zero areas marked]

- The center of k-space is zero w/ pure sin or cos image b/c avg brightness = 0

COMPLEX IMAGE

\[ I(x,y) = \cos(x) - i \sin(x) \]

\[ e^{-i\pi x} \]

![Image of complex image with center and zero areas marked]

FT OF COMPLEX IMAGE

- "Missing" spike results in single spike correlating with cos and sin

FT, FT⁻¹

non-Hermitian:
- K-space will only have Hermitian symmetry if image is real:

- complex conjugate (a complex num w/ sign flipped in imag. part) is equal to func value w/ neg arg:

\[ H(-x) = H^*(x) \]

![Diagram of FT and inverse FT of complex image]

* N.B., this is like what a gradient does!
GRADIENT COILS

- Gradient coils for x, y, z generate approximately a linear gradient in the strength of the z-component of the magnetic field $B_z$.

- For example, the x gradient coil induces a ramp in z-component of the magnetic field when moving in the x-direction:

$$B_{G,z} = G_x x$$

* Since a pure linear gradient of $B_{G,z}$ in only the x, y, or z directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the x- and y-direction ($B_{G,x}$ and $B_{G,y}$).

- The other magnetic field components are usually ignored because they are so small relative to $B_{G,z}$, since $B_{G,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{G,x}$, $B_{G,y}$, and $B_{G,z}$.

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms $B_{G,x}$, $B_{G,y}$ are known, can be included in the image processing.
slice select gradient on during RF stim

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle
- valid for small tip: $90^\circ \rightarrow 52\%$
- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} p(f) e^{-i 2\pi f t} \, df \]

- Time dependent RF stimulation (complex)

\[ B_1(t) = A \cdot f_w \cdot \text{sinc} \left( \pi f_w t \right) e^{-i 2\pi f_c t} \]

- Amplitude controlling flip angle
- Frequency width controlling slice thickness (N.B. wider \( f_w \) is narrower sinc)

Solve with: \( p(f) = \) frequency band:

- i.e., time-dependent complex \( (= \text{quadrature}) \) pulse waveform is Fourier transform of frequency spectrum of RF pulse

- Sinc envelope width inversely proportional to \( f_w \)
- Larmor oscillation at center freq.

Fourier Transform Pairs, Rules

- Convolution in one domain is multiplication in the other
- Convolution with delta funct. impulse moves function to impulse center

Fourier Transform Solution to: \( \frac{d}{dt} \)
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change freq</td>
<td>gradient changes freq.</td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF</td>
<td>same</td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs</td>
<td>same</td>
</tr>
<tr>
<td>4) FT of FID to get spectrum</td>
<td>FT of FID to get Δx offsets</td>
</tr>
<tr>
<td>[ \Delta f ] of Δf offsets</td>
<td>[ \Delta x ] offsets</td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"
  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>Signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq-domain</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because FT=FT⁻¹ (except sign change)
FREQUENCY ENCODING (1) avoid this mistaken intuition!

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction

\[ \text{precession} \uparrow \text{frequency} \uparrow B_z \uparrow \text{x-direction} \rightarrow \text{correct} \quad (\text{remember that strength of } G_x \text{ causes variation of slope of } B_z \text{ in x-direction}) \]

- Different frequency signals are mixed together and recorded as a 1-D signal over time

\[ \rightarrow \text{correct}, \text{ but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which converts back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq) is done on signal

\[ \rightarrow \text{correct} \]

- Spatial frequencies get confused/confused with precession frequencies

\[ \rightarrow \text{wrong}!! \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[ \text{conceptually wrong}!! \rightarrow \text{actualy converts spatial frequency to spatial position} \]

\[ \rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \rightarrow \text{the precession freq ramp is constant each timestep} \]

N.B.: gradient ramp does not need to be exactly the same as during recording.
FREQUENCY ENCODING (2)

- "Frequency"-encode gradient ($G_x$) turned on during
during echo causes precession rates
to immediately vary with $x$-position

- at beginning of gradient on, the phase of
  signal coming from each $x$-position is the same

  **Summed phase angle is what we measure**

- early after gradient on, phase advances (because
  of faster precession frequency) cause with greatest
  phase advance at largest $x$-position

- later during gradient on, phase advances cause
  multiple wraparounds of phase angle across space

- in practice, the lowest spatial frequency ($\delta = 0$)
  occurs in the middle of the gradient on time
  because the phase is " wound" negatively by
  a preparatory gradient (to the highest negative
  spatial frequency) before data collection occurs

  $\delta$ is spatial frequency

  $G_x$ in $x$-direction

  $G_x$ levels
  (= slope)

  actually

  $B_x$, $x$
FREQUENCY ENCODING (3)  why each datapoint is 1 spatial freq

Standard Fourier transform :  (Temporal freq \leftrightarrow time)

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-i \frac{2\pi ft}{\tau}} dt \]

"k" is often used instead of "f" for the frequency variable

Imaging equation :  (Spatial freq \leftrightarrow space)

\[ S(f) = \int_{-\infty}^{\infty} I(x) e^{-i \frac{2\pi fx}{\lambda}} dx \]

Sum across x of object

Signal strength at one x-position
(brightness of image point)

Spin density (spectral dens.)

Oscillations come from readout phase wrapping; where f is single spatial freq (e.g. 5) and x goes across object

END: \( G_x \)

SE: \( G_x (t-TE) \)

\[ f = G_x, \text{ that is, spatial freq depend on amount of time gradient was on (this f increases with time!)} \]

To make image, do inverse Fourier transform of recorded signal \( S(f) \)

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)
**Alternate Derivation (incl. Effects of $G_x$) Signal Eq**

- Oscillators at $w = \omega_B$ at each position (just $x$ for now)

\[ S(t) = M(x) e^{-i\phi(x)} dx \]

- By definition, freq. $w$ is rate of change of phase, $\phi$

\[ \frac{d\phi(x,t)}{dt} = w(x,t) = \omega_B(x,t) \quad \text{and} \quad \phi(x,t) = \int_0^t w(x,t) dt = \int_0^t \omega_B(x,t) dt \]

- Assuming phase initially $0$, $B$ affected by gradients

\[ B(x,t) = B_0 + G_x(t) \cdot x \]

So

\[ \phi(x,t) = \int_0^t \omega_B dt + \left[ \int_0^t G_x(t) dt \right] x \]

\[ = \omega_0 t + 2\pi k_x(t) x \]

$k$ is time integral of gradient waveform

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i\omega_0 t}$ from the first equation

\[ S(t) = \int_x M(x) e^{-i 2\pi k_x(t) x} dx \]

Amplitude of each oscillator, gradient-controlled phase
**Phase-encode Gradient** $G_y$

- Turn on gradient after excitation but before readout.
- Different levels of $G_y$:
  - Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
  - Higher spatial freq. (more phase wraps) in y-direction.
- Phase wraps persist after phase-encode gradient off.
- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase.

