MAGNET HARDWARE

- $B_0$ field from superconducting magnet
- RF transmit/receive
- gradient coils

$B_0 \rightarrow z$ (longitudinal)

$B_1 \rightarrow x, y$ (transverse)

(1) $B_0$ field

B1 field

superconducting coils in liquid helium
(no power required after current injected to bring up field using induction)

$1T = 10,000$ Gauss

$\frac{B_0}{2\pi} = 42.6$ MHz/T

(2) body gradient coils

(3) RF transmit body coil

RF receive-only head coils

RF transmitter (10 kW)

RF receiver

circularly polarized $B_1$ field rotating to $B_0$ at Larmor freq (B1 is several orders of magnitude smaller than $B_0$)

(4) RF receive-only head coils

- usual water cooled

- three one million watt amplifiers to add ramps to $B_0$ field
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (possibly odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field.

**Classical Picture**
- Current loop from spinning charge (right-hand rule).
- N.B.: Classically this would cause EM radiation, spin-down.

**Stern-Gerlach Experiment**
- Pass nuclei through strong magnetic field → split into just 2 beams.

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**Microscopic Picture**

<table>
<thead>
<tr>
<th>No Strong Magnetic Field</th>
<th>Strong Magnetic Field, $B_0$ = $\uparrow$</th>
<th>Strong $B_0$ Plus Oscillating $B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_0 = 0$</td>
<td>$B_0 = \uparrow$</td>
<td>Precessing vectors are bunched around circle</td>
</tr>
<tr>
<td>All vectors same length, random directions</td>
<td>Slight excess of &quot;up&quot; (3 ppm)</td>
<td>Precessing vectors are bunched around circle</td>
</tr>
<tr>
<td>$M^0 \neq 0$</td>
<td>$M^0 = M_z$</td>
<td>Bulk magnetization precesses</td>
</tr>
</tbody>
</table>

**Macroscopic Picture**

- Precession: distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

$2\pi B_0 = \omega = \gamma B_0$ (Larmor frequency).

- Gyromagnetic ratio (e.g., $1.5T$).

$M_z = |\vec{M}| = \gamma h^2 B_0 N_s$.

- Two non-constants:
  - $\gamma$ = gyromagnetic ratio
  - $h$ = Planck's constant

$M_0 = i \frac{\hbar}{3}$.

- Bulk equilibrium magnetization (parallel to $B_0$).

---

N.B.: Compared to top & gravity:
- Frictionless spin
- Signed gravity
- Can change precession dir.
- Can stick under floor.
- Neighbor bumping causes decay ($T_2$).

---

Left-hand rule:
- Thumb = $B_0$
- Fingers = precession
- Like top: precession faster w/ more gravity.
**Bloch Equation**

- Time-dependent behavior of $\vec{M}$ in the presence of an applied magnetic field (excitation & relaxation).

$$ \frac{d\vec{M}}{dt} = \vec{M} \times \vec{Y}_B - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_z^0) \hat{k}}{T_1} $$

- Change in magnet vector
- Precession: $B = B_0$
- Excitation: $B = B_0 + \delta B$

In the Larmor-rotating coordinate system, a tilt with a phase shift in a standard $B_1$ excitation is rotation around $x$-axis.

General case rotating (distinguish $B_1$ & $M$!)

- Longitudinal and transverse relaxations

$$ \frac{dM_z(t)}{dt} = - \frac{M_z(t) - M_z^0}{T_1} $$

$$ \frac{dM_x \cdot \hat{i}}{dt} = - \frac{M_x \hat{i}(t)}{T_2} $$

- Solution to equations above: time-dependent free precession e.g.

<table>
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<th>Lab frame: same!</th>
</tr>
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<tr>
<td>$M_z'(t) = M_z^0 (1 - e^{-t/T_1}) + M_z'(O) e^{-t/T_1}$</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>Rotating frame: same!</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_z'(t) = M_z^0 + [M_z'(O) - M_z^0] e^{-t/T_1}$</td>
</tr>
</tbody>
</table>

$M_x' \hat{i}(t) = M_x' \hat{i}(0) e^{-t/T_2}$

Initial condition

$M_L, M_M$ at time $t = 0$

- Leftover after pulse-decaying
- Re-grown from $M_L$

$M_z'(1) = 63\% M_z^0$

$M_y'(1) = 37\% M_y'(O)$

$M_x'(0) = 100\% M_x'(O)$
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Generalizes to any D
  \[ \|\vec{c}\| = \sqrt{(a_x + b_x)^2 + (a_y + b_y)^2} \]

- Multiple ways to multiply vectors: here are 3

**Dot Product**

(= Inner product)

(= "scaled projection onto")

\[ c = \vec{a} \cdot \vec{b} = [b_x \ b_y \ b_z] [a_x \ a_y \ a_z] = a_x b_x + a_y b_y + a_z b_z \]

- Generalizes to any D

\[ p = \|\vec{a}\| \cos \Theta \]

\[ c = p \|\vec{b}\| \quad \|\vec{b}\| = 1 \]

**Cross Product**

(= Outer product)

(can be generalized: see "geometric algebra")

\[ \vec{c} = \vec{a} \times \vec{b} = [0 \ b_z \ -b_y \]

\[ \begin{bmatrix}
  a_x \\
  a_y \\
  a_z
\end{bmatrix} = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]

(=vector)

Right hand rule: curl fingers from \( \vec{a} \) to \( \vec{b} \): thumb is \( \vec{c} \)

- Unique orthogonal

- Specific to 3D

\[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \sin \Theta \]

\( \Rightarrow \) max if orthogonal

**Complex Multiply**

(see also quaternions, geometric algebra generalization)

\[ \vec{c} = \vec{a} \cdot \vec{b} = [b_x - b_y] [a_x] = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]

(=vector)

- Angles add

- Magnitudes multiply

\[ \|\vec{c}\| = \|\vec{a}\| \|\vec{b}\| \]

\( \Rightarrow \) like real num
**Simple Matrix Operations**

**Basic Idea**
- A matrix \([\text{rotates/scales}]\) a vector
  \[
  \hat{b} = M \hat{a}
  \]

**3D Example**
- \[
  \begin{bmatrix}
  b_x \\
  b_y \\
  b_z
  \end{bmatrix}
  =
  \begin{bmatrix}
  M_{11} & M_{12} & M_{13} \\
  M_{21} & M_{22} & M_{23} \\
  M_{31} & M_{32} & M_{33}
  \end{bmatrix}
  \begin{bmatrix}
  a_x \\
  a_y \\
  a_z
  \end{bmatrix}
  \]

**Add Translate (after rotate/scale)**
- Commonly used "hack" for aligning vols
- A 4D matrix \([\text{rotates/scales}]\) a 3D vector then translates (4th D = 1)

**N.B.:** Have to keep track of order!!
- Rotate/scale then \(\neq\) Trans, then rot/scale
- Change rot component: untranslate, rot, retranslate

**3 Special Cases (3D):** Rotate around each major axis without changing length
  (Scale = 1.0)

- **Rotate around X-axis:**
  \[
  R_x(\alpha) =
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos\alpha & -\sin\alpha \\
  0 & \sin\alpha & \cos\alpha
  \end{bmatrix}
  \]
  e.g., 90° flip

- **Rotate around Y-axis:**
  \[
  R_y(\alpha) =
  \begin{bmatrix}
  \cos\alpha & 0 & -\sin\alpha \\
  0 & 1 & 0 \\
  \sin\alpha & 0 & \cos\alpha
  \end{bmatrix}
  \]
  e.g., 180° flip
to avoid add 180° phase after 90° flip on \(x\)

- **Rotate around Z-axis:**
  \[
  R_z(\alpha) =
  \begin{bmatrix}
  \cos\alpha & \sin\alpha & 0 \\
  -\sin\alpha & \cos\alpha & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  \]
  e.g., precession with \(B\phi\) along \(z\)

**General Case**
- **Rotate around general \(Z\)-axis:**
  \[
  R_z'(\alpha) = R_z(-\theta) R_y(-\phi) R_z(\alpha) R_y(\phi) R_z(\theta)
  \]
  (Quaternions are more efficient)
SOLUTIONS TO SIMPLE DIFFERENTIAL EQ.

**diff. eq.:** \[ dM_{x'y'}(t) = -\frac{M_{x'y'}(t)}{T_2} \]

**Solution:** \[ M_{x'y'}(t) = M_{x'y'}(0) \cdot e^{-t/T_2} \]

**Goal:**
1. Find \( e \) whose derivative satisfies diff. eq.
2. Also find soln (one of many) that passes thru init condition.

Since our diff. eq. is:
- Derivative of funct. = const. same funct.

\( \Rightarrow \) try exponential, since derivative \( (e^x)' = e^x \)

---

diff. eq. \( \frac{M(t)}{M'(t)} = \frac{1}{T_2} \cdot \frac{M(t)}{M'(t)} \)

One soln \( M(t) = e^{-t/T_2} \)

Take deriv. to check \( M'(t) = -\frac{1}{T_2} \cdot e^{-t/T_2} \)

OK!

Another soln \( M(t) = \text{const} \cdot e^{-t/T_2} \)

Take deriv. to check \( M'(t) = -\frac{1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} \)

OK!

Initial condition:
- Information added to soln (not from diff eq.):
  \[ M'(t) = M_{x'y'}(0) \cdot e^{-t/T_2} \]
- Magnetization immedi. after RF (B1) ends:
  \[ M_{x'y'}(0) \]

**Family of solutions**

---

\[ M(t) \]

\[ M'(t) \]

\[ M_{x'y'}(0) \]

\[ t \]
**Bloch Eq. - Matrix Version**

Differential Eq.:

\[ \frac{d\mathbf{\hat{M}}}{dt} = \mathbf{\hat{M}} \times \mathbf{\hat{B}}_0 \]

Solution:

\[ \mathbf{\hat{M}}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = \mathbf{R}_z(\omega t) \mathbf{\hat{M}}(0) \]

Include Relaxation

\[ \frac{d\mathbf{\hat{M}}}{dt} = \mathbf{\hat{M}} \times \mathbf{\hat{B}}_0 - \frac{M_x \mathbf{\hat{i}} + M_y \mathbf{\hat{j}}}{T_2} - \frac{(M_z - M_z^0) \mathbf{\hat{k}}}{T_1} \]

Differential Eq.:

\[ \frac{d\mathbf{\hat{M}}}{dt} = \begin{bmatrix} -\frac{1}{T_2} & \frac{\omega}{T_2} & 0 \\ -\frac{\omega}{T_2} & -\frac{1}{T_2} & 0 \\ 0 & 0 & -\frac{1}{T_2} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z^0 / T_1 \end{bmatrix} \]

Solution:

\[ \mathbf{\hat{M}}(t) = \begin{bmatrix} e^{-\frac{1}{T_2} t} & 0 & 0 \\ 0 & e^{\frac{1}{T_2} t} & 0 \\ 0 & 0 & e^{-\frac{1}{T_1} t} \end{bmatrix} \begin{bmatrix} \cos \omega t & \sin \omega t & 0 \\ -\sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1-e^{-\frac{1}{T_1} t}) \end{bmatrix} \]
**RF Field Polarization**

- Polarization (change of direction)

- Linearly polarized field
  \[ \vec{B}_1 = B_1 \cdot \cos \omega t \hat{x} \]
  Magnitude: \( \{ -1, 1 \} \cdot 1 \)

- N.B.: \( \vec{B}_1 \) adds to much larger \( \vec{B}_0 \)

- Circularly polarized field (quadrature)
  \[ \vec{B}_{1\text{ circ}} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  \[ = B_1 \cdot e^{-i\omega t} \]

  - In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF spin.

