Neuroimaging NOTES

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(122 pages)

MAGNET HARDWARE

1. **$B_0$ field** from superconducting magnet

2. Gradient coils

3. Body RF transmit/receive

4. RF receive-only

5. Shim coils
   - in gradients
   - $B_0 \rightarrow z$ (longitudinal)
   - $B_1 \rightarrow x, y$ (transverse)

- Opposite direction $B_0$ shield coils outside those not shown
- Superconducting coils in liquid helium
- No power required after current injected to bring up field using induction

- \[ \frac{1}{2\pi} = 42 \text{ MHz/T} \]
- \[ 1T = 10,000 \text{ Gauss} \]
- Earth: 0.25-0.65 G
- 25-65 AM

Max gradient:
- $80 \text{ mT/m}$
- $200 \text{ Tm/sec}$

- Non-superconducting water-cooled, external shield

- RF transmitter (30 kW)

- RF receiver

- Three 1.5 million watt amplifiers to add ramps to $B_0$ field

- Circularly polarized $B_1$ field rotating
- $B_1$ is several orders of magnitude smaller than $B_0$
**Spin & Precession**

- Nuclei act like a spinning sphere of matter with an embedded equatorial charge (nuclei w/ odd atomic weight or odd proton numbers).
- Moving charge creates magnetic field (classical picture).
- Current loop from spinning charge (right-hand rule).
- N.B.: Classically, this would cause EM radiation, spin-down.
- Stern-Gerlach experiment: pass silver atoms thru strong mag. field → split into just 2 beams.

**Microscopic picture**

- No strong magnetic field: $B_0 = 0$.
- Strong magnetic field: $B_0 = \uparrow$.
- Strong $B_0$ plus oscillating $B_1$.

**Macroscopic picture**

- All vectors some length, random directions
- "Up" (lo E) vs. "down" (hi E): slight excess of "up" (3 ppm), already precessing $x$, $y$ components still random
- Precessing vectors are "bunched" at any one moment around circle
- Bulk magnetization precesses

### Precession

- Distinguish precession (slow) from spin (fast).
- Treat classically, like spinning top.

$$2\pi f = \frac{\omega_0}{B_0} = \frac{Y}{B_0}$$  \hspace{1cm} \text{Larmor freq. (e.g., 63 MHz)}

$$\omega = \frac{h}{\gamma I (I + 1)/3}$$  \hspace{1cm} \text{Larmor freq. (e.g., 15 MRad/sec)}

### Bulk magnetization

- Bulk equilibrium magnetization (parallel to $B_0$)

$$M_z^0 = \frac{\vec{M}}{4 KT_s} = \gamma^2 h^2 B_0 N_s$$

- Two non-constants

$\gamma = \text{gyromagnetic ratio}$

$\hbar = \text{Planck's const.}$

$B_0 \rightarrow \text{i.e., } M_z^0 \text{ proportional to } B_0 \text{ strength}$

$N_s \rightarrow \text{i.e., } M_z^0 \text{ proportional to number spins}$

$K = \text{Boltzmann const.}$

$T_s = \text{abs. temperature sample}$
The Bloch Equation

- Time-dependent behavior of $\vec{M}$ in the presence of an applied magnetic field (excitation & relaxation)

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{V} - \frac{\vec{M}}{T_2}
\]

- In the Larmor-rotating coordinate system, a tilt with a phase shift from a standard $B_1$ excitation is rotation around $x$-axis

- Longitudinal and transverse relaxations

\[
\frac{dM_z(t)}{dt} = \frac{M_z(t) - M_0}{T_1}
\]
\[
\frac{dM_x(t)}{dt} = \frac{M_x(t)}{T_2}
\]

- Solution to equations above: time-dependent free precession e.g.'s

\[
M_z(t) = M_z^0 (1 - e^{-t/T_1}) + M_z'(0) e^{-t/T_1}
\]

\[
M_x(t) = M_x'(0) e^{-t/T_2}
\]

Given initial $M_z^0$, $M_x^0$
**VECTOR ADD, MULTIPLY**

- Adding vectors is easy
  \[ \vec{c} = \vec{a} + \vec{b} = [a_x + b_x, a_y + b_y] \]
  - Just add components (vector)
  - Applies to complex numbers

- Multiples ways to multiply vectors; here are 3

**Dot Product**

(= inner product)
(= "scaled projection onto")

\[ c = \vec{a} \cdot \vec{b} = [b_x, b_y, b_z] [a_x, a_y, a_z] = a_x b_x + a_y b_y + a_z b_z \]

(successor)

- Generalizes to any D

N.B. equals: \( \vec{a} \cdot \vec{a} \)
\[ \sqrt{a_x^2 + a_y^2 + a_z^2} \]

Length of \( \vec{a} \)

\[ c = \| \vec{a} \| \| \vec{b} \| \cos \theta \]

\( \Rightarrow \) zero if \( \vec{a}, \vec{b} \) orthogonal

**Cross Product**

(= outer product)
(can be generalized; see "geometric algebra")

\[ \vec{c} = \vec{a} \times \vec{b} = \begin{bmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{bmatrix} [a_x, a_y, a_z] = [a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x] \]

(vector)

Geometric algebra: bivector plane area

- Unique orthogonal specific to 3D

\[ \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \sin \theta \]

\( \Rightarrow \) max if orthogonal

**Complex Multiply**

(see also quaternions, geometric algebra generalization)

\[ \vec{c} = \vec{a} \cdot \vec{b} = \begin{bmatrix} b_x & -b_y \\ b_y & b_x \end{bmatrix} [a_x, a_y] = [a_x b_x - a_y b_y, a_x b_y + a_y b_x] \]

(vector)

\( \| \vec{c} \| = \| \vec{a} \| \| \vec{b} \| \)

\( \Rightarrow \) like real nums
**Effects of $\vec{M}$, $\vec{B}$, and $\Theta$ on PrecessionFreq.**

**Bloch 1st term**
\[
\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}
\]

- Cross prod. properties review:
  \[
  \left| \frac{d\vec{M}}{dt} \right| = |\vec{M}| |\vec{B}| \sin \Theta
  \]

**Starting condition**
- Now see effects of changing $\vec{M}$, $\vec{B}$, $\Theta$

**Change $\vec{M}$ length**
- \( \frac{d\vec{M}}{dt} \propto \) proportionally larger, so canals
  - Effect of larger $\vec{M}$
  - Same precession freq. as starting cond.

**Change $\Theta$ between $\vec{M}$ and $\vec{B}$**
- \( \frac{d\vec{M}}{dt} \) goes up (then down) as $\sin \Theta$
  - $\Theta$ vs. cross prod.
- But circumference also goes up as $\sin \Theta$, cancelling again
  - Opposite $= \sin \Theta = \text{radius}$
  - So circumference proportional to $\sin \Theta$
  - Same precession freq.