**2D Imaging Equation**

$$S(k_x, k_y) = \int \int I(x,y) \cdot e^{-i2\pi (k_x x + k_y y)} \, dx \, dy$$

- Signal recorded at single time point (one readout point).
- Complex signal (from phase-sensitive detection).
- Done by RF coil.
- Scalar (what we try to reconstruct).
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients.

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia"—they stay wherever the gradients last left them.
3-D IMAGING - two phase-encode gradients

- use $z$-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

$S(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, z) e^{-i2\pi (k_x x + k_y y + k_z z)} \, dx \, dy \, dz$

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)
- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice

$\therefore$ phase stripes created throughout volume vs. slice.

N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq encode gradient both affect phase the result is a rotation of phase "stripes" when the two add.

Phase encode $f=0$

Freq encode $f=4$

N.B.: Stripes have sharp edges from phase map (not sinusoid since $\phi$ from 2-comp quadrature!)

Stripes here represent complex value.

Phase of whole image summed to one (complex) number by RF coils.

3D phase encode w/ $G_y$ and $G_z$ starts rotated in y-z plane.
**Gradients Move k-space Location of Data Point**

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[
K = \mathbf{G} \int_0^{\text{recording time}} G(t) \, dt
\]

- Spatial freq recorded at t = recording time
- Gradient strength as function of t

\[K = \mathbf{G} t\]

\[K = \text{area under curve}\]

- All of the following gradients end up at the same point in k-space:

**Frequency-encode FID**

**Frequency-encode Gradient Echo**

**Frequency-encode Spin-Echo (plus gradient echo!!)**

**Phase-encode then Frequency-encode Gradient Echo**

**RF**

\[\mathbf{G}_x\]

\[\mathbf{G}_y\]

\[\mathbf{G}_z\]
**IMAGE RECONSTRUCTION**

\[
S(k_x, k_y) = \frac{1}{\sqrt{L_x L_y}} \int I(x, y) e^{i 2\pi (k_x x + k_y y)} dx \, dy
\]

\[
I(x, y) = \int S(k_x, k_y) e^{i 2\pi (k_x x + k_y y)} dk_x \, dk_y
\]

\[
\text{Ideally → image is real}
\]

\[
in \text{practice → complex}
\]

\[
\text{use amplitude image: } \tilde{I}(x, y) = \sqrt{I(x, y)}
\]

Adding exponents same as multiplying two \(e^{i2\pi k_x}\)'s

\[
\sum_{k_y} \sum_{k_x} S(k_x, k_y) e^{i 2\pi k_x x} e^{i 2\pi k_y y} dk_x \, dk_y
\]

Same as two sequential 1D FFT's (actual code)

\[
\sum_{k_y} \left[ \sum_{k_x} S(k_x, k_y) e^{i 2\pi k_x x} dk_x \right] e^{i 2\pi k_y y} dk_y
\]

In practice, finite number of samples, \(N\) and \(M\), are collected

\[k_x\text{ and } k_y\text{ directions of } K\text{-space (integral }\rightarrow\text{ discrete sum)}\]

\[
I(x, y) = \sum_{m=-M/2}^{M/2-1} \left[ \sum_{n=-N/2}^{N/2-1} S(n, m) e^{i 2\pi n \Delta k_x} \Delta k_x \right] e^{i 2\pi m \Delta k_y} \Delta k_y
\]

**Sampling interval in K-space**
**Sampling**

- must consider effects of sampling
  - limited points in $k$-space
  - limited in range of frequencies sampled ($k_{\min} \rightarrow k_{\max}$)
  - limited in rate of sampling ($\Delta k$)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

**Space**

**Spatial Frequency**

- correct reconstruction
- infinite frequency range
- infinitely fine sampling

- correct plus replicas
- infinite frequency range
- finite spacing of samples

[as above w/ blurring, ringing]

- as above w/ blurring, ringing
- finite frequency range
- finite spacing of samples

- aliasing occurs in spatial domain
- replicas overlap, causing wraparound

- finite frequency range
- too-wide spacing of samples

thus, finer sampling of same range of spatial freqs increase FOV
**UNDER/OVER SAMPLE**

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

\[ S_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x} \]

FOV (distance to repeat) is reciprocal of spatial frequency sampling interval.

Pixel size is FOV divided by K-space sample count.

---

**3 more examples (not incl. less samples to same spat. freq [bottom last page])**

**Basic Image**

**Same num samp. to 2X spat. freq.**

(i.e. gradients stronger or time ON longer)

\[ N=10 \]
\[ \Delta k_x=5 \]
\[ \text{FOV}=1 \]
\[ S_x=0.1 \]

**2X num. samples to same spat. freq.**

(i.e. gradients weaker or time ON shorter)

\[ N=20 \]
\[ \Delta k_x=2 \]
\[ \text{FOV}=2 \]
\[ S_x=0.05 \]

**2X number samples to 2X spat. freq.**

(i.e. gradients stronger or time ON longer)

\[ N=20 \]
\[ \Delta k_x=5 \]
\[ \text{FOV}=2 \]
\[ S_x=0.1 \]

- Basic image
- Square image
- Square pix
- X-pix half width
- Replicas intrude
  - Scanner makes square image
  - "wrap" occurs
- Square pix
  - twice X-pix count so FOV = 2X
  - this is "phase oversamp"
  - Scanner crops to square
  - replicas move out
- X-pix half width
  - twice X-pix count
  - same FOV
  - this is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Sample data freq
2. Result in image space

...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑... ←...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑... →...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑...