- **B1 generated/recorded by RF coil**
- **Magnetic field (vs. electric field)**

**Typical 90° flip**
(around x-axis)

- 180° flip (around opposite y-axis)
- 180° flip 20x weaker at 90°
**SIGNAL EQUATION**

\[ \Phi(t) = \int_{\text{obj}} \bar{B}(\vec{r}) \cdot \bar{M}(\vec{r}, t) \, d\vec{r} \]

magnetic flux thru coil (Scalar)

\[ V(t) = -\frac{\partial \Phi(t)}{\partial t} = -\frac{\partial}{\partial t} \int_{\text{obj}} \bar{B}(\vec{r}) \cdot \bar{M}(\vec{r}, t) \, d\vec{r} \]

Faraday law & Induction

Evaluate using same precision eqns. (Solution to Bloch) ignoring relaxation

- time deriv. inside
- rewrite \( \bar{M} \) as \( \cos \omega t \bar{M}_0 + \sin \omega t \bar{M}_x \)
- \( \tau \) in \( \bar{M}_x(h) \) is \( \text{free} \) of relaxation
- mult by \( i \rightarrow \text{add} \theta \)

- ignore the change in the \( z \)-component of \( \bar{M} \)

Since it changes so slowly compared to the free precession of \( x \)- and \( y \)-components: \( \omega_0 \gg \tau_1(\vec{r}) \)

- this is why we can only record transverse magnetization, \( M_{xy} \), but not longitudinal magnetization (\( M_z \) changes too slowly, so \( V(t) = \frac{\partial \Phi}{\partial t} \approx 0 \))

Laboratory frame Bloch Solution:

\( M_L \rightarrow \) same

\( M_r = M_{xy}(0) e^{i \omega_0 t} \)

\( M_{xy} \) is now scalar, direction is here

Spatially-dependent resonant freq in rotating frame — i.e., after subtraction of \( \omega_0 \)

\[ S(t) = \int_{\text{obj}} M_{xy}(\vec{r}, 0) e^{-i \omega_0 t} \, d\vec{r} \]

\[ \omega_0 = \text{freq} \rightarrow \text{radians/sec} \]

\[ w = \omega_0 + \Delta \omega \] (gradient)

\[ w_0 = \text{angle} \] (within, angle)

\[ w_0 + \phi = \text{angle} \]

\[ * \]

\( \omega t \) = \text{radians} \times \text{sec} = \text{radians} (\phi = /w_0 \text{deg})

Getting difference converts lab \( \rightarrow \) rotating frame

- i.e., at a single time point, RF signal is vector sum across object of local transverse magnetization vectors

Standard Signal Expression

\[ S(t) = \int_{\text{obj}} M_{xy}(\vec{r}, 0) e^{-i \omega_0 t} \, d\vec{r} \]

Phase angle in rotating frame

\[ wt = \text{radians/sec} \times \text{radians} (\phi = /w_0 \text{deg}) \]

- i.e., projection of \( \bar{M} \)-vector at each point onto coil magnetic field direction at each point, Summed across object
**Phase-Sensitive Detection**

- Method for moving very high-frequency Larmor oscillations down to tractable frequency range.

\[
V(t) \sim 63 \text{ MHz} \quad \xrightarrow{\text{multiply}} \quad \text{Low-Pass Filter} \quad \xrightarrow{} \quad S(t) \sim 50 \text{ kHz}
\]

Reference signal (63 kHz)

Demodulated signal \( \propto \) RF coil signal \( \propto \) reference (transmitter)

\[
\alpha \sin(w_0 + \Delta w)t \cdot \sin w_0 t
\]

\[
\Downarrow \quad \text{trig identity}
\]

\[
\sin a \sin b = \frac{1}{2} \left[ \cos(a-b) - \cos(a+b) \right]
\]

\[
\sin a \cos b = \frac{1}{2} \left[ \sin(a+b) + \sin(a-b) \right]
\]

\[
\alpha \frac{1}{2} \left[ \cos S wt - \cos (2w_0 + \Delta w)t \right]
\]

This signal is digitized

- One freq - freq domain
- Chirp - time domain

Signal
Reference
Demodulated
After filter

- Two signals are made from a single receiving RF coil
- A quadrature coil can be treated the same way (OK to combine after adding T/2 phase, then PSD)
- Quadrature coil has better S/N since noise in each part is uncorrelated (T/2 better)

- \( S(t) \) complex = \( e^{i \Delta w t} \) written as
FID - free induction decay, $T_2^*$

- FID (free induction decay) from an RF pulse w/ angle $\alpha$

$$S(t) = \sin \alpha \int_{w=-\infty}^{w=\infty} \rho(w) e^{-t/T_2^*(w)} e^{-i\omega t} dw$$

- For a single freq:

$$S(t) = M_z^0 \sin \alpha e^{-t/T_2} e^{-i\omega t}$$

- Max FID amplitude at $t=0$: $S(0) = M_z^0 \sin \alpha$

- If field inhomogeneous

$$S(t) = \frac{1}{\gamma} M_z^0 \sum \Delta B_0 \sin \alpha e^{-t/T_2^*} e^{-i\omega t}$$

- Where $1/T_2^* = 1/T_2 + \frac{1}{T_2'}$

- Decay rate: Transverse $\gamma B_0$ field

- Spin inhomogeneity $\Delta B_0$

- Unrecoverable "intrinsic"

- Recoverable

- NB, actually complex

- Typical time course 10's of msec

- vs.

- Each precess cycle, which is 10's of ns
ECHOES - spin echo

just after 90° x' pulse
f10 + fhi have same phase

relaxation +
phase dispersion of f10 + fhi
(both from B > B0)

- remember
RF just tips vector(s) while retaining length
relaxation includes tips and shrinks (and grows for echo)

- 180° x' pulse works, too, but echo will be +π phase (left side in figs above)
- echoes generated even if second pulse not 180° (see next)

- FID decay (and
echo growth/decay)
described by T2*,
from inhomogeneity

- reduction in
height of echo
compared to initial
described by T2,
echo fixes the start
ECHOES — Spin echo

\[ \alpha_1 - \tau - \alpha_2 \] (both pulses along y' for simplicity)

**Effect of \( \alpha \)-pulse**

\[
\begin{align*}
M_x' &\to M_x' \cos \alpha - M_z' \sin \alpha \\
M_y' &\to M_y' \\
M_z' &\to M_x' \sin \alpha - M_z' \cos \alpha
\end{align*}
\]

\[
\begin{align*}
M_x'' &\to (M_x' \cos \omega \tau + M_y' \sin \omega \tau) e^{-\tau/2} \\
M_y'' &\to (-M_x' \sin \omega \tau + M_y' \cos \omega \tau) e^{-\tau/2} \\
M_z'' &\to M_z' (1 - e^{-\tau/2}) + M_z' e^{-\tau/2}
\end{align*}
\]

**Effect of \( \tau \)-delay**

**Immediately after \( \alpha_1 \)-pulse**

\[
\begin{align*}
M_x'(w,0+) &= -M_z'(w) \sin \alpha_1 \\
M_y'(w,0+) &= 0 \\
M_z'(w,0+) &= M_z'(w) \cos \alpha_1
\end{align*}
\]

**For one isochromat of freq. \( \omega \)**

**After \( \tau \)-delay**

\[
\begin{align*}
M_x'(w,\tau) &= -M_z'(w) \sin \alpha, \cos \omega \tau e^{-\tau/2} \\
M_y'(w,\tau) &= M_z'(w) \sin \alpha, \sin \omega \tau e^{-\tau/2} \\
M_z'(w,\tau) &= M_z'(w) [1 - (1 - \cos \alpha) e^{-\tau/2}]
\end{align*}
\]

**Immediately after \( \alpha_2 \)-pulse** (no effect on \( M_y' \); rewrite \( x' \) and \( y \) eq's)

\[
\begin{align*}
M_x'(w,\tau) &= M_z'(w) \sin \alpha, \left( \sin^2 \frac{\alpha_2}{2} e^{i\omega \tau} - \cos^2 \frac{\alpha_2}{2} e^{i\omega \tau} \right) e^{-\tau/2} \\
&\quad - M_z'(w) \left[ 1 - (1 - \cos \alpha) e^{-\tau/2} \right] \sin \alpha_2
\end{align*}
\]

**Time dependent free precession around \( z' \)** (rewrite \( M_x', \text{eq}(w, \tau) \))

\[
M_x'(w,\tau) = M_x'(w,\tau+) e^{-(\tau - \tau)/T_2} e^{-i\omega(t - \tau)}
\]

**For a large num of freq's:**

\[
\begin{align*}
\text{terms } 1, 2 &\text{ are dephasing} \quad \text{\Rightarrow FID of echo} \\
\text{term } 3 &\text{ is nesphasing} \quad \text{\Rightarrow nesphase at } t = 2\tau
\end{align*}
\]

**Echo signal**

\[
\begin{align*}
S(t) &= \sin \alpha_1, \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T_2} e^{-i\omega(t - \tau)} dw \\
A_E &= \sin \alpha_1, \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-te^{-T_{E}/T_2}} dw
\end{align*}
\]

\[
\begin{align*}
S_1(t) &= \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/T_2} e^{-i\omega(t - \tau)} dw \\
S_2(t) &= \text{no / factor} \\
S_3(t) &= \text{multiply by } i \quad \text{\Rightarrow add } 90^\circ \text{ phase}
\end{align*}
\]

\[
\text{etc for } A_E \ldots \text{ like } A_E
\]
**Echo TRAINS** - spin-echo trains

- It's (too) easy to make echoes...

\[ E_n = \frac{3^{(n-1)} - 1}{2} \]

Echoes after end of n-th pulse
3 RF\(_i\) \rightarrow 4 echoes (here)
6 RF\(_i\) \rightarrow 121 echoes (!)

Secondary echo: \( SE_{i,2} \) acts like RF pulse \( \alpha_2 \) makes an echo from it

- Two more conventional two pulse spin echoes

Stimulated echo: combined effect of 3
\[ \alpha_1: M_L \rightarrow M_T \]
\[ \alpha_2: \text{ leftover } M_T \text{ flipped to } M_L \text{ (saved)} \]
\[ \alpha_3: \text{ flip saved } M_L \rightarrow M_T \text{ which can then begin to cancel delays} \]
\( 180^\circ \text{ FID}_2 \text{ and FID}_3 \); acts like 2-pulse echo

- A useful multi-echo sequence (CPMG) is a 90° followed by 180° at 2τ spacing

- Typically, 90° and 180° applied in different axes \( (x', \text{ then } y', \text{ etc.}) \)

which reduces phase errors due to imperfect 180° pulses (since slightly off rotation around \( y' \) affects phase less)
EXTENDED PHASE GRAPHS

- Using full Bloch eq. solutions is tedious 😊
- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ, 180^\circ$)
- Problem #1: $\alpha$ pulse rotates a position of transverse magnetization into a position that results in rephasing
- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

$\phi$ = phase displacement

\[ \begin{align*}
\text{Phase dispersion} & : \phi_i \\
\text{RF} & : \text{transverse (unflipped)} \\
& \rightarrow \text{longitudinal (no further phasechange)} \\
& \rightarrow \text{flipped transverse} \\
& \rightarrow \text{flipped longitudinal}
\end{align*} \]

Rule for effect of $\alpha$ RF pulse on transverse mag

\[ \begin{align*}
\phi & \rightarrow \text{RF transverse (RF pulse: $M_L \rightarrow M_T$)} \\
& \rightarrow \text{longitudinal}
\end{align*} \]

Longitudinal cannot be flipped to cause rephase

\[ \begin{align*}
\text{Rule for effect of $\alpha$ RF pulse on longitudinal mag} & \\
\rightarrow \text{echo when phase path crosses zero}
\end{align*} \]
**HYPER ECHOCES**

1. $\alpha_2 - 180^\circ - \alpha_2 = 180^\circ$
   - 3 solid lines
   - 1 dashed line

2. $\alpha_y - 180^\circ - \alpha_y = 180^\circ$

3. $\alpha - 180^\circ - \alpha = 180^\circ$

(N.B. mirror image coord syst vs. Bloch eq. (2001))

- ignore amplitude
- surface up the sphere then defines a 2D space that you can move around in using:
  - flip angle from $z'(\alpha)$
  - RF phase in $x'y'$

- "flip/phase" RF spin space

**Practical use**

- multi-echo example

- practical prob: 180° pulses deposit a lot of RF (6x 90°)
  - prob at high fields

- by arranging to get big echo in middle of k-space
  - can get by with much less RF power
**Gradient Echoes** - $T_2^*$, GE chains

- Initial negative gradient dephases spins.