**Change $\vec{B}$ length**
- \( \frac{d\vec{M}}{dt} \) goes up, proportional to $\vec{B}$
  - But circumference is same at starting cond.
- Increased precession freq. \( (\omega = \gamma B) \)
**Simple Matrix Operations**

**Basic Idea**
- A matrix \([\text{rotates/scales}]\) a vector

\[
\mathbf{b} = \mathbf{M} \mathbf{a}.
\]

**3D Example**

\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z
\end{bmatrix} =
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{21} & M_{22} & M_{23} \\
    M_{31} & M_{32} & M_{33}
\end{bmatrix}
\begin{bmatrix}
    a_x \\
    a_y \\
    a_z
\end{bmatrix}
\]

Add translate (after rotate/scale)
- Commonly used "hack" for aligning vols
- A 4D matrix \([\text{rotates/scales}]\) a 3D vector
- Then translates

\[
\begin{bmatrix}
    b_x \\
    b_y \\
    b_z \\
1
\end{bmatrix} =
\begin{bmatrix}
    3 \times 3 \\
\text{rot/scale} \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
    a_x \\
    a_y \\
1
\end{bmatrix}
\]

**N.B.:** Have to keep track of order!!
- Rotate/scale then translate \(\not=\) translate, then rotate/scale
d- Change rot component: untranslate, rot, retranslate

**3 Special Cases (3D):** Rotate around each major axis without changing length (Scale = 1.0)

- Rotate around \(x\)-axis:
  \[
  \mathbf{R}_x(\alpha) =
  \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\alpha & -\sin\alpha \\
0 & \sin\alpha & \cos\alpha
\end{bmatrix}
\]
  e.g., 90° flip

- Rotate around \(y\)-axis:
  \[
  \mathbf{R}_y(\alpha) =
  \begin{bmatrix}
\cos\alpha & 0 & \sin\alpha \\
0 & 1 & 0 \\
-\sin\alpha & 0 & \cos\alpha
\end{bmatrix}
\]
  e.g., 180° flip to avoid adding 180° phase after 90° flip on \(x\)

- Rotate around \(z\)-axis:
  \[
  \mathbf{R}_z(\alpha) =
  \begin{bmatrix}
\cos\alpha & \sin\alpha & 0 \\
-\sin\alpha & \cos\alpha & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
  e.g., precession with 3D along \(z\)

**General Case**
- Rotate around general \(z\)-axis:
  \[
  \mathbf{R}_z(\alpha) = \mathbf{R}_z(-\phi) \mathbf{R}_y(-\theta) \mathbf{R}_z(\phi) \mathbf{R}_y(\theta) \mathbf{R}_z(\phi)
  \rightarrow \text{(Quaternions are more efficient)}
  \]
  
  \[
  \mathbf{R}_z(\alpha) = \mathbf{R}_z(-\phi) \mathbf{R}_y(-\theta) \mathbf{R}_z(\phi) \mathbf{R}_y(\theta) \mathbf{R}_z(\phi)
  \rightarrow \text{(Quaternions are more efficient)}
  \]
Solutions to Simple Differential Eq.

Diff. Eq.: \[ dM_{xy}(t) = -\frac{M_{xy}(t)}{T_2} \]

Solution: \[ M_{xy}(t) = M_{xy}(O_0) \cdot e^{-t/T_2} \]

Goal:
1) Find \( e^x \) whose derivative satisfies diff. eq.
2) Also find \( \text{soln} \) (one of many) that passes thru init condition

\( \Rightarrow \) since own diff. eq. is derivative of funct. = const. funct.

\( \Rightarrow \) try exponential, since derivative \( (e^x)' = e^x \)

\( \text{denn.} \)

Diff. Eq.:
\[ M(t) = \frac{-1}{T_2} \cdot \frac{M(t)}{\text{const}} \cdot \text{same var} \]
OK - we have recovered only diff. eq.

N.B. this function is the "unknown" like the \( x \) in \( x + 1 = 3 \)

One Soln:
\[ M(t) = e^{-t/T_2} \]

Take deriv. to check
\[ M'(t) = \frac{-1}{T_2} \cdot \frac{M(t)}{\text{same as } M(t)} \]
N.B. same as \( M(t) \)

Another Soln:
\[ M(t) = \text{const} \cdot e^{-t/T_2} \]

Take deriv. to check
\[ M'(t) = \frac{-1}{T_2} \cdot \text{const} \cdot e^{-t/T_2} \]
N.B. again same as \( M(t) \)

So: any const OK!

e.g. \( \text{const} = 2 \) \( \text{const} = 0.5 \)

Initial condition:
Information added to soln (not from diff. eq.):
\[ M'(t) = M_{xy}(O_0) \cdot e^{-t/T_2} \]

Const = \( M_{xy}(O_0) \)
magnetization immedi. after RF (B1) ends
Verify Solution to T1 Regrowth

- Slightly more complex T1 soln compared to T2 soln

T2 soln verify (see prev)

T1 solution verify

\[
\frac{dM}{dt} = \frac{M_{xy}}{T2}
\]

\[
M'(t) = \frac{1}{T2} M(t)
\]

\[
M(t) = M_{xy}(0) e^{-t/T2}
\]

- Proposed solution

\[
M'(t) = \frac{1}{T2} M_{xy}(0) e^{t/T2}
\]

- Test by take deriv.

- Original diff eq.

\[
\frac{dM}{dt} = \frac{-(M_z - M_z^0)}{T1}
\]

\[
M'(t) = \frac{1}{T1} (M(t) - M_z^0)
\]

\[
M(t) = \frac{M_z^0 (1 - e^{-t/T1}) + M_z(0) e^{-t/T2}}{\text{init cond.}}
\]

\[
M'(t) = O + \frac{1}{T1} M_z^0 e^{-t/T1} - \frac{1}{T1} M_z(0) e^{-t/T2}
\]

- Chain rule as before

\[
M'(t) = \frac{1}{T1} (-M_z^0 e^{-t/T1} + M_z(0) e^{-t/T2})
\]

- Derivative in original T1 eq. says \(M(t)\) minus \(M_z^0\)

\[
M'(t) = \frac{1}{T1} \left( M(t) - M_z^0 \right)
\]

Solution → \(\left[ M_z^0 - M_z^0 e^{-t/T1} + M_z(0) e^{-t/T2} \right] \)

- Which equals our re-calculated derivative:

\[
M'(t) = \frac{1}{T1} \left( -M_z^0 e^{-t/T1} + M_z(0) e^{-t/T2} \right)
\]
**Bloch Eq. - Matrix Version**

**Differential Eq.:**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi
\]

\[
= \begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \begin{bmatrix}
\gamma B_\phi \\
-\gamma B_\phi \\
0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & \cos \omega t & \sin \omega t \\
-\sin \omega t & -\cos \omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = R_z(\omega t) \vec{M}(0)
\]

**Solution:**

\[
\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & -\cos \omega t & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} = R_z(\omega t) \vec{M}(0)
\]

**Include Relaxation**

\[
\frac{d\vec{M}}{dt} = \vec{M} \times \gamma \vec{B}_\phi - \frac{M_x I + M_y J}{T_2} - \frac{(M_z - M_z^0)}{T_1}
\]

**Differential Eq.:**

\[
\frac{d\vec{M}}{dt} = \begin{bmatrix}
\frac{1}{T_2} \gamma B_\phi & 0 & 0 \\
0 & \frac{1}{T_2} \gamma B_\phi & 0 \\
0 & 0 & \frac{1}{T_2}
\end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z^0/T_2 \end{bmatrix}
\]

**Solution:**

\[
\vec{M}(t) = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix} = \begin{bmatrix}
e^{t/T_2} & 0 & 0 \\
0 & e^{t/T_2} & 0 \\
0 & 0 & e^{t/T_2}
\end{bmatrix} \begin{bmatrix} M_x(0) \\ M_y(0) \\ M_z(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ M_z(0)(1-e^{-t/T_2}) \end{bmatrix}
\]
Excitation in the Rotating Frame

- Original Bloch eq. in laboratory frame
  \[ \frac{d\mathbf{M}_{\text{lab}}}{dt} = \mathbf{M} \times \mathbf{B} \]
  \[ \text{gradients} \]