X multiply × convolution

...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑... ←...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑... →...↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑↑...

= equals = equals

Useful FT's

Rect

\[ \text{Rect}\left(\frac{x}{W}\right) \xrightarrow{\mathcal{F}} W \text{sinc}(\pi W k) \]

Gaussian

\[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

Comb

\[ \sum_{n=-\infty}^{\infty} \delta(x-n\Delta k) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k-p\Delta k) \]

Limit approach to Fourier transform of comb

\[ \text{Fov} = \frac{1}{\Delta k} \]

\[ \Delta k = \frac{1}{\text{Fov}} \]
**POINT-SPREAD FUNCTION**

\[ \hat{I}(x) = \Delta k \sum_{n \in [N-k, N+k]} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \( S(m \Delta k) = 1 \)
- Substitute into \( \hat{I} \) to get PSF:
  \[ h(x) = \Delta k \sum_{n \in [-N, N]} e^{i 2\pi n \Delta k x} \]
- Simplify:
  \[ h(x) = \Delta k \frac{\sin \left( \frac{\pi N \Delta k x}{\sin \left( \pi \Delta k x \right)} \right)}{\sin \left( \frac{\pi \Delta k x}{N} \right)} \]
  - That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc.

Image

- Noise PSF modifies
  - Ideal (infinite \( k \))
  - Image

\[ \text{FT} \]

\[ \times \text{multiply} \]

\[ \text{Acquisition window (truncated hi spati. b)} \]
**GENERAL LINEAR INVERSE RECON FOR MRI**

\[
S(k_x) = \int_x I(x) e^{-i2\pi k_x x} dx
\]

Signal eq. \(\rightarrow\) fawd problem

\[
I(x) = \int_{k_x} S(k_x) e^{i2\pi k_x x} dk_x
\]

Recon eq. \(\rightarrow\) inv. problem

\[\Rightarrow s = Fi\]

Linear "forward solution" matrix, vectors have complex entries can build in any measurable priors

\[
F_{x,y,2} = g(x,y) e^{-\frac{(nT \pm m\Delta T + TE)t}{T2}} e^{\frac{-i\gamma B(nT \pm m\Delta T)}{\delta k_x}} e^{\frac{-i\gamma B(n\Delta k_x + n\Delta k_y)}{\delta k_y}}
\]

T2 decay, B0 error, T2 + phase

multi-coil

\[
\begin{bmatrix}
\frac{k_y}{k_x} & \frac{x,y}{k_x} \\
1 & F_{coil1} \\ 2 & F_{coil2}
\end{bmatrix}
\]

naturally incorporates undistorted field maps different sensitivities function for each coil contains additional info about some loc. but, need reference scan, lo-res ok (need phase corrections for each coil ?)

\[i = F^+s\] over-determined

\[
F^+ = (F^T F)^{-1} F^T
\]

More Reference inverse \(\rightarrow\) "small"

\[
(x,y)^2 \rightarrow 16 \times \text{ bigger for 4 coils}
\]

\[
F^+ = F^T (FF^T)^{-1}
\]

\[
(FF^T) \rightarrow \text{smaller}
\]

\[
i = [(FF^T)^{-1} F^T] s
\]

Slice-by-slice assume slice select swamps all
FAST SPIN ECHO (FSE)  
RARE, FSE, 3DFSE

- One 90° pulse followed by multiple 180° pulses (e.g., 8), each with a different phase-encode gradient.
- Each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips).

The "echo train"

Gx, Gy, Gz

Signal

Effective TE

The effective TE is the TE when center of k-space is collected (largest effect on contrast, largest echo).

Each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \)  
\( n = 1, 2, ..., M \)

By arranging to collect \( k_y = 0 \) early, PD-weighted instead of T2-weighted.

Different prominence of T2 decay in contrast-controlling \( k_y = 0 \) echo.

Possibility to convert different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train.

3DFSE — like 2D except wind/unwind added to thick slice select (w/cmiskers on 180°).
MULTI-SLAB 3DFSE, PROBLEMS

- Echo train e.g. 20
- etc to fill 3D k-space
- $G_z$ is "partition"
- $G_y$ is "phase encode"
- $G_x$ readout needs no pre-wind since $180^\circ$ does it
- $TE_{eff}$ is $90^\circ$ to echo thru center of k-space

- echoes die out quickly $\alpha e^{-t/\tau}$
- since echoes after $90^\circ$ limited to $<30$, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 \theta^2 A_f$
  $\Rightarrow 180^\circ$ pulses deposit 4-6x power of $90^\circ$

"multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

hard to get under 8 msec inter-echo spacing

limits speed of covering k-space
SINGLE-SLAB 3D FSE


- regular FSE (180° pulse train)

- sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)

- this "storage" in Z-axis preserves magnetization for longer time

- smaller flip angles allow much longer echo trains

- enough to collect whole plane of 3-D k-space +

- different than hyper echoes (not symmetric)

- contrast must consider STE

\[
\text{SE} = \sin \alpha, \sin^2 \alpha \frac{\sigma_3}{2} e^{-2\pi i/12} \\
\text{STE} = \frac{1}{2} \sin \alpha_3 \sin \alpha_2 \sin \alpha_1 e^{-7\pi i/12} e^{-2\pi i/12}
\]

- single-slab 3DFSE pulse seq,

variable flip angle (<1 msec)

hard (non-selective) pulse not 180°

\( RF_m \)

\( 40^\circ \)

\( 4 \) msec

\[ RF_{out} \]

\( G_x \)

\( G_y \)

\( G_z \)

\( 150 \) echo trains

\( \text{echo num} \)

\( \text{echo trains} \)

\( \text{long } TE_{eff} (T2) \)

\( \text{short } TE_{eff} (T2) \)

\( k_x \)

\( k_y \)

\( k_z \)

\( k_y \)

\( k_x \)

\( 26 \)

\( 16 \)

\( 6 \)

\( \text{train} \)

\( \text{don't collect} \)

\( \text{train} R \)