- After $t = T$ of positive gradient, spins rephase.

- Does not correct for $T_2^*$ inhomogeneities.

So the echo amplitude is

$$A_E = e^{-t/T_2^*}$$

- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay

$$
\frac{1}{T_2} < \frac{1}{T_2^*} < \frac{1}{T_2^{**}} \Rightarrow A_E = e^{-t/T_2^{**}}
$$

- Key difference between spin-echo (SE) and gradient echo (GE) is that $B_0$ inhomogeneities not canceled.

  $\Rightarrow$ Hence, echoes are $T_2^*$-weighted, not $T_2$-weighted $\Rightarrow$ more susceptible to inhomogeneities.

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get.

- EPI hardware $\Rightarrow$ 64 echoes
**IMAGE CONTRAST**  

**T1 Saturation-recovery (no echo, just FID)**

- Contrast (PD, T1, T2, T2*) depends on magnetization not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time.

- Simple saturation/recovery w/ no echo.

- Initial conditions:
  - $M_z$ before first pulse = $M_z^0$
  - $M_z = 0$ immediately after first pulse (i.e., 90° pulse)

- From Bloch eq, $M_z$ just before second pulse:
  $$M_z^{(2)}(O_+ = M_z^0 (1 - e^{-TR/T1}) + M_z^{(1)}(O_+) e^{-TR/T1}$$
  - $M_z$ before current pulse
  - $M_z$ regrowth-from-zero term
  - $M_z$ left-immed.-after-pulse term (i.e., decaying)

- Given:
  1. 90° pulse
  2. No $M_{xy}$ left

- Tip existing mag:
  $$M_z^{(n)}(O_-) = M_{xy}(O_+) = M_z^0 (1 - e^{-TR/T1})$$

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

- Assume this is 0 because we assume that $M_{xy}$ (transverse) completely decayed so that a 90° pulse doesn't generate any initial longitudinal.

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

$$I(r) = C P(r) (1 - e^{-TR/T1(r)})$$

- Spectral density $P(r)$ relates to density which underlies equilibrium $M_z^0$. 

- Reconst. spectral density.
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
- because they are brighter than all the rest
- because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
(e.g. at 3T, flip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec
  TR images reached
  after 8 images
IMAGE CONTRAST

IR (still just saturation-recovery - no echo)

- inversion recovery w/ no echo

RF

- 180° pulse reverses longitudinal magnetization
  \[ M_{z}^{(0)} = -M_{z} \]

- recovery to end of first TI from long part of Bloch eq.
  \[ M_{z}^{'} = M_{z}^{0} \left( 1 - 2e^{-\frac{t}{T_{1}/T_{1}}} \right) \]
  \( t = \text{from } 180^\circ \text{, since } M_{z} \text{ in second Bloch term} \)

- longitudinal then regrows from zero from first Bloch term only
  \[ M_{z}^{'} = M_{z}^{0} \left( 1 - e^{-\frac{(T_{R} - T_{I})}{T_{1}}} \right) \]

- after second 180°, just change sign again
  \[ M_{z}^{'} = -M_{z}^{0} \left( 1 - e^{-\frac{(T_{R} - T_{I})}{T_{1}}} \right) \]

- apply relaxation eq. again
  \[ M_{z}^{'} = M_{z}^{0} \left( 1 - e^{-\frac{T_{I}}{T_{1}}} \right) - M_{z}^{0} \left( 1 - e^{-\frac{(T_{R} - T_{I})}{T_{1}}} \right) e^{-\frac{T_{I}}{T_{1}}} \]

\[ M_{z}^{'} = M_{z}^{0} \left( 1 - 2e^{-\frac{T_{I}}{T_{1}}} + e^{-\frac{T_{R}}{T_{1}}} \right) \]

\[ \Rightarrow \text{ this is magnetization flipped to transverse, therefore made recordable} \]
**IMAGE CONTRAST**

**SE, IR-SE**

- **RF in** → **TE** → **TR**
- **G_x**, **G_y**, **G_z**
- **Slice Select** → **Phase Encode**
- **90° signal** → **180° causes echo**
- **Echo**

**Steady state mag (2nd TR) just before 90°**

\[ M_z'(O^-) = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{-TR}{T2}} \right) \]

- The echo signal (\( M_z' \)) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation.

\[ A_E = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T1}} + e^{-\frac{-TR}{T2}} \right) e^{-\frac{-TE}{T2}} \]

- If we assume TE much less than TR, then we can simplify:

\[ A_E = M_z^0 \left(1 - e^{-\frac{-TR}{T1}} \right) e^{-\frac{-TE}{T2}} \]

**SE-IR**

\[ A_E = M_z^0 \left(1 - 2e^{-\frac{-TI}{T1}} + e^{-\frac{-TR}{T2}} \right) e^{-\frac{-TE}{T2}} \]
GRE w/ small tip angle

- use basic longitudinal relaxation from Bluhm et. al again

\[
M_x(0^-) = M_x^0 (1 - e^{-TR/T1}) + M_x^{(n-1)}(0^+) e^{-TE/T1}
\]

- assume we have a small tip angle

\[
M_x^{(n)}(0^-) = M_x^0 (1 - e^{-TR/T1}) + M_x^{(n-1)}(0^-) \cos \alpha e^{-TR/T1}
\]

- assume we are in dynamic equilibrium

\[
M_z^{(n)}(O^-) = M_z^{(n-1)}(O^-) = M_z^{ss}(O^-)
\]

**Pre-pulse steady state**

\[
M_z^{ss}(O^-) = \frac{M_z^0 (1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}}
\]

**Post-pulse transverse magnetization**

\[
M_{xy}^{ss}(t) = \frac{M_z^0 (1 - e^{-TR/T1}) \cdot \sin \alpha e^{-2TE/T2}}{1 - \cos \alpha e^{-TR/T1}}
\]

**Gradient echo amplitude**

\[
A_E = \frac{M_z^0 (1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2}}{1 - \cos \alpha e^{-TR/T1}}
\]

TI contrast mainly depends on flip angle, not TR \( \cos \theta \approx 1 \) eliminates TI weighting since denominator is numerator.
- Saturate, wait for contrast \( z \), invert, wait for contrast \( x \), FLASH (continued)

A) \( M_2' (\text{just after } 90^\circ) = 0 \) (perfect 90°)

B) \( M_2' (\text{after } TD) = M_2^\circ (1 - e^{-TD/T1}) \) (Blach term 41)

C) \( M_2' (\text{just after } \text{invert}) = \cos \phi \ M_2^\circ (1 - e^{-TD/T1}) \)

D) \( M_2' (\text{after } TI) = M_2^\circ (1 - e^{-TI/T1}) + \left[ \cos \phi \ M_2^\circ (1 - e^{-TD/T1}) \right] e^{-TI/T1} \)

\[ = M_2^\circ \left[ 1 - \left[ 1 - \cos \phi (1 - e^{-TD/T1}) \right] e^{-TI/T1} \right] \]

Special case \( TI = TD \): \( M_2' = M_2^\circ \left[ 1 - e^{-TI/T1} \right]^2 \)

- after the final X pulse:

E) \( M_2' (\text{just after } \text{final}) = M_2^\circ \left[ 1 - [1 - \cos \phi (1 - e^{-TD/T1})] e^{-TI/T1} \right] \sin x \)

\( \Rightarrow \) using hard 130° ‘inversion’ can cancel hard alpha B1 inhomogeneities (Thomas et al., '05)
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}} \)
- Temporal SNR: \( \epsilon \text{SNR} = \frac{1}{\text{TR}} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:

\[
\text{CNR}_{AB} = \frac{S_A - S_B}{\sigma_n} = \frac{\text{SNR}_A - \text{SNR}_B}{\sigma_n}
\]

- Spin-echo:

\[
A_E = M_0 (1 - e^{-TR/T1}) e^{-TE/T2}
\]

- Gradient echo:

\[
A_E = \frac{M_0 (1 - e^{-TR/T1}) \sin \alpha}{1 - \cos \alpha e^{-TR/T1}} e^{-TE/T2}
\]

**General rules: Spin-echo, long TR GE**

<table>
<thead>
<tr>
<th>Proton-density weighted</th>
<th>TR (\text{long} ) (no T1 diffs)</th>
<th>TE (\text{long} ) (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR ( \approx T1 ) (big T1 diffs)</td>
<td>TE (\text{long} ) (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR ( \parallel ) (no T1 diffs)</td>
<td>TE ( \parallel T2 ) (big T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \frac{\Delta x \Delta y \Delta z \sqrt{N_{ac} N_x N_y N_z}}{\Delta t}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[
  \text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}
  \]

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(T_n\) effect on S/N
  \[
  \text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{\sqrt{128 \times 128}}{\sqrt{64 \times 64}} = 2 \text{ times better S/N}
  \]

- \# acquisitions, \(T_n\) better S/N
  \[
  \text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{\sqrt{2}}{\sqrt{1}} = 1.41 \text{ times better S/N}
  \]

- Larger timestep during readout, \(T\sqrt{\Delta t}\) better S/N
  \[
  \Delta t = \frac{1}{\text{BW}_{\text{read}}} \text{, digitization timestep during echo acquisition}
  \]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog low-pass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freq-detecting) \(\Delta t\)
- must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
COMPLEX ALGEBRA

Real/Imaginary

Add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)

Mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

Angle/Phase

Add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)

Mult: \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2 \cos (\phi_1 + \phi_2), A_1 A_2 \sin (\phi_1 + \phi_2))\) \(\rightarrow\) N.B.: 3rd kind of vector mult. different than dot product and cross product

Real to Complex Power

\[ e^{i\phi} = \begin{cases} \text{expand as series} \\ \text{recognize } \cos, \sin \text{ series} \end{cases} \]

\[ = \cos \phi + i \sin \phi \]

\[ = \cos \phi, \sin \phi \]

\[ = \text{vector on unit circle} \]

\[ e^{i\phi n} = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \]

Fourier Transform

\[ H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi if t} dt \]

\[ H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i \omega t} dt \]

Convolution Theorem

\[ F[g(x) \ast h(x)] = \mathcal{F}[g(x)] \cdot \mathcal{F}[h(x)] \]