- Add on-resonance B1 to TSP
  \[ \mathbf{B} = B1(t) \left( \cos w_0 t \hat{i} - \sin w_0 t \hat{j} \right) + Bp \hat{k} \]

- Matrix version
  \[ \mathbf{M} = \begin{bmatrix} M_x' \cr M_y' \cr M_z' \end{bmatrix} \]
  \[ \frac{d\mathbf{M}}{dt} = \begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} \]

- Substitution to convert to the rotating frame
  \[ \mathbf{M} = R_z(w_0 t) \cdot \mathbf{M}_{\text{rot}} \]
  \[ \mathbf{B} = R_z(w_0 t) \cdot \mathbf{B}_{\text{rot}} \]

- After substitution any off-resonance appears as residual Bp (Bz)
  (see off-res notes page)

- Rotating frame < on-resonance
  * basic excite, B1x-only no gradient
  \[ \text{rotating frame < on-resonance} \]
  \[ \text{basic excite, B1x-only no gradient} \]

- Rotating frame < off-resonance
  * general, B1x-only incl gradients
  \[ \text{gradient: } w(t) = \Gamma g_z z \]
  \[ \text{off-res: appears as residual } B_0 \text{, lifting } B1 \text{ vect.} \]

- Rotating frame < on-resonance
  * small tip approx.

- Rotating frame < no gradient
  \[ \text{small tip } M_x = \frac{m_0}{\gamma} \approx 0 \]

- Small tip < easier to solve!
**Bloch Eq. Summary**

\[
\frac{d\hat{M}}{dt} = \hat{M} \times \hat{\mathbf{B}} - \frac{M_x \hat{i} + M_y \hat{j}}{T_2} - \frac{(M_z - M_0) \hat{k}}{T_1}
\]

(Rel-frame)

(vector lengths not to scale!)

- Full lab-frame picture is complex:
  - 3 component of \(\frac{d\hat{M}}{dt}\) update vector
  - Larmor freq. component 7-9 orders magnitude larger than \(T_2\) or \(T_1\) decay
  - \(\hat{\mathbf{B}}_1\) is also rapidly wiggling

- Conceptual simplification in 4 stages:

1. **Lab frame**
   - Just precession
   - \(\hat{\mathbf{M}}\) stopped
   - That is, \(\mathbf{B}_0 = 0\)

2. **Rotating frame**
   - \(\hat{\mathbf{M}}\) also stopped
   - But \(\hat{\mathbf{M}} \times \hat{\mathbf{B}}\) still works!!
   - "Precess" around \(\hat{\mathbf{B}}_1\) axis

3. **Add \(\hat{\mathbf{B}}_1\)**
   - Slow precess, now around tilted \(\hat{\mathbf{B}}_{eff}\)

4. **Off-resonance**
   - Tilted plane
   - Apparent \(B_2\) component from residual precess. around \(z\) from off-resonance
RF FIELD POLARIZATION

- Polarization (change of direction) of magnetic field (vs. electric field)

- Linearly polarized field
  \[ \overrightarrow{B_1}(t) = B_1 \cdot \cos \omega t \]
  Magnitude \( \{1, 1, 1\} \cdot 1 \)

- N.B.: \( \overrightarrow{B_1} \) adds to much larger \( \overrightarrow{B_0} \)

- Circularly polarized field (quadrature)
  \[ \overrightarrow{B_1(\theta)} = B_1 (\cos \omega t \hat{x} - \sin \omega t \hat{y}) \]
  = \( B_1 \cdot e^{-j\omega t} \)

- In the rotating coordinate system, flipping around x-axis vs. y-axis is just difference in phase of RF spin

- Typical 90° flip (around x-axis)
- Typical 180° flip (around opposite y-axis)

- 180° flip regk
  ~6x power of 90°
**Phase-Sensitive Detection**

How we get rotating frame

- method for moving very high frequency Larmor oscillations down to tractable frequency range

V(t) → multiply → Low-Pass Filter → \( S(t) \)

\( \sim 123 \text{ MHz} \)

Reference Signal (123 MHz)

\( \sim 50 \text{ kHz} \)

Demodulated signal \( \propto \) RF coil signal \( \cdot \) reference (transmitter)

\( \propto \sin[(w_0 + 8w)t] \cdot \sin[w_0t] \)

\( \propto \frac{1}{2} \left[ \cos 8wt - \cos (2w_0 + 8w)t \right] \)

This signal is digitized

Filter this one out with low pass filter

One freq - freq domain

Chirp - time domain

Signal

Reference

Demodulated

After filter

(\( \rightarrow \) rotating frame!!)

- Two signals are made from a single receiving RF coil

- A quadrature coil can be treated the same way (OK to combine after adding \( \frac{\pi}{2} \) phase, then PSD)

- Quadrature coil has better S/N since noise in each part is uncorrelated (\( \frac{\pi}{2} \) better)
**FID - FREE INDUCTION DECAY, T2**

- Signal (FID) resulting from RF pulse w/angle $\alpha$
  \[ S(t) = \frac{\sin \alpha}{w = \infty} \int \rho(w) \cdot e^{-t/T_2(w)} \cdot e^{-i\omega t} \, dw \]
  - Recorded complex signal
  - $w$ = angular freq.
  - $\rho(w)$ spectral density funct.
  - $e^{-t/T_2}$ time-dep. rapid decay
  - $e^{-i\omega t}$ oscillations freq.

- An example spectral density ("Lorentzian inhomogeneity")
  \[ \rho(w) = M_0^2 \frac{\Delta B \phi}{\Delta w} \]
  - $w = \Delta B \phi$ (central freq.)
  - $\Delta w = \Delta B \phi$ (width)

- An example spectral density ("Lorentzian inhomogeneity")
  \[ \rho(w) = \frac{M_0^2}{\Delta w} \Delta B \phi \cdot \sin \alpha \cdot e^{-t/T_2} \cdot e^{-i\omega t} \]

- $\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$
  - $\rho(w) = \frac{M_0^2 \sin \alpha}{\Delta w} e^{-t/T_2}$
  - $e^{-t/T_2^*}$ overall decay rate including inhomogeneous $B\phi$

- N.B. center freq., not original integration variable

- Suggestive, since 100 million cycles per second
  - $\rho(w)$ is complex
  - $\rho(w)$ is not Lorentzian!
**ECHOES — spin echo**

- Just after 90° x' pulse: $f_{00} + f_{hi}$ have same phase.
- Relaxation + phase dispersion of $f_{00} + f_{hi}$ (both from $B > B_0$).
- Just after 180° y' pulse: (y' pulse like x', pulse but RF has +90° phase).
- Echo caused by re-phasing of $f_{00} + f_{hi}$ (w/ decay due to $T_2$).

- Remember: brief RF just tips vectors while retaining length.
- Relaxation includes tips and shrinks ($M_r$) and grows ($M_z$, echo).
- 180° x' pulse works, too, but echo will have $+\pi$ phase (left side in figs above).
- Echo generated even if second pulse not 180° (see next).

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- FID decay (and echo growth/decay) described by $T_2^*$ from inhomogeneity.
- Reduction in height of echo compared to initial described by $T_2$. Echo fixes the 'star'.