\( \text{don't collect} \)

\( \text{train} 26 \)

\( k_y \)

\( k_z \)

\( k_x \)

NB: time to exit k-space is \( \times 5X \)

apparent contrast time b/c of "storage"

(e.g. \( TE_{eff} = 585 \) ms looks like FSE \( TE = 140 \) ms)
FAST GRADIENT ECHO (GRASS, FLASH, SPIR, MPRAGE)

- small tip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

STEAZY-STATE COHERENT (GRASS, FISP)

- unwind phase from phase-encode Mx before next pulse (there because TR<TE)
- unwind read gradient, too

S = k \sin x \left[ \frac{1}{1 + \cos x + (1 - \cos x) T1/T2} \right] e^{-T1/T2}

T2/T1-weighted contrast (bright CSF)

Tissue T2/1:
brain 0.4
fat 0.3
CSF 0.7

STEAZY-STATE SPOILED (SPIR, FLASH)

- spoil with random gradient (but this still allows some $\alpha$ refocusing)
- spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- good gray-white contrast (T1-weighted)

NON-STEADY STATE, MAGNETIZATION-PREP

- preparation pulse \( \rightarrow \) strong T1-weighting
- contrast varies in spatial-frequency-dependent way

- longitudinal mag. not affect much by low angle pulses

- record $k_y = 0$ here
QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD

- Start with gradient echo signal: 
\[ S_\text{Ernst} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T_2}}{1 - \cos \alpha \cdot e^{-TR/T_2}} \]

Ernst eq. 
(max: \( \cos \alpha_c = e^{-TR/T_1} \))

- Simplify/linearize/estimate:
  \( TR \ll T_1 \)
  Linear approx. of exponentials
  Taylor expansion simplification of \( \sin, \cos \), drop small term.
  \[ S \approx A \cdot \frac{TR/T_1}{\alpha^2/2 + TR/T_1} \]

Helms et al. (200x)

- Solve for \( T_1 \) and \( A \) (proton density) given signals for 2 diff flip angles.

\[ T_1\text{est} = 2TR \cdot \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2/\alpha_2 - S_1/\alpha_1} \]

\[ A\text{est} = S_1S_2(\alpha_2/\alpha_1 - \alpha_1/\alpha_2) \]

\[ (\alpha_2/\alpha_1 - \alpha_1/\alpha_2) \]

- Problem: flip angle varies a lot at 3T (e.g., 25%) from nominal/ requested (e.g., flip series).

- Collect spin-echo and stimulated echo (FPI)

  \[ S_\text{SE} = k \cdot \sin^3 \alpha \cdot e^{-TE/T_2} \]

  \[ S_{STIM} = k/2 \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T_2} \cdot e^{-TM/T_1} \]

  \[ \alpha = \arccos \left( \frac{S_{STIM} \cdot e^{-TM/T_1}}{S_{SE}} \right) \]

  Jiru & Klose (2006)
**Echo Planar Imaging (EPI)** (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

RF pulse sequence:

- RF (radio frequency) pulse
- Slice select
- 

- $G_x$, $G_y$ gradients
- "blip" waveform makes the loud "beep".
- Repeat till 64 echoes

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (zero freq, center of k-space).

- Therefore, the recording point ($\Delta t$) in k-space (= spin phase stripe pattern) stays wherever the $x$ and $y$ gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the $G_y$ "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.

$k$-space traversal:

- Individual time points of recording of one echo.
SPIN ECHO EPI

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of oxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For $TE \approx 100$ ms, spins diffuse 10's of $\mu m$, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extravascular spins — intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extra w/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

- Over half of SE-BOLD at IST is venous...
- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.

- "Asymmetric spin-echo EPI" arranges for the spin echo to occur 2 msec before the gradient echo, which gives more $T_2^*$-weighting (for $k_y=0$ echo).

- The 180° pulse rephasing reduces the $T_2^*$ signal, which is why the partially rephased asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising:
  - Signal to noise is higher so we can take spin echo hit.
  - Contribution from venous blood is reduced, since blood $T_2$ is so short, we can let it decay away before recording.
- Coil fall-off intuitively contains info about location of same brain location imaged by different coils w/ different fall-offs, but what does this look like in k-space?

- Slow variation in RF-field fall-off (e.g. 1-4 cyles/FOV) causes a blur in acquired data in k-space (N.B. not addition!)

- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space - at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq:

  image domain \[ \text{image ("brain")} \]

  \[ \text{\( \times \text{(multiply)} \)} \]

  spatial freq. domain \[ \text{FT} \]

  \[ \text{(equals)} \]

  acquired image \[ \text{FT} \]

  \[ \text{(equals)} \]

- N.B. inverse FT of k-space data "smeared" in spatial freq space is sharp image w/ fall-off (not blurred img.)

- "Smeared" means normally orthogonal spatial freq's leak to adj. freqs.

- GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center across multiple coils

- SENSE - general linear inverse approach

- N.B.: neither would work unless normally orthogonal spatial freqs. are well-blurred!
Pulse-se\textsuperscript{2} SMS/MULTIBAND/blipped CAIPI

\[ V_b \]
\[ RF_{in} \]
\[ V_y \]
\[ G_y \]
\[ G_x \]
\[ RF_{out} \]

- Excite multiple slices at once
- Function of \( G_z \) blips is to shift slices in \( G_y \) direction
- This occurs because for given slice, a phase pedestal is added to the entire slice
  \[ \text{this } \text{"Fourier Shift Theorem"} \]
  \[ \text{[N.B. : different than } B_0 \text{ defect-induced incremented phase errors]} \]
- Problem w/ all up \( G_z \) blips \( \rightarrow \) phase error builds up

\textbf{Trick \#1}
- Start w/ 2 slices, one at \( z = 0 \), other above
  \[ \text{\( \rightarrow \) if } 180^\circ \text{ phase shift used, blip up/down same! (no effect at } z = 0) \]
  \[ \text{\( \rightarrow \) i.e., move top or bottom replica} \]

\textbf{Trick \#2}
- For multiple slices not all at \( z = 0 \), phase no longer same for even/odd
  \[ \text{\( \rightarrow \) but can add phase to equilibrate to k-space before recomb.} \]

\textbf{Trick \#3}
- For more than 2 slices:
  \[ \text{1st even odd even odd etc} \]
MULTI BAND/BLIPPED CHIRP (cont.)