\(\rightarrow\) because of FFT, faster if kernel not small

Convolution

\[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x-z) \, dz \]

Cross-convolution

\[ f(x) = g(x) \otimes h(x) = \int_{-\infty}^{\infty} g(z) \cdot h(x+z) \, dz \]

- For arbitrary amplitude, multiply
  \[ A e^{i\phi} \]

- Phase is integral of freq. variable
  \[ \phi = \int \omega \, dt \]

Shorthand for a unit vector (2D)
painting in the direction of \(\phi\)
- How to calculate $H(f)$ for one $f$ $(f=3)$:
Fourier transform (1b)

- $e^{i\phi} = \cos \phi + i \sin \phi$
- $e^{-i\phi} = e^{i(-\phi)}$
- $= \cos (-\phi) + i \sin (-\phi)$
- $= \cos \phi - i \sin \phi$

- $\cos$ is an "even" function, $\sin$ is an "odd" function

An orthogonal decomposition

- think of discretely sampled $\sin(bx)$, $\cos(bx)$ as vectors
- $\text{Corr}(\vec{v}_1, \vec{v}_2) \equiv \text{projection of } \vec{v}_1 \text{ onto } \vec{v}_2 \equiv \vec{v}_1 \cdot \vec{v}_2$

\[
\begin{align*}
\text{Corr}(\cos bx, \sin bx) &= 0 \\
\text{Corr}(\sin bx, \sin bx) &= 0 \\
\text{Corr}(\cos bx, \sin bx) &= 0
\end{align*}
\]

- in the continuous case, orthogonal functions defined as: $\int_{x=hi}^{x=lo} f(x) g(x) \, dx = 0$
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- Start with spike in image domain
- Take example of spike at $x = 0$
- \[ \cos(x), \cos(2x), \cos(kx) \text{ all equal 1.}\] There are all freqs correlate w/ spike at $x = 0$
- If spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates
- To see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the $e^{-2\pi i k x}$ cos and sin at location of spike

\[ \text{successively higher freqs} \]
\[ \text{real component of FT} \]
\[ \text{cos (even), sin (odd)} \]
\[ \text{positive spikes same dist from origin: } \Rightarrow \text{pick cos's} \]
\[ \text{positive & negative spikes, same dist: } \Rightarrow \text{pick sin's} \]

- This is one way of thinking about what one point in K-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse FT)
FOURIER TRANSFORM OF AN IMAGE (2)

(1) real image → imaginary image s → (zero) → spatial freq

(2) amplitude image s → phase image s → spatial freq

(3) complex vectors → zero vectors → spatial freq

- 3 equivalent representations of image & spatial freq. space
FOURIER TRANSFORM OF AN IMAGE

- what a single k-space point looks like in image space (polar coordinates $A, \phi$ instead of $r, i$)

- Cartesian dimension of k-space — x- and y- spatial freq

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin and cos — don't confuse $k_x, k_y$ w/ sin, cos!

N.B.: increasing one 1D component increases the spatial freq of the 2D wave and rotates it
Fourier Transform of Image (4)

3 equivalent representations of complex numbers in image space and spatial-freq. space (k-space)

- example: cosinusoid in image space, then shifted in x-dir

**Real Image**

\[ I(x, y) = \cos(x) \]

**FT of Real Image**

\[ FT \text{ of } I(x, y) \]

- Large values only here:
  - \( k_x = 0, k_y = 0 \)
  - \( k_x = 1, k_y = 0 \)
  - \( k_x = -1, k_y = 0 \)

**Example:**

1. \( I(x, y) = \cos(x) \)
   - Max pos
   - Max neg

2. \( I(x, y) = \cos(x - \pi/4) \)
   - Halfway between \( \cos \) and \( \sin \)
   - Shifted 45° to right

**Note:**

- N.B.: an example of the "Fourier Shift Theorem" (see below)
- 45° rotation compared to complex above

Real component less than above because rot:
FOURIER TRANSFORM OF IMAGE (5)

- (cont.) center of k-space (real image)
- complex image

**REAL IMAGE**

\[ I(x, y) = 1 + \cos(x) \]

\[ \begin{array}{c}
0 \\
\end{array} \]

\[ \begin{array}{c}
\text{zero}
\end{array} \]

\[ \begin{array}{c}
\text{max}
\end{array} \]

\[ \begin{array}{c}
\text{complex}
\end{array} \]

\[ \begin{array}{c}
\text{A}
\end{array} \]

\[ \begin{array}{c}
\phi
\end{array} \]

\[ \begin{array}{c}
\text{zero}
\end{array} \]

**FT OF REAL IMAGE**

- center of k-space:
- \[ H(k) = \int h(x) e^{-i\pi k x} \]
- \[ \text{avg image brightness} \leq 1 \text{ (real)} \]
- \[ \text{positive center k-space} \]

\[ \begin{array}{c}
r
\end{array} \]

\[ \begin{array}{c}
c
\end{array} \]

\[ \begin{array}{c}
\text{A}
\end{array} \]

\[ \begin{array}{c}
\phi
\end{array} \]

\[ \begin{array}{c}
\text{zero}
\end{array} \]

\[ \text{complex} \]

\[ \text{FT, FT}^{-1} \]

**COMPLEX IMAGE**

\[ I(x, y) = \cos(x) - i \sin(x) \]

\[ e^{-i x} \]

\[ \begin{array}{c}
\text{A}
\end{array} \]

\[ \begin{array}{c}
\phi
\end{array} \]

\[ \text{flat} \]

\[ \text{complex} \]

**FT OF COMPLEX IMAGE**

- "missing" spike results in single spike correlating with \( \cos \) and \( \sin \)
- non-Hermitian:
- k-space will only have Hermitian symmetry if image is real:
- complex conjugate (\( \text{complex num w/ sign flipped in imag. part} \))
- is equal to function value \( w \text{ avg arg} \):
- \[ H(-x) = H^*(x) \]

\[ \begin{array}{c}
r
\end{array} \]

\[ \begin{array}{c}
c
\end{array} \]

\[ \begin{array}{c}
\text{A}
\end{array} \]

\[ \begin{array}{c}
\phi
\end{array} \]

\[ \begin{array}{c}
\text{zero}
\end{array} \]

\[ \text{complex} \]

[Note: this is like what a gradient does! ]
**Gradient Coils**

- Gradient coils for \( x, y, z \) generate approximately a linear gradient in the strength of the \( z \)-component \( B_z \) of the magnetic field \( B_0 \).

- For example, the \( x \) gradient coil induces a ramp in \( z \)-component of the magnetic field when moving in the \( x \)-direction:

\[
B_{g,z} = G_x x
\]

* Since a pure linear gradient of \( B_{g,z} \) in only the \( x, y, z \) directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the \( x \)- and \( y \)-directions \( (B_{g,x} \text{ and } B_{g,y}) \) and the other magnetic field components are usually ignored because they are so small relative to \( B_{g,z} \), since \( B_{g,z} \) is added to \( B_0 \), and since \( B_0 \) is much stronger than \( B_{g,x}, B_{g,y}, \) and \( B_{g,z} \).

- Since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- The Maxwellian terms \( B_{g,x}, B_{g,y} \), are known; can be included in the recon. process.

\[ \Delta \phi G_x(x) \approx -\frac{x^2 G_0^2 t}{2B_0} \]
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim

$$f = \left( B_0 + B_{G_z} \right)$$

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

in practice, Gaussian pulse envelope good too

$$= \frac{\sin(\theta)}{\theta}$$

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

valid for small tip: $90^\circ \rightarrow 52^\circ$

- in practice: adjust to max, use crusher to kill spurious transverse on $180^\circ$
**PULSES FOR SLICE SELECTION** Fourier approach details

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ B_1(t) \propto \int_{-\infty}^{\infty} \tilde{p}(f) e^{-i2\pi ft} df \]

- Time dependent RF stimulation (complex)
- Frequency selection function

**Solve with:** \( \tilde{p}(f) = \text{frequency band} \)

\[ B_1(t) = A \cdot \tilde{f}_w \cdot \text{sinc} \left( \pi \frac{f_w t}{T} \right) e^{-i2\pi f_c t} \]

- Amplitude controlling flip angle
- Sinc envelope determined by freq. width \( f_w \)
- Modulation (complex) at center freq. \( f_c \)
- N.B. wider \( f_w \) is narrower sinc

---

**Fourier Transform Pair, Rules**

- Convolution in one domain is multiplication in the other
- Convolution with delta function impulse moves function to impulse center

**Fourier Transform Solution to:** \( \frac{1}{\sqrt{t}} \)

**Frequency** \( \leftrightarrow \) **Time**

\[ \text{convolution} \]

\[ = \text{equals} \]

\[ \times \text{multiply} \]
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) chemical shift change ( \rightarrow ) gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF ( \rightarrow ) same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs ( \rightarrow ) same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum ( \rightarrow ) FT of FID to get ( \Delta x ) offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading

  - e.g., phase-encoding (turning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican turn"

  - idea is that data is not a set of samples of a time domain signal generated by multiple chemical-shift like frequencies

  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations, (which are analogous to multiple time points)

  - i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>MRI</th>
</tr>
</thead>
<tbody>
<tr>
<td>samples of oscillations in time-domain ( \rightarrow ) frequency-domain spectrum of shifts</td>
<td></td>
</tr>
<tr>
<td>samples of spatial freq. in freq. domain ( \rightarrow ) spatial object (like a time-domain signal)</td>
<td></td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because \( \text{FT} \approx \text{FT}^{-1} \) (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction

\[ \text{precession} \uparrow B_m \uparrow \text{in } x\text{-direction} \]

\[ \rightarrow \text{correct (remember that strength of } G_x \text{ causes variation of slope of } B_m \text{ in } x\text{-direction)} \]

- Different frequency signals are mixed together and recorded as a 1-D signal over time

\[ \rightarrow \text{correct, but remember, we are recording summed local magnetization vectors after de-modulation} \]

- A Fourier transform, which converts back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

\[ \rightarrow \text{correct} \]

- Spatial frequencies get confused/confused with precession frequencies

\[ \rightarrow \text{wrong}!! \]

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

\[ \rightarrow \text{conceptually wrong}!! \]

\[ \rightarrow \text{FT actually converts spatial frequencies to spatial position} \]

\[ \rightarrow \text{the spatial frequency increases for each time point in the readout} \]

\[ \rightarrow \text{the precession freq ramp is constant each timestep} \]
FREQUENCY ENCODING (2)

- "Frequency" encode gradient ($G_x$) turned on during
  during echo causes precession rates to immediately vary with $x$-position

- at beginning of gradient on, the phase of signal coming from each $x$-position is the same
  summed phase angle is what we measure

- early after gradient on, phase advances (because of faster precession frequency) arise with greatest
  phase advance at largest $x$-position

- in practice, the lowest spatial frequency ($\delta = 0$) occurs in the middle of the gradient on time
  because the phase is "wound" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

- later during gradient on, phase advances cause multiple wraparounds of phase angle across space

- early after gradient on, phase advances (because of faster precession frequency) cause greatest
  phase advance at largest $x$-position