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**Diagram notes:**

- Rotating coords
- N.B.; 2. Blends eigs in one voxel.
- Arrows below plane.
- FID decay (and echo growth/decay) described by $T_2^*$ from inhomogeneity.
- Reduction in height of echo compared to initial described by $T_2$. Echo fixes the 'star'.
**ECHOES — spin echo**

\[ \alpha_1 - \tau - \alpha_2 - \tau \] (both pulses along \( y' \) for simplicity)

**effect of \( \alpha_y \) pulse**

\[
\begin{align*}
M_x' &\rightarrow M_x' \cos \alpha - M_z' \sin \alpha \\
M_y' &\rightarrow M_y' \\
M_z' &\rightarrow M_x' \sin \alpha - M_z' \cos \alpha \\
\end{align*}
\]

\( \Rightarrow \) (etc for \( \alpha_x, \alpha_2 \))

**general transforms**

**effect of \( \tau \) delay**

**precession**

\[
\begin{align*}
M_x' &\rightarrow (M_x' \cos \omega \tau + M_y' \sin \omega \tau)e^{-\tau/2} \\
M_y' &\rightarrow (-M_x' \sin \omega \tau + M_y' \cos \omega \tau)e^{-\tau/2} \\
M_z' &\rightarrow M_z'(1 - e^{-\tau/2}) + M_z'e^{-\tau/2} \\
\end{align*}
\]

---

**immediately after \( \alpha_1 \) pulse**

\[
\begin{align*}
M_x'(w,0) &= -M_z'(w) \sin \alpha_1 \\
M_y'(w,0) &= 0 \\
M_z'(w,0) &= M_z'(w) \cos \alpha_1 \\
\end{align*}
\]

**for one isochromat of freq. \( w \)**

---

**after \( \tau \) delay**

\[
\begin{align*}
M_x'(w,\tau) &= -M_z'(w) \sin \alpha_1 \cos \omega \tau e^{-\tau/2} \\
M_y'(w,\tau) &= M_z'(w) \sin \alpha_1 \sin \omega \tau e^{-\tau/2} \\
M_z'(w,\tau) &= M_z'(w) [1 - (1 - \cos \alpha_1) e^{-\tau/2}] \\
\end{align*}
\]

**immediately after \( \alpha_2 \) pulse (no effect on \( M_y' \); rewrite \( y' \); combine \( x \) and \( y \) eqs.)**

\[
\begin{align*}
M_x'(w,\tau) &= -M_z'(w) \sin \alpha_1 \left( \sin^2 \frac{\alpha_2}{2} e^{-i\omega \tau} - \cos^2 \frac{\alpha_2}{2} e^{i\omega \tau} \right) e^{-\tau/2} \\
&\quad - M_z'(w) [1 - (1 - \cos \alpha_1) e^{-\tau/2}] \sin \alpha_2 \\
\end{align*}
\]

---

**time dependent**

**free precession around \( z' \)**

(rewrite \( M_{x'y'}(w,\tau) \))

---

**for a large num of freq's:**

\[
\left[ \text{terms 2 & 3 are dephasing} \rightarrow \text{FID of echo} \right] \\
\text{term 1 nphasing} \rightarrow \text{nephase at } t = 2\tau
\]

---

**echo signal**

\[ S(t) = \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2} e^{-i\omega(t-\tau)} dw \]

**peak ampl**

\[ A_E = \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \int_{-\infty}^{\infty} \rho(w) e^{-\tau/2} e^{-i\omega(t-\tau)} dw \]

\[ 90^\circ_y - \tau - 90^\circ_y \]

\[ S_1(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2} e^{-i\omega(t-\tau)} dw \]

\[ 90^\circ_y - \tau - 180^\circ_y \]

\[ S_2(t) = \frac{\text{no } 1/2 \text{ factor}}{\text{multiply by } i \rightarrow \text{add } \pi/2 \text{ phase}} \]

---

\[ 90^\circ_y - \tau - 180^\circ_y \]

\[ S_3(t) = \frac{1}{2} \int_{-\infty}^{\infty} \rho(w) e^{-t/2} e^{-i\omega(t-\tau)} dw \]

\[ = M_z^0 \sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-\tau/2} \]

\[ \text{echo amplitude, ignoring freq. dependence of } T_2 \]

---

\[ e.g. \text{special case} \]

\[ \alpha_2 = 180^\circ, \quad T_2 = 100 \]

---

\[ \text{etc for } A_E \ldots \text{like } \]
Echo TRAINS - spin-echo trains

- It's (too) easy to make echoes...

\[ E_n = \frac{3^{(n-1)} - 1}{2} \]

Echoes after end of nth pulse
3 RFs \( \rightarrow \) 4 echoes (here)
6 RFs \( \rightarrow \) 12 echoes (!)

Secondary echo:
- \( SE_{1,2} \) acts like RF pulse
- \( \alpha_3 \) makes an echo from it
- Two more conventional two pulse spin echoes

Stimulated echo: combined effect of 3
- \( \alpha_1 \): \( M_L \rightarrow M_T \)
- \( \alpha_2 \): leftover \( M_T \) flipped to \( M_L \) (saved)
- \( \alpha_3 \): flip saved \( M_L \) \( \rightarrow M_T \) which can then begin to cancel delays (after being held "in limbo" between \( 180^\circ \), \( FID_2 \) and \( FID_3 \)); acts like 2-pulse echo

- A useful multi-echo sequence (CPMG) is a \( 90^\circ \) followed by \( 180^\circ \) at \( 2\tau \) spacing

- Typically, \( 90^\circ \) and \( 180^\circ \) applied in different axes (x', then y', z'...), which reduces phase errors due to imperfect \( 180^\circ \) pulses (since slightly off rotation around y' affects phase less)
**Extended Phase Graphs**

- Using full Bloch eq. solutions is tedious 😊

- Pictorial method for visualizing effects of series of $\alpha$ pulses (vs. easier to visualize $90^\circ, 180^\circ$)

- Problem #1: $\alpha$ pulse rotates a portion of transverse magnetization into a position that results in rephasing and another portion into $M_L$

- Problem #2: third pulse can uncover and rephase transverse magnetization temporarily saved in longitudinal

Rule for effect of $\alpha$

- RF pulse on transverse mag

Rule for effect of $\alpha$

- RF pulse on longitudinal mag

Echo when phase path crosses zero

$\phi$ dispersion

RF transverse (unflipped)

Longitudinal (no further phase change)

$\uparrow$

RF transverse (RF pulse: $M_L \rightarrow M_T$)

Longitudinal

$\uparrow$

 longitudinal cannot be flipped to cause rephase

$\phi$ dispersion

RF transverse (unflipped)

Longitudinal (no further phase change)

$\uparrow$

RF transverse (RF pulse: $M_L \rightarrow M_T$)

Longitudinal

$\uparrow$

 longitudinal cannot be flipped to cause rephase
3 - Pulse Echo Amplitudes

- Assume $M^0_z = 1$

RF transmit

RF receive

<table>
<thead>
<tr>
<th>Echo</th>
<th>Time</th>
<th>Amplitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SE_{1,2}$</td>
<td>$t = 2\tau_1$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} e^{-2\tau_1/T_2}$</td>
</tr>
<tr>
<td>$2^\circ$</td>
<td>$t = 2\tau_2$</td>
<td>$-\sin \alpha_1 \sin^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2\tau_2/T_2}$</td>
</tr>
<tr>
<td>$2^\circ$ (&quot;secondary&quot;)</td>
<td>$t = 2\tau_1 + 2\tau_3$</td>
<td>$\sin \alpha_1 \sin^2 \frac{\alpha_2}{2}$</td>
</tr>
<tr>
<td>STE (&quot;stimulated&quot;)</td>
<td>$t = 2\tau_1 + \tau_2$</td>
<td>$\frac{1}{2} \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 e^{-\tau_2/T_1} e^{-2\tau_2/T_2}$</td>
</tr>
<tr>
<td>$SE_{2,3}$</td>
<td>$t = \tau_1 + 2\tau_2$</td>
<td>$[1 - (1 - \cos \alpha_1) e^{-\tau_1/T_1}] \sin \alpha_2 \sin^2 \frac{\alpha_3}{2} e^{-(\tau_1 + 2\tau_2)/T_2}$</td>
</tr>
<tr>
<td>$SE_{1,3}$</td>
<td>$t = 2(\tau_1 + \tau_2)$</td>
<td>$\sin \alpha_1 \cos^2 \frac{\alpha_2}{2} \sin^2 \frac{\alpha_3}{2} e^{-2(\tau_1 + \tau_2)/T_2}$</td>
</tr>
</tbody>
</table>