- Relation between leave-one-out aliasing and nominally fully-sampled SMS

- Leave alternate lines out wraps image
- SENSE/GRAPPA to fix block coil view swears K-space data
- Nominally, w/ SMS we record every line of K-space
- But equivalent to leave alternate out b/c our multi-slice FOV was not big enough

- Slice-GRAPPA
  - reg GRAPPA - recon missing lines
  - slice-GRAPPA - recon multiple K-spaces, i.e. not for each overlapped slices by training on fully-sampled data at beginning of scan

- Inter-slice "leakage block"
  - When training GRAPPA kernel on fully-sampled data, also minimize inter-slice leakage (split-slice-GRAPPA)
  - Can also do regular GRAPPA on top of this reason: for diffusion, loss in S/N from undersampled cancelled by shorter TE readout
  - Gain from reduced image distortion from shorter readout
**Echo Volume Imaging (EVI)**

- Multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

---

**Diagram**

- RF in
  - slab selected

- \( G_z \)
  - phase-encode blips

- \( G_y \)

- \( G_x \)

- RF out

---

- **First partition**
  - plane

- **Second partition**
  - plane

- Center 4 \( K_x, K_y \) for this \( K_z \) partition: TE off

- Finished the first partition

---

- Entire k-space must be filled before 3D image is reconstructed

- Since entire volume is excited each shot, potentially higher S/N

- Must use smaller flip angle to avoid killing \( M_L \) since entire volume excited every partition (e.g., every 80 msec)

---

- Main issue is movement artifact since data assembled from many shots over several secs

- Breathing-induced BP problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant Δk_x

- sinusoids in both G_x and G_y give spiral k-space trajectory

RF

G_z

G_y

G_x

Sig

90°

- constant angular velocity goes too fast at large k_x, k_y
- constant linear velocity better but impractical near k_x=0, k_y=0
- compromise: start constant angular, end constant linear
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)

Possible to present sign
- High, uniform contrast, but lots of waiting (T1), high BW

RF

180° (prep.) TI ~ 700 ms

G_x

G_y

G_z

Sig.

3D k-space

("stack of spirals")

Spiral interleaves

k_x interleaves

k_z echoes

Echoes \( \rightarrow \) (after one 90°)
PHASE ERRORS & ECHO-CENTERING ERRORS

- anything that causes a deviation of the $B_2$ field strength from the expected value $(B_{0,z} + G_{x,z}X + G_{y,z}Y + G_{z,z}Z)$ changes precession frequency and therefore, expected phase angle.

- incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

- correct with shimming and $B_0$-mapping/phase un-mapping before reconstruction.

**Fourier shift theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x - x_0) = \int e^{-i2\pi k_x x} S(k_x) e^{i2\pi k_x x} dk_x$$

- if realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.

- since echo is in spatial frequency domain, this is frequency shift.

- spatial frequency shift results in wrapping in phase image after reconstruction.

- magnitude image unchanged.

**Fourier freq. shift theorem**

Freq. shift in freq. domain causes phase shift in spatial freq. space.
FAST SCAN ARTIFACTS  EPI vs. Spiral

**EPI**
- \(G_x\) readout gradient strong \(\rightarrow\) field defects smaller percentage less deformation of \(k_x\) (vertical stripe components)
- \(G_y\) "blips" are weak and total \(G_y\) record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the \(x\)-direction, for example, wraps and unmaps phase as a function of \(x\)-position
- but \(G_x\) big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates

- for a given \(x\)-position, the strength of the spurious gradient is constant, so the accumulation of phase error results in shift in the \(y\)-direction (\(k\)-space spin-stripe displ.)
- the phase error causes a shift in the \(y\)-direction proportional to \(x\)-gradient strength (shear) but no blurring (N.B. Shift varies w/\(x\)-position) \(y_{x} \rightarrow y_{y} \)

**Spiral**
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error (= more shearing)
- for spurious \(x\)-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq. \(y_{x} \rightarrow y_{y} \)

- for defects with more complex contains in the \(y\)-direction (than linear, as above) the vertical shifts (in EPI) will vary with \(y\)-position, and may result in signals from different \(y\)-positions being reconstructed on top of each other
- Localized B0 defects often arise from air pockets embedded in tissue
  - Air in middle/outer ear → indentation in inferior temporal lobe
  - Cupped olfactory epithelium → orbitofrontal d/c x, ant, thal. compression

- Localizing signal
  - Collect one data (k-space) point
  - 4 cycles of phase in y-dir
  - Complex multiply
    - Correlate sin/cos with brain
  - Brain structure sampled with distorted stripes

  One complex number

- Reconstruction
  - From distorted data points

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by B0 defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of ΔB0 needed to correct images
  1) Fieldmap method: <multiple TE's to est ΔB0 from 1/T1 slope
  2) Point-spread-function: <extra phase encode to estimate PSF (should be δ-function)
  - Deconvolve distorted image in phase-encode direction

**N.B.** Image shift only occurs if shift span, fgr sampled w/ successively later echoes times (see next page)
LOCALIZED $\Phi$ DEFECT, EFFECT ON RECON

- when local $\Phi$ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- if each successive k-space recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spatial freq. offset in k-space

- a k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- however, w/ EPI, static $\Phi$ defect causes more and more local displacement of image phase stripes for each additional ky line

- that is, later lines have greater spatial freq. offset
- effectively stretches k-space in ky direction
- same num samples to higher spatial freq.
- shrinks FOV (squished voxels — see FOV page)

- when image is reconstructed, region with local $\Phi$ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) static local $\Delta\Phi$ defect
  2) successive increases in phase error for successive spatial freq. measurements during long EPI readout
  3) small size of ky phase encode blips relative to $\Phi$ defect (much less of this effect in freq. encode direction)

- Respiration (which affect $\Phi$) in 3D FLASH might cause similar effect within k2 partition (if successive spatial freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impress a linear variation onto the $z$-component of the $B$ field, $B_z$, in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (e.g., printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently for 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!)