- in practice, the lowest spatial frequency ($\delta = 0$) occurs in the middle of the gradient on time
  because the phase is "wound" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs

\[ G_x \rightarrow t \]

\[ B_x \text{ in } x\text{-direction} \]

\[ G_x \text{ levels (} = \text{slope)} \]

\[ \text{actually} \]

\[ \uparrow B_x, x \]

\[ \text{at beginning of gradient on, the phase of signal coming from each } x\text{-position is the same} \]

\[ \text{summed phase angle is what we measure} \]

\[ \text{early after gradient on, phase advances (because of faster precession frequency) arise with greatest} \]

\[ \text{phase advance at largest } x\text{-position} \]

\[ \text{in practice, the lowest spatial frequency (} \delta = 0 \text{) occurs in the middle of the gradient on time} \]

\[ \text{because the phase is "wound" negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs} \]

\[ \begin{align*}
\text{Single time point} \\
\text{Early} \\
\downarrow \\
\uparrow \\
\text{Single time point} \\
\text{Later} \\
\downarrow \\
\end{align*} \]
**Frequency Encoding (3)**

Why each data point is 1 spatial freq

### Standard Fourier Transform

\[
H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i 2\pi \omega t} dt
\]

- **k** is often used instead of "f" for the frequency variable

### Imaging Equation

\[
S(\nu) = \int_{-\infty}^{\infty} I(x) e^{-i 2\pi \nu x} dx
\]

- Sum across x of object
- Oscillations come from readout phase wrapping, where \(f\) is single spatial freq (e.g., 5) and \(x\) goes across object

To make image, do inverse Fourier transform of recorded signal \(S(\nu)\)

get this single readout point by summing signal across x-position (RF coil records sum)

- Signal strength at one x-position
  - Brightness of image point
  - Spin density (spectral density)

- even though variable is \(\nu\), it represents one time period during readout

- don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each \(x\) position)

- RF is single spatial freq (e.g., 5)
- \(x\) goes across object

END: \(G_x(t-TE)\)

SE: \(G_x(t-TE)\)

\(f = G_x\), that is, spatial freq depend on amount of time gradient was on (this \(f\) increases with time!)

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ALTERNATE DERIVATION (incl. effects of $G_{x}(t)$) SIGNAL EQ

- Oscillators at $w = \gamma B$ at each position (just $x$ for now)

$$S(t) = M(x) e^{-i \phi(x)} dx$$

- By definition, freq., $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) \quad \text{and} \quad \phi(x,t) = \int_{0}^{t} w(x,t) dt = \int_{0}^{t} B(x,t) dt$$

- Assuming phase initially $0$, $B$ affected by gradients

$$B(x,t) = B_0 + G_x(t) x$$

$$\phi(x,t) = \gamma \int_{0}^{t} B_0 dt + \left[ \int_{0}^{t} G_x(t) dt \right] x$$

$$= \omega_0 t + 2\pi k_x(t) x \quad \text{k is time integral of gradient waveform}$$

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i \omega_0 t}$ from the first equation

$$S(t) = \int_{x} m(x) e^{-i 2\pi k_x(t) x} dx$$

Amplitude of each oscillator

Gradient-controlled phase
**Phase-encode Gradient ($G_y$)**

- Turn on gradient after excitation but before readout.

- Different levels of $G_y$
  
  $B_{z,y}$
  
  $y$
  
  $y$
  
  $y$

- Higher levels of $G_y$ (slope of $B_z$ in y-direction!)
  
  Higher spatial freq. (more phase wraps) in y-direction.

- Phase wraps persist after phase-encode gradient off.

- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase.

### 2D Imaging Equation

$$S(k_x, k_y) = \iiint \int \mathbf{I}(x, y) \cdot e^{-i2\pi (k_x x + k_y y)} \, dx \, dy$$

- Signal recorded at single time point (one readout point).
  
  Complex signal (from phase-sensitive detection).

- Image (strength of magnetization at each x-y point).
  
  Done by RF coil.

- Phase (vector of unit length and particular angle which is function of $G_x$ and $G_y$).
  
  Phase angle (of scalar magnetization) in rotating frame, set by gradients.

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia" — they stay wherever the gradients last left them.
3-D IMAGING — Two phase-encode gradients

- Use z-gradient for 2nd phase-encoding instead of slice selection
- Excitation of whole slab (slice-select is whole brain)
- Simple spin-echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [SPGR])

\[ S(k_x, k_y, k_z) = \text{signal recorded at single point of readout} \]

\[ I(x, y, z) = e^{-i2\pi (k_x x + k_y y + k_z z)} \, dx\, dy\, dz \]

- i.e., freq-encode phase, 1st phase-encode phase, and 2nd phase-encode phase all just add (= 3D rotation of phase stripes)

- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
  → phase stripes created throughout volume vs. slice:
  
  N.B., this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

- Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

N.B.: Stripes have sharp edges from phase wrap (not sinusoid since q from 2-comp quadrature!)

- Successive read out steps:
  - More rotation
  - Higher spatial freq

3D phase encode w/ G_y and G_z starts rotated in y-z plane

- 3D phase encode w/ G_y and G_z starts rotated in y-z plane
Gradients move k-space location of data point

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point:

\[ k = \frac{G}{t} \int_0^t G(t) \, dt \]

simple form of integral w/ boxcar gradient

(k is area under curve)

- all of the following gradients end up at the same point in k-space:

Frequency-encode FID

Frequency-encode gradient echo

Frequency-encode spin-echo (plus gradient echo!!)

Phase-encode then frequency encode gradient echo

Frequency-encode echo

Phase-encode echo

Frequency-encode gradient echo

Phase-encode gradient echo

Frequency-encode spin-echo
**IMAGE RECONSTRUCTION**

\[ S(k_x, k_y) = \int \int I(x, y) e^{i 2\pi (k_x x + k_y y)} dx dy \]

\[ I(x, y) = \int \int S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} dk_x dk_y \]

In practice, finite number of samples, \( N \) and \( M \), are collected

- \( k_x \) and \( k_y \) directions in \( k \)-space (integral \( \rightarrow \) discrete sum)

\[
I(x, y) = \sum_{|x| < \frac{1}{\Delta k_x}} \left[ \sum_{|n| < \frac{N}{2}} S(n, m) e^{i 2\pi \frac{n \Delta k_x}{N} x \Delta k_x} e^{i 2\pi \frac{m \Delta k_y}{N} y \Delta k_y} \right] \]
**SAMPLING**

- must consider effects of sampling
  - limited points in k-space
  - limited in range of frequencies sampled ($k_{\text{min}}$ to $k_{\text{max}}$)
  - limited in rate of sampling ($\Delta k$)

- N.B. aliasing less familiar when result of limited frequency domain sampling than limited space or time domain sampling

- correct reconstruction
  - infinite frequency range
  - infinitely fine sampling

- correct plus replicas
  - infinite frequency range
  - finite spacing of samples

- as above w/ blurring, ringing
  - finite frequency range
  - finite spacing of samples

- aliasing occurs in spatial domain
  - replicas overlap, causing wraparound

- thus, finer sampling of same range of spatial freqs increases FOV
UNDER/OVER SAMPLE

Basic Image

Spatial freq.

- Basic image
  - Square pix
  - X-pix half width
  - Y-pix half height
  - Square makes replica size
  - This is "pixel overlap"
  - Repliass make square
  - This is decrease to square
  - X-pix half width
  - Repliass make square

Some examples (not ind. less samples to same spatial freq. using last paper)

- X2 number samples
  - To X2 spatial freq
- Some num. samples
  - To X2 spatial freq

FOV = \frac{1}{DK_x}

S_x = \frac{FOV_x}{N_x}

Pixel size is FOV divided by x-scope sample count

More examples

3 more examples
Fourier Transform Solution to Replicas

1. Sample data in spatial freq space
2. Result in image space

\[ \text{Sample data freq} \quad \xrightarrow{\mathcal{F}} \quad \text{Result in image space} \]

\[ \text{x multiply} \quad \xrightarrow{*} \quad \text{Convolution} \]

\[ \text{...} \quad \xrightarrow{\mathcal{F}} \quad \text{...} \quad \xrightarrow{=} \quad \text{...} \]

\[ \text{...} \quad \xrightarrow{\mathcal{F}} \quad \text{...} \quad \xrightarrow{=} \quad \text{...} \]

Useful FTs

- Rect
  \[ \text{Rect}(x) \xrightarrow{\mathcal{F}} W \text{sinc}(\pi Wk) \]

- Gaussian
  \[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

- Comb
  \[ \sum_{n=-\infty}^{\infty} \delta(x - n\Delta k) \xrightarrow{\mathcal{F}} \delta(k) \sum_{p=-\infty}^{\infty} \delta(k - p\Delta k) \]

Limit approach to Fourier transform of comb

\[ \text{FOV} = \frac{1}{\Delta k} \]

\[ \Delta k = \frac{1}{\text{FOV}} \]
**POINT-SPREAD FUNCTION**

\[ \hat{I}(x) = \Delta k \sum_{n \in \{N/2, N/3\}} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to \( S \)-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]

- Substitute into \( \hat{S} \) to get PSF:
  \[ h(x) = \Delta k \sum_{n \in \{N/2, N/3\}} e^{i 2\pi n \Delta k x} \]

- Simplify:
  \[ h(x) = \Delta k \frac{\sin \left( \pi N \Delta k x \right)}{\sin \left( \pi \Delta k x \right)} \to \text{periodic} \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in \( k \)-space is an image sinc. All outside centers at sinc zero-crossings if \( k_{max} = X_{max} \)

---

Now PSF modifies ideal (infinite \( k \)) image

* convolve = ringing

\[ \text{FT} \]

\[ \times \text{ multiply} \]

acquisition window (truncated hi spatiol)
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int_x I(x) e^{-i 2\pi k_x x} dx \]  
Signal eq. \to forward problem

\[ I(x) = \int_k S(k_x) e^{i 2\pi k_x x} dk_x \]  
Recon eq. \to inv. problem

\[ s = Fi \]

\[ s = \begin{bmatrix} x \to & y \to \end{bmatrix} \begin{bmatrix} F & i \end{bmatrix} i \]

Linear forward solution.

Matrix/vector have complex entries. Can build in any measurable priors.

\[ F_{x,y} = g(x,y) e^{-(nT \pm m\Delta T + T_E)/T_2} e^{-i \gamma B(y)(nT \pm m\Delta T)} e^{i (n\Delta k_x x + m\Delta k_y y)} \]

coil gain at location

T2 decay

Error

T2 + phase

Multi-coil

\[ s = \begin{bmatrix} x \to & y \to \end{bmatrix} \begin{bmatrix} F_{coil 1} & i \end{bmatrix} \begin{bmatrix} F_{coil 2} \end{bmatrix} \]

Naturally incorporates undistorted field map.

Different sensitivity function for each coil.

Contains additional info about source location.

But, need reference scan, lo-res OK.

(need phase corrections for each coil?)

\[ i = F^+ s \quad \text{over-determined} \]

More precise inverse

\[ F^+ = (F^T F)^{-1} F^T \]

(x,y)^2 \to "small"

\[ (x,y)^2 \to 16 \times \text{bigger} \quad \text{for 4 coils} \]

\[ i = [(F^T F)^{-1} F^T] s \]

Slice-by-slice assume slice select swamps RBG.
FAST SPIN ECHO (FSE)  RARE, FSE, 3DFSE

- one 90° pulse followed by multiple 180° pulses (e.g., 8) each with a different phase-encode gradient.
- Each phase "winder" is "unwound" because leftover phase would be re-focused away by 180° (vs. EPI where it persists between blips).