- $T_2$-dependence in STE (but also $SE_{2,3}$) from temporary "storage" of $M_T$ in $M_L$, then recovery by third pulse
**Hyper Echoes**

1. Ignoring amplitude, both ends of the sphere are spin echo!
   - $\phi_z = 180^\circ$ and $\phi_y = 180^\circ$

2. Rotation of $\vec{M}$ around tilted axis in transverse $x-y$ plane by RF with flip $\alpha$ and phase $\phi = P(\alpha, \phi)$
3. Rotation around $z$ by phase evolution due to freq offset $\omega$ (Bohr offset) and time $t = \gamma(t) + \phi$

- Three symmetries:
  - Solid lines: phase evolution by RF flip or RF
  - Dashed lines: just $180^\circ$ again

**Practical Use**

- Multi-echo example
- Can also use to prepare, then separate read-out

- Practical prob: $180^\circ$ pulses deposit a lot of RF ($6 \times 90^\circ$) -> prob at high fields

- by arranging to get big echo in middle of k-space can get by with much less RF power
GRADIENT ECHOCES - T2*, GE chains

- Initial negative gradient dephases spins
- After t = T of positive gradient, spins rephase
- Does not correct for T2* inhomogeneities
  - So echo amplitude is
    \[ A_E = e^{-t/T2*} \]
- The initial "FID" is not "free" since it is being actively de-phased by gradient, so FID decay
  \[ \frac{1}{T2} < \frac{1}{T2*} < \frac{1}{T2**} \Rightarrow A_E = e^{-t/T2**} \]

- Key difference between spin-echo (SE) and gradient echo (GE)
  - Initial negative gradient dephases spins
  - There are no inhomogeneities cancelled
  - Hence, echoes are T2* weighted, not T2 weighted => more susceptible to inhomogeneities

- Echo trains possible w/ gradient echo (CPMG-like)

- The faster the gradients are switched, the more echoes you get
- EPI hardware => 64 echoes
IMAGE CONTRAST

T1 Saturation-recovery (no echo, just FID)

- Contrast (PD, T1, T2, T2*) depends on magnetization
  not getting back to equilibrium, and then differences in how far away each tissue type is at measurement time

RF

\[
\begin{align*}
M_z \text{ longitudinal magnetization} & \quad M^0_z \\
\text{before first pulse} & \quad \text{M}_z^0 \left(1 - e^{-TR/T1}\right) + \left[\text{ignored}\right]
\end{align*}
\]

\[M_z \xrightarrow{\text{"steady state" after here}} M_z^0\]

- Simple saturation/recovery w/ no echo

- Initial conditions:
  \[M_z \text{ before first pulse} = M_z^0\]
  \[M_z = 0 \text{ immediately after first pulse (i.e., 90° pulse)}\]

- From Bloch eq, \(M_z\) just before second pulse:

\[
M_z^2(O_-) = M_z^0 \left(1 - e^{-TR/T1}\right) + M_z^\text{regrowth-from-zero} \left(e^{-TR/T1}\right)
\]

\[\text{M}_z \text{ before second pulse} \quad \text{M}_z \text{ "regrowth-from-zero" term} \quad \text{M}_z \text{ "left-immediately after pulse" term (U.S. decaying)}\]

- Given:

  (1) 90° pulse
  (2) no \(M_{xy}\) left

\[\rightarrow \text{pure tip: } M_{xy} = M_z\]

- Tip existing mag

\[
M_z^2(O_-) = M_{xy}^2(O_+) = M_z^0 \left(1 - e^{-TR/T1}\right)
\]

- That is, the not-completely-regrown longitudinal magnetization, which depends on T1, but which we cannot record, is completely converted to recordable transverse magnetization.

\[I(r) = C \cdot \rho(r) \left(1 - e^{-TR/T1(r)}\right)\]

- Assume immediate recording of signal

- Spectral dens \(p(r)\) & p.density: underlying equilib. \(M_z^0\)
**IMAGE CONTRAST**

Why imperfect 90° takes multiple flips til steady state

- initial fMRI images are usually discarded (why?)
- because they are brighter than all the rest
- because multiple flip required before steady state

N.B.: B1 imperfections guarantee this situation will occur
(e.g. at 3T, tip angle varies almost 25% across brain)

- at 3T, steady state
  for typical 1-2 sec TR images reached after 8 images
- inversion recovery w/ no echo

- 180 deg pulse reverses longitudinal magnetization
  \[ M_z' = -M_z \]

- recovery to end of first TI from long. part of Bloch eq.
  \[ M_z' = M_z^0 \left(1 - 2e^{-\frac{TI}{T_1}}\right) \rightarrow \text{flipped into transverse by second pulse (180° 90°)} \]
  \[ \text{from 180°, since } -M_z \text{ in second Bloch term} \]
  \[ \text{ignore!} \]

- longitudinal then regrows from zero
  \[ M_z' = M_z^0 \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) \]
  \[ \text{from first Bloch term only} \]

- after second 180°, just change sign again
  \[ M_z' = -M_z^0 \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) \]
  \[ \text{2nd Bloch term} \]

- apply relaxation eq. again
  \[ M_z' = M_z^0 \left(1 - e^{-\frac{TI}{T_1}}\right) - M_z^0 \left(1 - e^{-\frac{(TR-TI)}{T_1}}\right) e^{-\frac{TI}{T_1}} \]

\[ M_z' = M_z^0 \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_1}}\right) \]

\[ \text{--- this is magnetization flipped to transverse, made recordable} \]
**IMAGE CONTRAST**

- Steady state mag (2nd TR) just before 90°
  \[
  M_z (O) = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}} + e^{-\frac{TR}{T_2}}\right)
  \]

- The echo signal (\(M_z^e\)) unlike in simple saturation-recovery FID has an additional delay before it is recorded, so we have to take account of transverse mag relaxation
  \[
  A_E = M_z^0 \left(1 - 2e^{-\frac{(TR-TE/2)}{T_1}} + e^{-\frac{TR}{T_2}}\right) e^{-\frac{TE}{T_2}}
  \]

- If we assume TE much less than TR, then we can simplify:
  \[
  A_E = M_z^0 \left(1 - e^{-\frac{TR}{T_1}}\right) e^{-\frac{TE}{T_2}}
  \]

- Similar equation for SE-IR
  \[
  A_E = M_z^0 \left(1 - 2e^{-\frac{TI}{T_1}} + e^{-\frac{TR}{T_2}}\right) e^{-\frac{TE}{T_2}}
  \]
**Contrast - 4**