- This can result in errors approaching 1 cm in function-structure overlays.

- Different coil designs have different directions of distortion (!)

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- These effects do not build up over time in phase-encode direction, since they only occur when gradients are turned on.

- These distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm)

Linear shim coils impose gradients in $x$, $y$, and $z$
Higher order shims impose higher order spherical harmonic field components (e.g., $z^2$)

Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field

Local resonance offsets caused by $B_0$ defects estimated from images
$\Rightarrow$ e.g., sample phase at multiple echo times

Fit defective field using combination of fields generated by shim coils. Then add these corrections to base shim currents

$\Rightarrow$ this only corrects spatially gradual field defects
$\Rightarrow$ local defects due to air in sinuses much higher order than shims

After shimming, field map measured again

Image voxel displacements calculated from resonance offset map are used to un warp the reconstructed magnitude image

For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects)
- 1D navigator
  - B\phi drift problem
    - slow up/down drifts in B\phi continuously occur
    - a pedestal in B\phi is pedestal in phase (not gradient)
    - which causes spatial shift (Fourier shift theorem)
    - in EPI, mainly affects phase-encode dir b/c small flip angle readout
    - result is successive volumes drift in phase encode dir
  - Gradient balance problem
    - unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
    - causing N/2 (Nyquist) ghosting \Rightarrow another phase error

- 3D navigator: collect 3D sphere in k-space
  - rotation of object \Rightarrow rotation of k-space amplitude pattern
  - translation of object \Rightarrow phase shift of k-space phase (Fourier shift)
  - sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

Walsh et al. (2002) MRM

\[
\begin{align*}
\text{RF} & \rightarrow 90^\circ \\
G_2 & \rightarrow \text{crush} \\
G_y & \rightarrow \text{100% points} \\
G_x & \rightarrow \text{4 (msec)} \\
Sg & \rightarrow \text{8} \\
\end{align*}
\]

\[
\begin{align*}
x(n) & = \sin\left(\frac{\pi n}{N} \sin^{-1}z(n)\right) / (1 - z^2(n)) \\
y(n) & = \cos\left(\frac{\pi n}{N} \sin^{-1}z(n)\right) / (1 - z^2(n)) \\
2(n) & = \frac{2n - N - 1}{N} \\
\end{align*}
\]

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR of EPI images (27 ms vs. 2.4 sec)
- may need to smooth rot,trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES

- Receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  \( \Rightarrow \) variations can be used (cf. GRAPPA, SENSE) and/or corrected

- Transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  \( \Rightarrow \) potentially worse (why local transmit is still in progress)
  \( \Rightarrow \) usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields (\( \approx \) higher RF frequencies)
  is less uniform:
  1) decreased RF wavelength (closer to size of head) at higher freq.
  2) increased permittivity (\( \varepsilon \)) and conductivity (\( \sigma \)) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- Different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

  Normalization ("pre-scan normalize")
  - record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)
  - divide small coil body coil at each voxel to determine receive field
  - use receive field to normalize main image(s)

  \[ \text{[see also: } T_1, \text{ MP2RAGE, } T_1/T_2 \]
**DIFFUSION - WEIGHTED IMAGING**

- **Simple diffusion weighting**
  - RF: $A \rightarrow 90^\circ \rightarrow T \rightarrow$ (or $t$)
  - $G_x \rightarrow (\text{or just } S)$
  - $G_y$, $G_z$, readout
- "Apparent diffusion coefficient" map
- To get large $b$, need $G \uparrow \& \uparrow S$ (need big $G$'s)
- Long $S$ gives spurious $T_2$-weighting
- Can use stimulated echoes: $90^\circ$ RF $\rightarrow$ $S_1$, $90^\circ$ RF $\rightarrow$ $S_2$
  - Transverse $\rightarrow$ diffusion lobe 1 $\rightarrow$ Park in length direction $\rightarrow$ back to transverse $\rightarrow$ diffusion lobe 2

1) **Anisotropic Diffusion (Gaussian)**
- Measure $D$ along multiple axes
- Have to measure tensor, not scalar
  - Even for determining one primary direction
- $D = \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix}$
  - Since $\mathbf{D}$ is symmetric, only need 6 measurements
- $\Rightarrow$ i.e., paired diffusion lobes applied along 6 axes

**Diffusion Surface (Non-Gaussian)**
- Need to measure diffusion in many directions (>6) to properly characterize even 2 main directions

2) **Length Scale by multiple $b$-values**
- Fit line to semilog signal as funct of $b$
- If not straight line: multi-exponential, e.g.
  - High ADC/fast extra vs low ADC/slow/intercellular
  - $S = A_1e^{-bD_1} + A_2e^{-bD_2}$

- Fiber tract mapping
  - An ill-posed prob.
  - E.g., constrain both ends and center point (!?)
  - Need vis. areas! test

**Classical diffusion coefficient**
- $D = \frac{\sigma_x}{72DT}$
- Ex: brain $D = 0.001 \text{ mm}^2/\text{sec}$
**PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ**

- **Spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)**
  - Allows longer TE
  - Clips M2 so rephase gradient has same sign as dephase

- **Eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion**

- "Doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/particle time constants)
  - Also, keep phasors orthogonal to diffusion-encoding gradients

---

**Phase dispersion (6 echo)**

\[ y_{TRSE} = 0 = y_1 - y_2 - y_3 + y_4 \]

---

**Twice-refocused Spin-echo**

(for center k-space)

---

**Nagy et al. (2014) MRM**
**PERFUSION - ARTERIAL SPIN LABEL**

- Basic idea: Tag blood below area of interest; collect control & tagged images assuming directional input flow.

  - Continuous ASL (CASL) — Continuously tag a plane
due to gradient; blood gets adiabatically inverted as it passes through location w/coned resonant freq.

  - Pulsed ASL (PASL) — e.g., EPISTAR, FAIR, PICORE, QUIPPS II
    - Tag block of tissue below slice(s)

- Small differences between control and tag (~1%) require accurate balancing of control & tag images.