2DFSE

RF

G2

Gy

Gx

Signal

- The "effective TE" is the TE when center of k-space is collected (largest effect on contrast, largest echo).
- Each subsequent echo has more T2 decay: \( E_n = e^{-nTE / T2} \) for \( n = 1, 2, \ldots, M \).
- By arranging to collect \( G_y = 0 \) early, PD-weighted instead of T2-weighted.

possible to correct different T2-weighting of echoes by estimating T2 curve from \( G_y = 0 \) echo train.

- 3DFSE — like 2D except wind/unwind added to thick slice select (w/emphasis on 180°).
MULTI-SLAB 3DFSE, PROBLEMS

- echo train e.g. 20
- etc to fill 3D k-space
- $C_{z_2}$ is "partition"
- $G_y$ is "phase encode"
- $G_x$ readout needs no pre-wood since $180^\circ$ does it
- $T_{Eeff}$ is $90^\circ$ to echo thru center of k-space
- echoes die out quickly $\propto e^{-t/\tau}$
- since echoes after $90^\circ$ limited to $<30$, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 \theta^2 \Delta f$
  $\Rightarrow$ $180^\circ$ pulses deposit 4-6x power of $90^\circ$

- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries - esp. movement
- multislab requires slice selective RF pulses $\Rightarrow$ longer than non-selective 'hard' pulses

- 4 ms RO
- hard to get under 8 msec inter-echo spacing
- limits speed of covering k-space
SINGLE-SLAB 3D FSE

- Regular FSE (180° pulse train)

- Sub-180° pulses cause each successive pulse to also generate a stimulated echo (STE)

- This "storage" in Z-axis preserves magnetization for longer time

- Smaller flip angles allow much longer echo trains

- Enough to collect whole plane of 3-D k-space

- Different than hyper echoes (not symmetric)

- Contrast must consider STE

\[
SE = \sin\alpha, \sin^2 \alpha e^{-\frac{2\pi}{\tau_2}}
\]

\[
STE = \frac{1}{2} \sin^2 \alpha \sin^2 \alpha \sin^2 \alpha e^{-\frac{2\pi}{\tau_2}}
\]

Note: to collect k-space is \( \approx 5X \)

Apparent contrast time b/c of "storage" (e.g., TE_eff = 585 ms, looks like FSE TE = 140 ms)
**Fast Gradient Echo (FISP, SPGR, MPRAGE)**

- Small flip so TR can be greatly reduced (e.g. 10 msec, less than T2)
- 'leftover' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

**General RF Sequence Diagram**

- Select (balanced)
- Phase
- Read

**Steady-State Coherent (GRASS, FISP)**

- Unwind phase from phase-encode M\(_T\) before next pulse (there because TR<TE)
- Unwind read gradient, too

\[ S = k \sin \alpha \left[ \frac{1}{1 + \cos \alpha + (1-\cos \alpha) T_1/T_2} \right] e^{-T_2/T_1} \]

- T2/T1-weighted contrast (bright CSF)
- Tissue T2/T1
- Brain 0.41
- Fat 0.3
- CSF 0.7

**Steady-State Spoiled (SPGR, FLASH)**

- Spoil with random gradient (but this still allows some \(\alpha\) refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast (T1-weighted)

**Non-Steady State, Magnetization-Prep (MPRAGE)**

- Preparation-pulse → Strong T1-weighting
- Contrast varies in spatial–freq-dependent way

- Longitudinal mag. not affect much by low angle pulses
- Take k-space filled for this k\(_z\) (i.e., zero
  G\(_y\) phase encode)

- Move in k\(_y\) space
- Record k\(_y\) = 0 here
QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD

- Start w/ gradient echo signal e.g., dropping T2 decay: \( e^{-TE/T2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T2}}{1 - \cos \alpha \cdot e^{-TR/T2}}
\]

Ernst eq.

(max: \( \cos \alpha_e = e^{-TR/T2} \))

\( \quad \uparrow \quad \text{"Ernst angle"} \)

- Simplify/linearize/estimate

\( TR \ll T1 \)

linear approx. of exponentials

Taylor expansion simplification of \( \sin, \cos \), drop small term

Helms et al. (2008)

\[
S \approx A \cdot \frac{\alpha \cdot TR/T1}{\alpha^2/2 + TR/T1}
\]

(maxis \( \alpha^2/2 \approx TR/T1 \))

- Solve for TD and

A (proton density) given

signals from 2 diff flip angles

\[
T1_{\text{est}} = 2 \cdot TR \cdot \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2 \alpha_2 - S_1 \alpha_1}
\]

\[
A_{\text{est}} = S_1 S_2 \left( \alpha_2/\alpha_1 - \alpha_1/\alpha_2 \right) \left( \alpha_2/\alpha_1 - \alpha_1/\alpha_2 \right)
\]

- Problem: flip angle varies a lot at 3T (e.g., 25%) from nominal/requested (e.g., flip series)

- Collect spin-echo and stimulated echo (SE, EPI)

\[
S = k \cdot \sin^3 \alpha \cdot e^{-TE/T2}
\]

\[
S_{\text{SE}} = k \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2}
\]

\[
S_{\text{STIM}} = k_2 \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TE/T2 \cdot TM/T1}
\]

\[
\alpha = \arccos \left( \frac{S_{\text{STIM}} \cdot e^{-TM/T1}}{S_{\text{SE}}} \right)
\]

Jiru & Klose (2006)
ECHO PLANAR IMAGING, EPI (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes.

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq., center of k-space).

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it.

- That explains why successive y phase-encode steps are achieved without changing the size of the Gay "blips".

- Echoes are T2*-weighted (gradient echo).

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes.
**SPIN ECHO EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxyhemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing.

- The excess of oxyhemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect.

- Spin echo corrects (cancels) static $T_2^*$ ($T_2'$) dephasing, incl. deoxy

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing.

- Diffusion exposes spins to different fields (reducing gradient echo dephasing).

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels.

- For $TE \approx 100$ ms, spins diffuse $10^5$ of $\mu m$, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time.

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here).

- This argument only works for extravascular spins. Intravascular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells measure intra/extra v/ bipolar pulse which kills signal in faster moving blood in moderate and larger vessels.

Over half of SE-BOLD at 1ST is venous...
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence
- "Spin-echo EPI" uses a 180° pulse to add a single spin-echo to the contrast-controlling gradient echo through the center of k-space
- "asymmetric spin-echo EPI" arranges for the spin-echo to occur t msec before the gradient echo, which gives more T2*-weighting (for ky=0 echo)

- the 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used
- at higher fields, spin echo EPI is more promising
  - signal to noise is higher so we can take spin echo hit
  - contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording
COIL FALL-OFF / UNDERSAMPLE / GRAPPA / SENSE

- Coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs.
  But what does this look like in k-space?

- Slow variation in RF-field fall-off (e.g., 1-4 cyc/FOV) causes a blur in acquired data in k-space (N.B. not addition!)

- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space - at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq:
  Image domain
  \[
  \text{image ("brain")} \quad \text{FT} \quad \text{spatial freq. domain} \quad \text{FT} \quad \text{acquired image}
  \]

- N.B. inverse FT of k-space data "smeared" in spatial freq.
  Space is sharp image w/ fall-off (not blurred image)

- "Smeared" means normally orthogonal spatial freq's leak to adj. freq's.

[GRAPPA - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center across mult. coils]

[SENSE - general linear inverse approach]

- N.B.: Neither would work unless normally orthog. spatial freq's blurred!
SMS/MULTIBAND/blipped CAMPI

RF_{in} + A\cdot b_i

\begin{align*}
G_z & \\
G_y & \\
G_x & \\
RF_{out} &
\end{align*}

- Excite multiple slices at once
- Function of $G_z$ blips is to shift slices in $G_y$ direction

- This occurs because for given slice, a phase pedestal is added to the entire slice.
  \[ \Rightarrow \text{this is "Fourier Shift Theorem"} \]
  \[ \Rightarrow [\text{N.B.: different than } B_0 \text{-defect-induced incremented phase errors}] \]
- Problem w/ all up $G_z$ blips $\Rightarrow$ phase error builds up

**Trick #1**
- Start w/2 slices, one at $z=0$, other above
  \[ \Rightarrow \text{if } \pi (180^\circ) \text{ phase shift used, blip up/down same! (no effect at } z=0) \]
  \[ \Rightarrow \text{i.e., move top or bottom replica} \]

**Trick #2**
- For multiple slices not all at $z=0$, phase no longer same for even/odd
  \[ \Rightarrow \text{but can add phase to equilibrate to } k\text{-space before recon.} \]

**Trick #3**
- For more than 2 slices:
  \[ \begin{array}{c}
  1^{st} \\
  \text{even} \\
  \text{odd} \\
  \text{even} \\
  \text{odd} \\
  \end{array} \Rightarrow \text{etc} \]
MULTI BAND/BLIPPED C/PI

relation between leave-one-out aliasing and nominally fully-sampled SMS

- leave alternate lines out wraps image
- SENSE/GRAPPA to fix block coil view swears
K-space data
- nominally, w/ SMS we record every line of K-space
- but equivalent to leave alternate out b/c our multi-slice
  FOV was not big enough

- slice - GRAPPA
  - reg GRAPPA -> recon missing lines
  - slice GRAPPA -> recon multiple K-spaces
    i.e., not for each overlapped slices
    by training on fully-sampled data
    at beginning of scan

- interslice leakage block
  - when training GRAPPA kernel on fully-sampled data,
    also minimize interslice leakage (split-slice-GRAPPA)
  - can also do regular GRAPPA on top of this
    reason: for diffusion, loss in S/N from undersample
    cancelled by shorter TE readout
  - gain from reduced image distortion from shorter readout
**ECHO-VOLUME IMAGING EVI**

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo, too)

---

**Diagram:**

- Slab select
- Phase-encode blips
- Partitions (2nd phase encode)
- Center 6 $k_x, k_y$ for this $k_z$ partition: $T_E$ off
- Finished the first partition
- Second partition

---

- Entire k-space must be filled before 3D image is reconstructed
- Since entire volume is excited each shot, potentially higher S/N
- Must use smaller flip angle to avoid killing $M_L$ since entire volume excited every partition (e.g., every 80 msec)

---

- Main issue is movement artifact since data assembled from many shots over several secs
- Breathing-induced $B_0$ problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/ trapezoids (less eddy currents)

earlier EPI hardware like this : sinusoidal gradient waveform from resonant circuit w/ non-uniform sampling to get constant $\Delta k_x$

- sinusoids in both $G_x$ and $G_y$ give spiral $k$-space trajectory

- constant angular velocity goes too fast at large $k_x$, $k_y$

- constant linear velocity better but impractical near $k_x=0$, $k_y=0$

- compromise: start constant angular, end constant linear

**Constant angular velocity**

$$w(t) = w_0 \tau$$

$$k(t) = A t e^{i w_0 t}$$

$$G(t) = \frac{1}{\tau} \frac{d}{dt} k(t)$$

$$= A e^{i w_0 t} + i A w_0 e^{i w_0 t}$$

$$G_x(t) = A \cos w_0 t - A^2 w_0 \sin w_0 t$$

$$G_y(t) = A \sin w_0 t + A^2 w_0 \cos w_0 t$$

**Constant linear velocity**

$$w(t) = w_0 \tau t$$

$$k(t) = A \tau e^{i w_0 \tau t}$$

$$G(t) = \frac{1}{\tau} \frac{d}{dt} k(t)$$

$$= \frac{A}{\tau} e^{i w_0 \tau t} + \frac{A}{2} w_0 e^{i w_0 \tau t}$$

$$G_x(t) = \frac{A}{\tau} \cos w_0 \tau t + \frac{A}{2} w_0 \cos w_0 \tau t$$

$$G_y(t) = \frac{A}{\tau} \sin w_0 \tau t + \frac{A}{2} w_0 \sin w_0 \tau t$$

$G_x$

$G_y$
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)
  → all echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- Possible to present sign
- High, uniform contrast, but lots of waiting (TI), high BW

180° (prep1) → 700 μsec

RF

G_{z}

G_{y}

G_{x}

Sig.