**IMAGE CONTRAST**

GRE w/ small tip angle

- Use basic longitudinal relaxation from Bloch eq, again
- Assume $M_{x'y'}(O_-) = 0$ → transverse dephased before next pulse

$$M_{2'}(O_-) = M_o(1 - e^{-TR/T2}) + M_{2'}(O_+) e^{-TR/T1}$$

- Assume we have a small tip angle:
  $$M_{z_x} \cos \alpha \rightarrow M_{z_x}$$
  $$M_{2'}(O_+) = M_z(0_-) \cos \alpha$$

$$M_{2'}(O_-) = M_o(1 - e^{-TR/T1}) + M_{2'}(O_-) \cos \alpha e^{-TR/T1}$$

- Assume we are in dynamic equilibrium:
  $$M_{2'}(O_-) = M_{2'}(O_-) = M_{2'}^{ss}(O_-)$$

**pre-pulse**
**steady state**
**longitudinal**

**post-pulse**
**transverse**
**magnetization**

$$M_{2'}^{ss}(O_-) = \frac{M_o(1 - e^{-TR/T1})}{1 - \cos \alpha e^{-TR/T1}}$$

$$M_{x'y'}(t) = \frac{M_o(1 - e^{-TR/T1}) \cdot \sin \alpha e^{-\frac{t}{T_2^*}}}{1 - \cos \alpha e^{-TR/T1}}$$

**gradient echo amplitude**

$$A_e = \frac{M_o(1 - e^{-TR/T1}) \cdot \sin \alpha e^{-\frac{TE/T_2^*}}}{1 - \cos \alpha e^{-TR/T1}}$$

**T1 contrast mainly depends on flip angle, not TR → cos \(\theta\) = 1 → eliminates T1 weight since denominator = numerator**
- Saturate, wait for contrast $T_1$, invert, wait for contrast $T_2$ FLASH (center out)

A) $M_z'_{(just \ after \ 90^\circ)} = 0 \ (\text{perfect } 90^\circ)$

B) $M_z'_{(after \ TD)} = M_z^0 \left(1 - e^{-TD/T_2}\right) \ (\text{Blach term \ #1})$

C) $M_z'_{(just \ after \ invert)} = \cos \phi \ M_z^0 \left(1 - e^{-TD/T_2}\right)$

D) $M_z'_{(after \ TI)} = M_z^0 \left(1 - e^{-TI/T_2}\right) + \left[\cos \phi \ M_z^0 \left(1 - e^{-TD/T_2}\right)\right] e^{-TI/T_2}$

E) $M_z'_{(after \ \text{first \ pulse})} = M_z^0 \left[1 - \left[1 - \cos \phi \left(1 - e^{-TD/T_2}\right)e^{-TI/T_2}\right]\sin \alpha\right]$
MAGNETIZATION TRANSFER CONTRAST

- protons in macromolecules & bound to membranes have wide range of resonant freqs ("bound") \( \Rightarrow T_2 = 1 \text{ msec} \)
- free protons in blood, CSF, water have narrower range of resonant freqs ("free") \( \Rightarrow T_2 = 50 \text{ msec} \)

- mag transfer pulse sequence
  1) off-center freq pulse to hit "bound" (but don't hit water too hard)
  2) wait for magnetization transfer from saturated
     longitudinal \( M_z \) of "bound" \( \rightarrow M_z \) of "free"
  3) result of transfer \( \Rightarrow \) attenuation

\( \Rightarrow \) N.B. this always happens a little (cf. T1-weighted, T2-weighted)
   something to keep in mind if hard pulse (wide freq)

- used to increase contrast in TOF
  TOF (not explained) bright vessels from inflow fresh spins
  MT - contrast added: suppress tissue but not blood

- view w/ MIP: maximum intensity projection along lines

max max \( \Rightarrow \) view as movie
**SIGNAL-TO-NOISE, CONTRAST-TO-NOISE**

- Signal-to-noise defined as: \( \text{SNR} = \frac{\text{avg obj signal}}{\text{s.d. non-object region}} \)
- Temporal SNR: \( \text{ESNR} \)
- "Contrast" is a difference
- Contrast-to-noise ratio:
  \[ \text{CNR}_{AB} = \frac{\overline{Y}_A - \overline{Y}_B}{\sigma_n} = \frac{\text{SNR}_A - \text{SNR}_B}{\sigma_n} \]

**Spin-echo:**
\[ A_E = M_z (1 - e^{-TR/T1}) e^{-TE/T2} \]

**Gradient echo:**
\[ A_E = \frac{M_z (1 - e^{-TR/T1}) \sin \alpha e^{-TE/T2\lambda}}{1 - \cos \alpha e^{-TR/T1}} \]

**General rules:** Spin-echo, long TR GE

<table>
<thead>
<tr>
<th>Proton density weighted</th>
<th>TR (\gg) T1 (no T1 diffs)</th>
<th>TE (\ll) (no T2 diffs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T1-weighted</td>
<td>TR (\approx) T1 (big T1 diffs)</td>
<td>TE (\ll) (no T2 diffs)</td>
</tr>
<tr>
<td>T2-weighted</td>
<td>TR (\approx) T2 (big T2 diffs)</td>
<td>TE (\ll) (no T2 diffs)</td>
</tr>
</tbody>
</table>
SIGNAL-TO-NOISE S/N

- Generalized dependence of SNR on 3D imaging parameters

\[
\text{SNR/voxel} \propto \Delta x \Delta y \Delta z \sqrt{N_{r}N_{x}N_{y}N_{z}\Delta t}
\]

- Size (volume) of voxels (with the number of voxels held constant), linear effect on S/N
  \[\text{e.g., } 3 \times 3 \times 3 \text{ mm} \rightarrow 4 \times 4 \times 4 \text{ mm} \rightarrow \frac{64}{27} = 2.37 \text{ times better S/N}\]

- More voxels (with size of voxels, \(\Delta t\) per read step constant), \(1^n\) effect on S/N
  \[\text{e.g., } 64 \times 64 \rightarrow 128 \times 128 \rightarrow \frac{128 \times 128}{64 \times 64} = 2 \text{ times better S/N}\]

- # acquisitions \(1^n\) better S/N
  \[\text{e.g., } 1 \text{ acq} \rightarrow 2 \text{ acq} \rightarrow \frac{12}{11} = 1.09 \text{ times better S/N}\]

- Larger timestep during readout, \(\sqrt{\Delta t}\) better S/N

\[
\Delta t = \frac{1}{\text{BW}_{\text{read}}}, \quad \text{digitization timestep during echo acquisition}
\]

- \(\text{BW}_{\text{read}}\) determined by cutoff freq, analog lowpass filter
- \(\Delta t\) controls BW because low-pass cutoff has to be set higher for smaller (higher freqdetecting) \(\Delta t\)
- must filter out freq's > \(f_{\text{max}} = \frac{1}{2\Delta t}\) because they alias
**Complex Algebra**

- **real/imaginary**
  - add: \((r_1, i_1) + (r_2, i_2) = (r_1 + r_2, i_1 + i_2)\)
  - mult: \((r_1, i_1) \times (r_2, i_2) = (r_1 r_2 - i_1 i_2, r_1 i_2 + i_1 r_2)\)

- **angle/phase**
  - add: \((A_1, \phi_1) + (A_2, \phi_2) = (A_1 \cos \phi_1 + A_2 \cos \phi_2, A_1 \sin \phi_1 + A_2 \sin \phi_2)\)
  - multiply (non-commutative): \((A_1, \phi_1) \times (A_2, \phi_2) = (A_1 A_2, \phi_1 + \phi_2)\)
  - divide: \((A_1, \phi_1) \div (A_2, \phi_2) = (A_1 / A_2, \phi_1 - \phi_2)\)