- Contrast problems:
  - Transit delays: biggest confounding factor
  - Relaxation rate: diffuses
  - Renal clearance (vs. microvascular, which get stuck)

- QUIPPS II — Quantitative Perfusion

  1. Pre-saturate spins in target slices
  2. Tag - 180° pulse below slices
due to control above slices (to control off-resonance)

  3. Saturate tagged block to end tag (TI)
    - Both tag and control
    - Can use train of thin slices pulses at top of tag band

  4. EPI of spiral images of target slices (T1)
    - Image most distal slice first to cancel delays
    - Fast between slice so imaging excitation doesn't get interpreted as flow

\[
\Delta M = \text{flow} \times [2M_0 \text{ TI}, e^{-T1/2}]
\]

- Can extract flow and BOLD

- Alternate tag and control, GRE TE=30 ms

- Dual echo spiral
  - K=0 early => hi S/N flow
  - TE = 50 ms => BOLD

- Adjacent subsections minimize movement artifact.
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g. 1H) by surrounding electron orbitals

- e.g., acetic acid:

  oxygen attracts electron so less shielding of target nucleus

- how we get chemical shift spectrum:

  Larmor oscillations are multiplied (PSD) by center freq to obtain Δf (not MHz high freq)

  data before FT is a series of time-domain samples of the mix of shifted-freq offsets

  FT turns data into short spectrum

  Pulse Sequence

  - since we are already using phase (=freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

  - use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal

  - and FT-it like chemists do!

<table>
<thead>
<tr>
<th></th>
<th>signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMR</td>
<td>time domain oscillation</td>
<td>FT shift freq spectrum</td>
</tr>
<tr>
<td>MRI</td>
<td>spatial shift freq samples</td>
<td>spatial object (like time domain signal)</td>
</tr>
</tbody>
</table>

N.B.: opposite "direction" of FTs!
Phase-encoded Stimulus & Analysis

Periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

Calculate significance
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, low freq (= movement)

Smooth
Vector average of complex significance (A, φ) with that at nearest neighboring surface points

Display
- Plot phase using hue and saturation to indicate significance

Delay correction
- Record responses to opposite directions of stimulus (ccw/lcw, in/out, up/down)
- Vector average after reversing angle of one
  - Penalizes inconsistent more than just avg of angles

Typically 0.5 - 5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

Real

Imaginary

FFT, convert to A, φ

\[ t = \text{total TRs} \]

\[ \text{freq} = \frac{\text{total TRs}}{2} \]

Reversed CCW
Vector average
CCW significance (complex)
CONVOLUTION

\[ f(x) = g(x) * h(x) = \int_{-\infty}^{+\infty} g(z) \cdot h(x - z) \, dz \]

- definition of convolution

Why we reverse:

Impulse response function (HRF)

Impulses (ERP design)

N.B. cross-correlation is the convolution except no reversal, \( h(x+z) \) instead of \( h(x-z) \)

How to calculate convolution for this time point (only 3 terms in integral - all other zero)
### General Linear Model

\[ \hat{\mathbf{y}} = \mathbf{Xh} + \mathbf{Sb} + \mathbf{n} \]

- goal is to solve for the hemodynamic response functions, \( h \)

\[ \begin{bmatrix} \mathbf{y} \\ \mathbf{data} \end{bmatrix} = \begin{bmatrix} \mathbf{X} \\ \mathbf{h} \end{bmatrix} + \begin{bmatrix} \mathbf{S} \\ \mathbf{n} \end{bmatrix} \]

\( n \) = design + HDR + drifts + weights + noise

\( \mathbf{y} \) = raw data

\( \mathbf{X} \) = matrix of predictors

\( \mathbf{h} \) = hemodynamic response functions

\( \mathbf{S} \) = signal matrix

\( \mathbf{n} \) = noise

- multiple conditions

\( \hat{\mathbf{h}} \) = maximum likelihood estimate

1) assume white noise, solve for \( \hat{\mathbf{h}} \)

2) \( \hat{\mathbf{h}} = (\mathbf{X}^T \mathbf{P}_s \mathbf{X})^{-1} \mathbf{X}^T \mathbf{P}_s \mathbf{y} \)

   or

   \( \hat{\mathbf{h}} = (\mathbf{X}_\perp \mathbf{X}_\perp)^{-1} \mathbf{X}_\perp \mathbf{y} \)

   where \( \mathbf{X}_\perp = \mathbf{P}_s \mathbf{X} \)

3) significance (how to construct F-ratio)

\[ F = \frac{N - K - L}{K} \begin{bmatrix} y^T (\mathbf{P}_{xs} - \mathbf{P}_s) y \\ y^T (\mathbf{I} - \mathbf{P}_{xs}) y \end{bmatrix} \]

- see diagram next page for geometric interp
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions ($S$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

$\hat{y} = X\hat{h} + \hat{e}$  
$X\hat{h} = P_x\hat{y}$

Projection matrix, $P_x$, operates on $\hat{y}$ to give projection of data into experiment space, $X$.

- When nuisance functions, $S$, are considered, problem: $S$ may not be orthogonal to $X$.

$S$ may not be orthogonal to $X$ for example: linear trend not orthogonal to std. block design.

Remember: "orthogonal" means dot prod. = 0, corr = 0.

Geometric Picture

(Liu et al., 2001, Neuroimage)

$X_S$: space of data modeled by all reference and nuisance

Oblique projection onto nuisance ($E_y$)

Orthogonal projection onto reference ($P_y$) (data explained by reference)

Orthogonal projection onto reference + nuisance ($P_{xs}$)

Data orthogonal to nuisance

Error ($\epsilon$) not explained by reference and nuisance (F-denoise).