3D k-space
("stack of spirals")

Loop order

Spiral interleaves
k_{z} interleaves
k_{z} echoes

Echoes → (after one 90°)
**Phase Errors & Echo-Centering Errors**

- Anything that causes a deviation of the $B_2$ field strength from the expected value $(B_{0,z} + G_{y,z} x + G_{y,z} y + G_{2,z} z)$ changes precession frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

---

**Fourier Shift Theorem**

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int_{k_x} e^{-i2\pi k_x x} S(k_x) e^{i2\pi k_x x} dk_x$$

- Correct w/ shimming and $B_0$-mapping/phase unwrapping before reconstruction.

---

**Echo Centering Error**

- If realignment of all spins $(k_x = k_y = 0)$ doesn't occur at the middle of read gradient, echo is shifted.

- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction.
FAST SCAN ARTIFACTS

EPI vs. Spiral

brain-induced field defects lead to phase errors

EPI
- $G_x$ readout gradient strong $\rightarrow$ field defects smaller percentage, less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weaker and total $G_y$ record time much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the $x$-direction for example, unwraps phase as a function of $x$-position
- but $G_x$ big, so effect on freq.-encode direction is much less than on phase-encode direction, where error accumulates
- for a given $x$-position, the strength of the spurious gradient is constant, so the accumulation of phase error results in a shift in the $y$-direction ($k_y$-space spin-stripe disp) (N.B. Shift varies w/ x-position)

Spiral
- with center-out spirals phase errors accumulate in a radial direction
- thus, higher spatial frequencies have more error (= more shearing)
- for spurious $x$-direction gradient as above, there is a radial blurring, rather than a vertical shift because higher frequency phase stripes misaligned relative to low spatial freq.
- for defects with more complex contours in the $y$-direction (than linear, as above) the vertical shifts (in EPI) will vary with $y$-position, and may result in signals from different $y$-positions being reconstructed on top of each other
- localized B\(\Phi\) defects often arise from air pockets embedded in tissue
  - air in middle/outer ear \(\rightarrow\) indentation in inferior temporal lobe
  - air under olfactory epithelium \(\rightarrow\) orbitofrontal dx, ant, thal. compression

- collect one data (k-space) point
- 4 cycles of phase in y-dir (2 gradient)
- localized B\(\Phi\) defect
- complex multiply (= correlate sin/cos with brain)
- brain structure sampled with distorted stripes
  - one complex number

- reconstruction from distorted data points
- \[ \text{undistorted stripes used by inverse FFT} \]
- \[ \text{amplitude and phase of this component} \]
- \[ \text{same for 5 cycles} \]
- \( = \) local upward displacement
- \( = \) image phase (phase encode dir)

- same defect makes leftward dent in vertical phase stripes
- spatial information can be lost when continuous changes in phase are flattened by B\(\Phi\) defect
- shifts can pile multiple pixels on top of each other into one bright pixel

- local estimates of B\(\Phi\) needed to correct images
  1) fieldmap method: \(<\) shift each image pixel proportional to B\(\Phi\) in phase-encode dir.
  2) point-spread-function: \(<\) extra phase encode to estimate P.S.F. (should be \(\delta\)-function)
  \(\) deconvolve distorted image in phase-encode direction
LOCALIZED $B\phi$ DEFECT, EFFECT ON RECON (2)

- When local $B\phi$ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.)

- If each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space

- A k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error)

- However, with w/ EPI, static $B\phi$ defect causes more and more local displacement of image phase stripes for each additional ky line

  - That is, later lines have greater spat. freq. offset
  - Effectively stretches k-space in ky direction
  - Same num samples to higher spatial freq. shrinks FOV (squishes voxels — see FOV page)

- When image is reconstructed, region with local $B\phi$ defect shifted oppositely

- Thus, local shift effect due to combination of 3 things:
  1) Static local $\Delta B\phi$ defect
  2) Successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) Small size of ky phase encode blips relative to $B\phi$ defect (much less of this effect in freq. encode direction)

- Respiration (which affect $B\phi$) in 3D FLASH might cause similar effect within $k_2$ partition (if successive spat. freqs.)
Gradient Non-Linearities

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impose a linear variation onto to $z$-component of the B-field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (e.g., printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.

  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently:
  - For 3-D scans (all directions), 2-D scans (just in-plane), and EPI scans (no corrections!).

- This can result in errors approaching 1 cm in function-structure overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

  - These can also be corrected since the $B_x$ and $B_y$ components are known.

  - That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHIMMING AND $B_0$-MAPPING

- Passive iron shims inserted to flatten $B_0$ field.
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the $B_0$ field (a few ppm good).
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc.) (= several hundred ppm).

- Linear shim coils impose gradients in x, y, and z.
- Higher order shims impose higher order spherical harmonic field components (e.g., $z^3$).

- Secondary use is to compensate for inhomogeneities caused by introducing the sample into the $B_0$ field.

- Local resonance offsets caused by $B_0$ defects estimated from images (e.g., sample phase at multiple echo times.

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents.
  - This only corrects spatially gradual field defects.
  - Local defects due to air in sinuses much higher order than shims.

- After shimming, field map measured again.

- Image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image.

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is strong relative to defects).
**NAVIGATOR ECHOES**

- **1D navigator**
  - Bφ drift problem
  - slow up/down drift in Bφ continuously occur
  - a pedestal in Bφ is pedestal in phase (not gradient)
  - which causes spatial shift (Fourier shift theorem)
  - in EPI, mainly affects phase-encode dir b/c small slice/sag readout
  - result is successive volumes drift in phase encode dir

**Gradient balance problem**

- unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
- causing N/2 (Nyquist) ghosting
- another phase error

- **3D navigator**: collect 3D sphere in k-space

  - rotation of object → rotation of k-space amplitude pattern
  - translation of object → phase shift of k-space phase (Fourier shift)

  - sample at sufficient radius to pick up high spatial freq features
  - N.B.: excite whole volume
  - do N,S hemispheres separately (less T2*, cancel EPI-like error accumulation)

  [Welch et al. (2002) MRM]

  \[ z(n) = \frac{2n - N - 1}{N} \]

  \[ x(n) = \sin(\pi n \sin^{-1} z(n)) / (1 - z^2(n)) \]

  \[ y(n) = \cos(\pi n \sin^{-1} z(n)) / (1 - z^2(n)) \]

  (skip poles → slow rate too high)

- can be used for prospective motion correction (rotate, translate w/gradient)
- better estimate, because of speed, than full TR & EPI images (27 ms vs. 2.4 sec)
- may need to smooth rot, trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES $\mathbf{B}_0$, inhomogeneities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  \[ \Rightarrow \text{variations can be used (cf. GRAPPA, SENSE) and/or corrected} \]

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  \[ \Rightarrow \text{potentially worse (why local transmit is still in progress)} \]
  \[ \Rightarrow \text{usu. fixed by using a large transmit coil (e.g. body coil)} \]

- RF penetration at higher fields ($\propto$ higher RF frequencies)
  is less uniform:
  \[ 1) \text{decreased RF wavelength (closer to size of head) at higher freq.} \]
  \[ 2) \text{increased permittivity ($\varepsilon$) and conductivity ($\sigma$) at higher field} \]

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

  normalization ("pre-scan normalize")
  - record lo-res volume (b/c coil falloff is smooth)
    through both body coil and small coil(s)
  - divide small body coil at each voxel to determine receive field
  - use receive field to normalize main image(s)

[see also: qT1, MP2RAGE, T1/T2]
Diffusion - Weighted Imaging

Simple diffusion weighting

RF

\[ G_z \]

\[ G_y \]

\[ G_x \]

- Spins acquire phase during first \( \Delta t \)
- Spins diffuse (= move) along gradient by time \( T \), signal is lost because negative \( \Delta t \) doesn't re-phase
- Attenuation: \[ A(\Delta t) = e^{-b\Delta t} \]
where \( b = (yG\Delta t)^2 (T = \Delta t/3) \)

"Apparent diffusion coefficient" map
- To get large \( b \), need \( G \) \& \( \Delta t \) (need big \( G \)'s)
- Long \( \Delta t \) gives spurious \( T_2 \)-weighting
- Use stimulated echoes:
- Anisotropic Diffusion (Gaussian)
  - Measure \( D \) along multiple axes
  - Have to measure tensor, not scalar
    \( \Rightarrow \) even for determining one primary direction
  \[ D = \begin{bmatrix} U_x \ U_y \ U_z \end{bmatrix} \begin{bmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{bmatrix} \begin{bmatrix} U_x \ U_y \ U_z \end{bmatrix} \]
  - Since \( D \) is symmetric, only need 6 measurements
  \[ D = u^T \times D \times u \]

Diffusion Surface (Non-Gaussian)
- Need to measure diffusion in many directions
- E.g., icosahedral subtellesation (126)
- Required for even two fiber directions

Fiber tract mapping
- An ill-posed problem
- E.g., constrain both ends and center point (!)
- Need vis. areas test
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- Spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)
  - Allows longer TE

- Eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/partic. time constants)
  - (also, keep vectors orthogonal to diffusion-encoding gradients)

- Induced by onset and offset of gradients

- Nagy et al. (2014) MRM

- Phase dispersion (6 echo)

- Twice-refocused Spin-echo (for center k-space)
PERFUSION - ARTERIAL SPIN LABEL

- Basic idea: Tag blood below area of interest
  - collect control & tagged image
  - assume directional input flow!

Continuous ASL (CASL) -
  - continuously tag a plane
  - gradient on, blood gets adiabatically inverted as it passes through location w/constant resonant freq.

Pulsed ASL (PASL) -
  - e.g., EPSTAR, FAIR, PICORE, QUIPPS II
  - tag block of tissue below slice(s)

- Small diffs between control and tag (~1%)
  - requires accurate balancing of control & tag images

- Contrast problems:
  - transit delays = biggest confounding factor
  - relaxation rate diffs
  - venous clearance (vs. microvessels, which get stuck)

Solutions for quantitative:
  - insert delay so all spins arrive into low velocity capillaries
  - kill end of tag to reduce spatial variation of tag

QUIPPS II - quantitative perfusion

1) Pre-saturate spins in target slices

2) Tag - 180° pulse below slices
   - Control - 180° pulse above slices
   - (to control off-resonance)

3) Saturate tagged block to end tag (TI,)
   - Both tag and control
   - Can use train of thin slices
   - Pulses at top of tag band

4) EPI of spiral images of target slices (T₁)
   - Image most distal slice last to cancel delays
   - Fast between slice so imaging excitation don't get interpreted as flow

\[
\Delta M \approx \text{flow} \times \left(2M_0 \text{TI}, e^{-T_1/T_1A}\right)
\]

Con extract flow and BOLD

1) Alternate tag and control; GRE TE = 30 ms
   - Control + tag = BOLD
   - Control - tag = flow

2) Dual echo spiral
   - k=0 early \(\Rightarrow\) hi S/N flow
   - TE = 30 ms \(\Rightarrow\) BOLD
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g., $^1$H) by surrounding electron orbitals

- e.g., acetic acid:

  - oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:

  - Larmor oscillations are multiplied (PSD) by center freq to obtain $\Delta f$ (not Hz; high freq)

- data before FT is a series of time-domain samples of the mix of shifted-freq offsets

- FT turns data into "shift spectrum"

N.B.: opposite "direction" of FTs!