- **complex to real power**: \((A, \phi)^n = (A^n, n\phi)\)

- **e to \(i\phi\)**
  - expand as series
    - recognize \(\cos\), \(\sin\) series
    - \(e^{i\phi} = \cos \phi + i \sin \phi\)
  - **the real "e" to "purely imaginary" power**
    - \(e^{i\phi} = \cos \phi + i \sin \phi\)
    - vector on unit circle
  - \(e^{i\phi n} = (\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi\)

**Fourier Transform**

- \(H(f) = \int h(t) e^{-2\pi i ft} dt\)
- \(H(\Theta f) = \int h(t) e^{i 2\pi f t} dt\)

**Convolution Theorem**

- \(F[g(x) \ast h(x)] = G(k) \ast H(k)\)
  - because of FFT, faster if kernel not small

**Convolution**

- \(f(x) = g(x) \ast h(x) = \int g(z) \cdot h(x-z) dz\)
- \(E(x) = g(x) \otimes h(x) = \int g(z) \cdot h(x+z) dz\)

- **Cross-correlation**
  - \(f(x) \ast g(x) = \int f(z) \cdot g(x-z) dz\)

- **the Fourier transform of two functions multiplied by each other equals the convolution of the Fourier transform of each function**
Fourier transform (I)

\[ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i \frac{2\pi ft}{f_0}} dt \]

- How to calculate \( H(f) \) for one \( f \) (\( f=3 \)):
  (real signal: only need 2 correlations)

\[
\begin{align*}
\text{real signal} & \quad \rightarrow \quad \text{real frequency domain} \\
\text{imaginary signal (zero)} & \quad \rightarrow \quad \text{imaginary frequency domain}
\end{align*}
\]

\[
\begin{align*}
\cos & \quad \rightarrow \quad \text{integrate/sum these multiplies across all } t \\
\sin & \quad \rightarrow \quad \text{i.e., two correlations}
\end{align*}
\]

\[
\begin{align*}
3 & \quad \rightarrow \quad \text{real frequency domain} \\
3 & \quad \rightarrow \quad \text{imaginary frequency domain}
\end{align*}
\]

\[ b t \text{ is a phase angle} \]
\[ b \cdot t = \omega t \quad \text{cyc/sec \cdot sec = cyc} \]

Fourier integral terms written out

\[
\begin{align*}
h(t) \cos(2\pi ft) + h(t) \sin(2\pi ft) \\
h(t) \sin(2\pi ft) + h(t) \cos(2\pi ft)
\end{align*}
\]

4 correlations

Fourier integral w/real signal

\[
\begin{align*}
h_r(t) \cos(2\pi ft) - h_r(t) \sin(2\pi ft)
\end{align*}
\]

2 correlations

like correlating with \( \sin \) and \( \cos \) (at each freq) so we get phase (at each freq.)
Fourier transform (1b)

\[ e^{i\phi} = \cos \phi + i \sin \phi \]
\[ e^{-i\phi} = e^{i(-\phi)} \]
\[ = \cos (-\phi) + i \sin (-\phi) \]
\[ = \cos \phi - i \sin \phi \]

- \( \cos \) is an "even" function, \( \sin \) is an "odd" function

An orthogonal decomposition

- think of discretely sampled \( \sin(6x) \), \( \cos(6x) \) as vectors

\[ \text{Corr} (\vec{V}_1, \vec{V}_2) \equiv \text{projection of } \vec{V}_1 \text{ onto } \vec{V}_2 \equiv \vec{V}_1 \cdot \vec{V}_2 \]

\[
\begin{align*}
\text{Corr} (\cos b_1 x, \sin b_1 x) &= 0 \\
&= \sin \text{ & cos of same frequency are orthogonal} \\
&= \sin 2x \quad \cos 2x \\
\text{Corr} (\sin b_1 x, \sin b_2 x) &= 0 \\
&= \text{different integer freqs of } \sin (\text{or } \cos) \text{ are orthogonal} \\
&= \sin 2x \quad \sin 3x \\
\text{Corr} (\cos b_1 x, \sin b_2 x) &= 0 \\
&= \text{as above}
\end{align*}
\]

- in the continuous case, orthogonal functions defined as:

\[
\int_{x=\text{hi}}^{x=\text{lo}} f(x) g(x) \, dx = 0
\]
UNDERSTANDING INVERSE FOURIER TRANSFORM AS ANOTHER CASE OF CORR W/ COS, SIN

- start with spike in image domain
- take example of spike at \( x = 0 \)

\[
\begin{bmatrix}
\cos(x), \cos(2x), \cos(kx)
\end{bmatrix}
\]

all freq's correlate w/ spike at \( x = 0 \)

- if spike is moved away from zero, the frequency spectrum obtained by correlating cos with spikes oscillates

- to see why this is, since spike is all zero except at spike, the dot product for a given frequency only depends on the value of the \( e^{-j2\pi kx} \) cos and sin at location of spike

real component of \( \text{FT} \)

- pos, pair (real) spikes same dist from origin
- pos/neg, pair (imaginary) spikes, same dist orig.
- one spike at distance from origin

\( \rightarrow \) this is one way of thinking about what one point in \( k \)-space means, via correlating it w/ cos's, sin's to get periodic result in image space (inverse \( \text{FT} \))
FOURIER TRANSFORM OF AN IMAGE (2)

(1) real image  imaginary image
   (Zero)  (Zero)
   X ->  X ->

Fourier Transform

Inverse Fourier Transform

(2) amplitude image  phase image
   (Zero)  (Zero)
   X ->  X ->

What you see on screen

view complex vectors directly

(3) complex vectors  zero vectors
    X ->

-3 equivalent representations of image & spat. freq. space
FOURIER TRANSFORM OF REAL IMAGE (2)

- what a single k-space point looks like for real image (polar coordinates $A, \phi$ instead of $r, \theta$)

**image space**

- offset of stripes is k-space phase
- brightness of stripes proportional to k-space amplitude

**k-space** (spatial freq. space)

- distance from center is stripe spacing
- angle of point perpendicular to angle of stripes

inverse Fourier transform

(should be all zero,
not same as "stripe phase above")

(value from 0 to 360°)

- Cartesian dimension of k-space — x- and y- spatial freq.

N.B.: each dimension of spatial freq. space (k-space) from correlation w/ sin, cos — don’t confuse $k_x, k_y$ w/ sin, cos!

N.B. — increasing one 1D component increases the spatial freq of the 2D wave and rotates it
Fourier Transform of Image (4)

3 equivalent representations of complex numbers
in image space and spatial-freq. space (k-space)
- example: cosine in image space, then shifted in x-dir

Real Image

\[ I(x, y) = \cos(x) \]

FT of \( I(x, y) \)

\[ k_x = 1, \quad k_y = 0 \]
\[ k_x = -1, \quad k_y = 0 \]

FT of \( I(x, y) \)

\[ \text{FT} \]
\[ \text{FT}^{-1} \]

Halfway between cos and sin
(Shifted 45° to right)

Complex

Max pos

Max neg

N.B.: an example of the "Fourier Shift Theorem" (see below)

45° rot compared to complex above
**FOURIER TRANSFORM OF IMAGE (5)**

- (cont.) center of k-space (real image)
- complex image

### REAL IMAGE

\[ I(x, y) = 1 + \cos(x) \]

[Diagram showing real image with center marked as zero, with max and zero labels.

**FT OF REAL IMAGE**

\[ H(k) = \int \! h(x) \cdot e^{-i2\pi kx} \, dx \]

- avg image brightness = \( I(\text{real}) \)
- positive center k-space

[Diagram showing FT and inverse FT with complex and zero labels.