$[(I - P_{ys})y]$
Segmentation & Surface Recon

1) MNI auto-Talairach → generates 4x4 matrix

- make average brain target (blurry)
- blur target (furthest), blur single brain (a bit), gradient descent on xcorr
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut-off, only 2 points near center of brain!
  → better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")

- histogram of pixel values in 10 mm thick T1R slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  → interpolated scaling factor 1 to T1R
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D
  find points in 5x5x5 within 10% of WM, get new scale for them
  build Voronoi to interpolate scales set above
  soap-bubble-smooth Voronoi boundaries (3 iterations)
  re-scale each voxel

3) Skull Stripping (output: "brain")

- "shrink-wrap" algorithm
- start with ellipsoidal template
- minimize brain penetration and curvature
  curvature: spring force (from center-to-neighbor vector sum)
- brain penetration
  apply force along surface normal that prevents surface from entering gray matter
SEGMENTATION & SURFACE RECON

- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrink-wrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ r_{\text{center}}(t+1) = r_{\text{center}}(t) + F_{\text{smooth}}(t) + F_{\text{MRI}}(t) \]

for one vertex

**Rule for each vertex, \( \hat{r}_{\text{center}} \)**

\[ F_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (I - n_{\text{center}}n_{\text{center}}^T) \cdot (r_{\text{neigh}} - r_{\text{center}}) \]

identically 3×3 stronger than normal (0.5)

through distribution to: neighbor vertex minus projection of neighbor onto normal = tangential!

\[ + \lambda_{\text{normal}} \left[ \sum_{\text{neigh}} (n_{\text{center}}n_{\text{center}}^T) \cdot (r_{\text{neigh}} - r_{\text{center}}) - \frac{1}{\text{#vertices}} \sum_{v} \sum_{\text{neigh}} (n_{v}n_{v}^T) \cdot (r_{\text{neigh}} - r_{v}) \right] \]

projection of a neighbor vertex vector onto normal in the direction of the normal (\(n\)) is squared (as above) so we get a vector out (not a scalar)

\[ F_{\text{MRI}} = \lambda_{\text{MRI}} n_{\text{center}} \frac{30}{d} \max \left[ 0, \tanh \left[ I (r_{\text{center}} - d n_{\text{center}}) - I_{\text{thresh}} \right] \right] \]

for all terms \( F_{\text{MRI}} = 0 \) if any are zero

Intensity MRI data

Snapshot of surface and "core sample" from one vertex

Outsides (dark)

Skin (light)

Skull (dark-light; dark)

GM

WM

Surface (moving outward)

etc
SEGMENTATION & SURFACE RECON

4) Non-isotropic filtering (output: "win")
   - "floss" and "speckle"
   - preliminary hard threshold:
     - find ambiguous/boundary voxels
       \( \leq \) 20\% or more of 26 immediate neighbors different
   - find plane of least variance
     for each direction (from icosahedral superfine tessellation)
     consider 5x5x5 volume around 1 voxel
     find plane of least variance in this hemisphere
     median filter w/ hysteresis
     \( \leq \) if 60\% of within-slab differ, reverse classification
     \( \Rightarrow \) "flosses" sulci without blurring

5) Find cutting planes
   midbrain [callousum, to separate hemispheres (SG)]
   midbrain, to avoid fill into cerebellum (HT)
   Talairach to start:
   fill WM in SG or HT till min area

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out — for each hemisphere
   - up/down cycles within each plane
   - plane-by-plane
     "wormhole filter" (3x3x3 = center + 26)
     \( \Rightarrow \) fill (unfilled) voxel if 60\% neighbors differ
     \( \Rightarrow \) eliminates structures within 1-D structure
7) Surface Tessellation (output: rh.orig, lh.orig)

- variable num neighbors possible!
- quads to triangles

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  -> so cross-products oriented properly

- long list of values associated with each numbered vertex
  e.g. [position (orig, morphed),
       area (orig, morphed),
       curvature (intrinsic, Gaussian),
       "sulciness" (summed 1 movement during unfolding),
       cortical thickness,
       fMRI data, EEG/MEG dipole strength]

- separate fMRI data set must be aligned, sampled
  - fMRI voxels larger
    Sample at each surface vertex
    nearest-neighbor "soap bubble" smoothing
to interpolate data into hi-res mesh

- some quantities only well-defined on surface
  gradient of magnitude of cortical map measure (e.g., eccentricity)

3D data set $\rightarrow$ $I(x,y,z)$

$\downarrow$

vertex list (coords)

$\quad x_1, y_1, z_1$
$\quad x_2, y_2, z_2$

$\vdots$

face list (vertex nums)

$\quad a_1, b_1, c_1, d_1$
$\quad a_2, b_2, c_2, d_2$

$\vdots$

$\vdots$

$\vdots$

fMRI $\rightarrow$ surface
initial sample
same value here

interpolate on surface
SEGMENTATION & SURFACE RECON

- smoothing/inflation/WM, pial done as derivative of energy functional

\[
J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}}
\]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- scalar curvature error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[
J_{\text{normal}} = \frac{1}{2 \ #\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right]^2
\]

\[
J_{\text{tangential}} = \frac{1}{2 \ #\text{vert}} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ t_x^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right]^2 + \left[ t_y^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right]^2
\]

\[
J_{\text{image}} = \frac{1}{2 \ #\text{vert}} \sum_{\text{centers}} \left[ I_{\text{center}}^{\text{targ}} - I(\mathbf{r}_{\text{center}}) \right]^2
\]

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-intersect test

\[
- \frac{\partial J}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} [I_{\text{center}}^{\text{targ}} - I(\mathbf{r}_{\text{center}})] \nabla I(\mathbf{r}_{\text{center}})
\]

- go the opposite direction of largest scalar error for one vertex

\[
+ \sum_{\text{neighbors}} \lambda_{\text{normal}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] \mathbf{n}_{\text{center}}
\]

- unit normal vector scaled by dot product

\[
+ \sum_{\text{neighbors}} \left[ t_x^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] t_x^{\text{center}} + \left[ t_y^{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neigh}}) \right] t_y^{\text{center}}
\]

- x-component of tangential
- y-component of tangential

N.B.: eq. 9 in Dale, Fischl & Sereno different — and incorrect!
- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"

- Add term to energy function: "sulcus-ness" error: \((S_{\text{out}} - S_{\text{targ}})^2\)

- Bootstrap morph to one brain
  - Make any target
  - Remorph to any target

- Each sub's native surf has diff # vertices

- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

- Average surface made from folded/inflated avg coords
  - Folded: loses area from sulcal crinkles (fr average "inflated")
  - Inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")

- Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

\(\Rightarrow\) N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)