<table>
<thead>
<tr>
<th>Signal</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMR</td>
<td>time domain FT of Oscillation Samples (shift)</td>
</tr>
<tr>
<td>MRI</td>
<td>spatial FT of spatial object (like time domain signal)</td>
</tr>
</tbody>
</table>

Pulse Sequence

- since we are already using phase (freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal

$\rightarrow$ and FT it like chemists do!
PHASE-ENCODED STIMULUS & ANALYSIS

- map polar angle
- map frequency
- map eccentricity
- map prox/distal axis, Road maps

Periodic stimuli (phase-encoded) - e.g., 8 cycles at 64 sec/cycle

Calculate significance
- ratio between amplitude at stimulus frequency (= signal)
  and average of amplitudes at other frequencies (= noise)
- ignore harmonics, low freq (= movement)

Smooth
- vector average of complex significance (A, \( \phi \)) with that at nearest neighbor surface points

Display
- plot phase using hue and saturation to indicate significance

Delay correction
- record responses to opposite directions of stimulus (ccw/cw, in/out, up/down)
- vector average after reversing angle of one
- penalizes inconsistent more than just avg of angles

Typically 0.5 - 5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

FFT, convert to A, \( \phi \)

Real

Imaginary

Freq = total TR's / 2

Display reversed CCW

Vector average

CW significance

CCW significance (complex)
CONVOLUTION

\[ f(x) = g(x) \ast h(x) = \int_{-\infty}^{+\infty} g(z) \cdot h(x-z) \, dz \]

- definition of convolution

How to calculate one term:
- Sum across all \( z \) to get the value of the convolution at \( x \)
- Move kernel to calculate next \( x \)

Why we reverse:

Impulse response function (HRF)

Impulses (ERP design)

How to calculate convolution for this time point (only 3 terms in integral – all other zero)
GENERAL LINEAR MODEL

\[ \vec{y} = \vec{X} \hat{h} + \vec{S} \hat{b} + \vec{n} \]

- goal is to solve for the hemodynamic response functions, \( h \)

\[ \begin{align*}
\text{data} & = \text{design} \cdot \text{HDR} \cdot \text{drift} \cdot \text{weights} + \text{noise} \\
\text{y} & = \begin{bmatrix} \text{h} \\ \text{X} \end{bmatrix} + \begin{bmatrix} \text{S} \\ \text{n} \end{bmatrix}
\end{align*} \]

multiple conditions

1) assume white noise, solve for \( \hat{h} \)

2) \( \hat{h} = (X^T P_{s}^T X)^{-1} X^T P_{s}^T y \) where \( P_{s}^T = I - S (S^T S)^{-1} S^T \)

or

\( = (X^T X)^{-1} X^T y \) where \( X = P_{s}^T X \)

3) significance (how to construct F-ratio)

\[ F = \frac{N-K-\ell}{K} \begin{bmatrix} y^T (P_{XS} - P_{S}) y \\ y^T (I - P_{XS}) y \end{bmatrix} \]

\( \Rightarrow \) see diagram next page for geometric interp
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- with no nuisance functions ($S$), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance

\[ \hat{y} = \hat{X} \hat{h} + \hat{e} \]

projection matrix, $P_x$, operates on $\hat{y}$ to give projection of data into experiment space, $\hat{X}h$

- when nuisance functions, $S$, are considered, problem: $S$ may not be orthogonal to $X$

for example: linear trend not orthogonal to std. block design

Geometric Picture

(Liu et al. 2001, Neuroimage)

$X_S$: space of data modeled by all reference and nuisance

oblique projection onto reference ($E_x y$)

orthogonal projection onto reference ($P_y y$) (data explained by reference)

orthogonal projection onto nuisance ($P_s y$) (data explained by nuisance)

error ($e$) not explained by reference and nuisance ($F$ denoted as $[(I - P_s)y]$)

how much more of data you can explain by adding reference functions ($F$ numerator) $[(P_x - P_s)y]$

same as projection onto reference only in special case where $S \perp X$
SEGMENTATION & SURFACE RECON
Talairach, Normalize, Strip Skull

1) MNI auto-Talairach → generates 4x4 matrix
   - make average brain target (blurry)
   - blur target (further), blur single brain (a bit), gradient descent on xcorr
   - repeat w/ less blurring of avg target and current brain
   - problems: variable neck cut off, only 2 points near center of brain!
     ➡️ but much better than standard! < fit to bounding box

2) Intensity Normalization (output: "T1")
   - histogram of pixel values in 10 mm thick T1R slices
   - smooth histogram
   - peak find to get initial estimate of white matter
   - discard outlier peaks across slices
   - fit splines to peaks across slices ➡️ interpolated scaling factor 1 to T1R
   - scale each pixel so WM peak is 110
   - refine estimate to interpolate in 3D
   - find points in 5x5x5 within 10% of WM, get near scale for them
   - build Voronoi to interpolate scales onset above
   - soap-bubble-smooth Voronoi boundaries (3 iterations)
   - re-scale each voxel (5-10 times)

3) Skull Stripping (output: "brain")
   - "shrink-wrap" algorithm
   - start with ellipsoidal template ➡️ sub-tessellated icosahedron
   - minimize brain penetration and curvature
     - curvature: spring force ➡️ force along surface normal that
     - brain penetration
       (from center-to-neighbor vec sum)
     - decompose into 1 and tangential (local normal from summed normalized cross products)
SEGMENTATION & SURFACE RECON

- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the "something")
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno Surface refinement)

\[
\mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + \mathbf{F}_{\text{smooth}}(t) + \mathbf{F}_{\text{MRI}}(t)
\]

\[ \mathbf{F}_{\text{smooth}} = \lambda_{\text{tang}} \sum_{\text{neigh}} (I - \mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) \]

\[ + \lambda_{\text{normal}} \left( \sum_{\text{neigh}} (\mathbf{n}_{\text{center}} \mathbf{n}_{\text{center}}^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_{\text{center}}) - \frac{1}{\#\text{vertices}} \sum_v \sum_{\text{neigh}} (\mathbf{n}_v \mathbf{n}_v^T) \cdot (\mathbf{r}_{\text{neigh}} - \mathbf{r}_v) \right) \]

\[ \mathbf{F}_{\text{MRI}} = \lambda_{\text{MRI}} \mathbf{n}_{\text{center}} \max_{d} \left[ 0, \tanh \left[ I \left( \mathbf{r}_{\text{center}} - d \mathbf{n}_{\text{center}} \right) - I_{\text{thresh}} \right] \right] \]

Snapshot of surface and "core sample" from one vertex
4) Non-isotropic filtering (output: "win") — "floss" and "speckle"

- preliminary hard thresholds: output = blood vessels (bright) also truncated to black
- find ambiguous/boundary voxels 
  \[ \geq 20\% \text{ or more of 26 immediate neighbors different} \]
- find plane of least variance
  for each direction (from icosahedral super-tessellation)
  consider 5x5x5 volume around 1 voxel
  find plane of least variance in this hemisphere
  medium filter w/ hysteresis
  \( \Rightarrow \) if 60% of within-slab differ, reverse classification
  \( \Rightarrow \) "flosses" sulci without blurring

5) Find cutting planes

midbrain, callosum, to separate hemispheres (SAG)
midbrain, to avoid fill into cerebellum (HOR)

Talairach to start;
fill WM in SAG or HOR till min area

6) Region-growing to define connected parts (output: "filled")

- inside-out, outside-in, inside-out — for each hemisphere
- up/down cycles within each plane
- plane-by-plane
- "wormhole filter" (3x3x3 = center + 26)
  \( \Rightarrow \) fill (unfilled) voxel if 66% neighbors differ — eliminates structures within I-D structure
7) Surface Tessellation (output: rh.orig, lh.orig)

- Find filled voxels bordering unfilled
- Make ordered list of neighboring vertices
  → So cross-products oriented properly

- Long list of values associated with each numbered vertex
  e.g.: position (orig, morphed)
  area (orig, morphed)
  curvature (intrinsic, Gaussian)
  "Sulcussness" (summed 1 movement during unfolding)
  cortical thickness
  fmri data
  EEg/MEG dipole strength

- Separate fmri data set must be aligned, sampled
  - Fmri voxels larger
  - Sample at each surface vertex
  - Nearest-neighbor "soap bubble" smoothing to interpolate data into hi-res mesh
- Some quantities only well-defined on surface
  → Gradient of magnitude of cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON

- smoothing/inflation_WM, pial done as derivative of energy functional

\[
J = J_{\text{tangential}} + \lambda_{\text{normal}} J_{\text{normal}} + \lambda_{\text{image}} J_{\text{image}}
\]

- total scalar error (fixed by minimizing)
  - scalar tangential error (fixed by redistributing vertices)
  - scalar curvature error (fixed by reducing curvature)
  - scalar image error (fixed by moving toward target image value)

\[
J_{\text{normal}} = \frac{1}{2 \#\text{vert}} \sum \sum [n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}})]^2
\]

- across all vertices
  - \(\frac{1}{2}\) so no derivative coefficient

\[
J_{\text{tangential}} = \frac{1}{2 \#\text{vert}} \sum \sum \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right]^2 + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right]^2
\]

- across vertices of one vertex
  - \(t^x, t^y\): direction unit normal to vertex
  - project vector to neighbor onto \(x, y\) plane

\[
J_{\text{image}} = \frac{1}{2 \#\text{vert}} \sum \left[ I_{\text{targ}}(r_{\text{center}}) - I^*(r_{\text{center}}) \right]^2
\]

- mean of pixels labeled WM in 5 mm neighborhood
  - \(I_{\text{targ}}\) for WM: global - small num for CSF - like

\[\frac{\partial J}{\partial r_{\text{center}}} = \lambda_{\text{image}} (I_{\text{targ}} - I(r_{\text{center}})) \nabla I(r_{\text{center}})\]

- go in opposite direction of largest scalar error for one vertex

\[+ \sum \lambda_{\text{normal}} \left[n_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}})\right] n_{\text{center}} \]

- unit normal vector scaled by dot product

\[+ \sum \left[ t^x_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) t^x_{\text{center}} + \left[ t^y_{\text{center}} \cdot (r_{\text{center}} - r_{\text{neighbor}}) \right] t^y_{\text{center}}\right] \]

- take directional derivative of energy functional (find steepest uphill)

- move each vertex in the opposite (negative) direction w/ self-intersect test

N.B.: eq. 9 in Birds, Fischer & Sarrac different — and incorrect!
**SULCUS-BASED CROSS-SUB. ALIGN**

- Use summed perpendicular vertex movement during inflation as per-vertex measure of "sulcus-ness"
- Add term to energy function: "sulcus-ness" error: $\left( S_{\text{orig}} - S_{\text{target}} \right)^2$
- Bootstrap morph to one brain, make any target remorph to any target

Each sub's native surf has diff # vertices
- Interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)

Average surface made from folded/inflated avg coords
- Folded: loses area from sulcal crinkles (ft. average "inflated")
- Inflated: retains orig area, correct sulc gyrus ratio ("inflated_avg")

Can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

$\Rightarrow$ N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. sub.)