### COMPLEX IMAGE

\[ I(x, y) = \cos(x) - i \sin(x) \]

[Diagram showing complex image with flat and zero labels.

**FT OF COMPLEX IMAGE**

- ("missing" spike results in single spike correlating with cos and sin)
  - N.B.: this k-space is non-Hermitian
  - k-space will only have Hermitian symmetry if image is real

- Hermitian symm. when complex conjugate (\( \bar{I} \)) is equal to function w/imag arg:

1D: \( H(k) = H^*(k) \)

2D: \( H(-k_x, k_y) = H^*(k_x, k_y) \)

[Diagram showing complex labels with note: N.B. this is also exactly what a gradient does to image space!]

- spike only on one side of k-space

- N.B. this is like what an artifact "spike" does: tho it would have real, phase
**FOURIER TRANSFORM OF IMAGE**

- (cont.) x- and y-spatial freqs.
- special case: real image from sum of reals
  
  **REAL IMAGE**

\[ I(x,y) = \cos(x) + \cos(y) \]

N.B. adds but doesn't rotate stripes

\[ I(x,y) = \cos(x+y) \]

rotates stripes!

\[ \text{FT} \quad \text{FT}^{-1} \]

\[ \text{FT} \quad \text{FT}^{-1} \]

- Remember, single k-space point transforms to complex img.
- but if Hermitian symmetry, imaginary components cancel

- Since all we want in image space reconstruction is real component, can just add real components of complex vectors at each image space point for every complex image corresponding to each k-space point

- N.B.: the k-space phase will affect offset of real-valued image space cosinusoid

- Therefore for real-valued image, we can visualize inverse FT as real-valued sum of offset real-valued cosinusoids

- N.B. cannot do this with MRI k-space data since phase errors (incl. multiple wraps) mess up real component - must use amplitude img
**Gradient Coils**

- gradient coils for $x, y, z$ generate approximately a linear gradient in the strength of the $z$-component of the magnetic field $B_z$.

- for example, the $x$ gradient coil induces a ramp in $z$-component of the magnetic field when moving in the $x$-direction:

$$B_{G,x} = G_x x$$

*Since a pure linear gradient of $B_{G,z}$ in only the $x, y, z$ directions is not possible according to Maxwell equations, each gradient coil also induces a magnetic field that has components in the $x$- and $y$-direction ($B_{G,x}$ and $B_{G,y}$).*

- the other magnetic field components are usually ignored because they are so small relative to $B_{G,z}$, since $B_{G,z}$ is added to $B_0$, and since $B_0$ is much stronger than $B_{G,z}, B_{G,y}, B_{G,y}$.

- since standard reconstruction methods assume the existence of "non-Maxwellian" gradient fields, spatial distortion is introduced.

- the Maxwellian terms $B_{G,x}$, $B_{G,y}$ are known; can be included in the recon. process.

\[ \Delta \phi G_x(x) \approx -\frac{xG_x^2}{2B_0} \]
SLICE SELECTION ($G_z$)

- slice select gradient on during RF stim

\[ f = \frac{x}{2\pi} \left( B_0 + B_{G_z} \right) \]

- protons here can only be excited by a narrow band of radio frequencies

- to apply a pulse containing a narrow band of frequencies, we use a sinc pulse envelope (Fourier transform of a narrow freq. band)

\[ = \sin(x)/x \]

- this excites protons in a narrow slab

- since the slice-selection gradient introduces (space-dependent) phase shifts (see freq encode) these have to be removed by a post-excitation rephasing $z$-gradient

- approximation from assuming tip occurs instantaneously in middle

- valid for small tip: 90° $\rightarrow$ 52°

- in practice: adjust to max, use crusher to kill spurious transverse on 180°
PULSES FOR SLICE SELECTION

- Fourier transform approach to slice-selective pulse (linear approx. even tho tipping is non-linear)

\[ \tilde{B}_1(t) \propto \int_{f=-\infty}^{f=\infty} p(f) \cdot e^{-i2\pi ft} \, df \]

frequency selection function

\[ \tilde{B}_1(t) = A \cdot f_w \cdot \text{sinc}(\pi f_w t) \cdot e^{-i2\pi ft_c} \]

- Amplitude controlling flip angle (controls slice width)
- Sinc envelope determined by freq. width \( f_w \) at center freq. \( f_c \)
- Modulation (complex)
- Larmor oscill., at center freq.

Sinc envelope width inversely proportional to \( f_w \)

\( f_w \) = free width

\[ \text{Larmor oscill., at center freq.} \]

\[ \text{Sinc envelope width inversely proportional to } f_w \]

### Fourier Transform Pairs, Rules

- Multiplication in one domain equals convolution in other:

\[ \mathcal{F}\{g(t) \cdot h(t)\} = G(f) \ast H(f) \]

- Convolution with delta function equals other function to impulse center

### Fourier Transform Solution to: \[ \tilde{B}_1(t) \]
SLICE SELECT RF PULSES

Interleaved Acquisition \rightarrow better S/N b/c imperfect slice profile
\[ \text{spin history prob if motion} \]

Common RF pulses
- non-selective pulse ("hard" pulse)
- standard slice select sinc
- Gaussian
  \[ \text{pulses need to be "apodized" (have "foot" removed)} \]
  \[ \rightarrow \text{multiply by function so begin/end of pulse is differentiable} \]

Fat Saturation
- fat protons have chemical shift causing resonant freq offset
- add phase offset not due to gradients, RF
  \[ \text{fix by off-water-resonance 90° (saturation) pre-pulse centered on fat freq} \]
  \[ \rightarrow \text{need high quality (narrow-freq) pulse to avoid saturate water!} \]

How To
1) Fat sat pulse
2) wait T2 so fat signal decays, but no T1 regrowth of fat
3) RF stim for water "protons - of interest"

Adding Another Gradient Tilts Slice

\[ \text{with 3 gradients on, can excite arbitrary angle plane} \]
\[ \text{translate plane by changing either gradient amplitude} \]
\[ \text{or RF freq band: } \]
WHY "FREQUENCY-ENCODING" IS A MISNOMER

- comes from original analogy in Lauterbur (1973):

<table>
<thead>
<tr>
<th>Spectroscopy</th>
<th>Imaging</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) chemical shift change freq. ⇒ gradient changes freq.</td>
<td></td>
</tr>
<tr>
<td>2) stimulate w/ broadband RF ⇒ same</td>
<td></td>
</tr>
<tr>
<td>3) time-sample FID containing multiple freqs ⇒ same</td>
<td></td>
</tr>
<tr>
<td>4) FT of FID to get spectrum ⇒ FT of FID to get Δx offsets</td>
<td></td>
</tr>
</tbody>
</table>

- this is technically correct (FT of FID) but highly misleading
  - e.g., phase-encoding (tuning a different gradient ON and OFF before recording FID) seems to be something completely different since OFF gradient can't affect freqs in FID

- the "k-space" perspective is a "Copernican Turn"
  - idea is that data is not a set of samples of a time-domain signal generated by multiple chemical-shift like frequencies
  - rather, it is a set of samples of a frequency-domain signal, each sample generated by multiple spatial locations (which are analogous to multiple time points)

- i.e., the 'direction' of the FT (Fourier transform) is reversed:

<table>
<thead>
<tr>
<th></th>
<th>Signal</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>spectroscopy</td>
<td>samples of oscillations in time-domain</td>
<td>FT → frequency-domain spectrum of shifts</td>
</tr>
<tr>
<td>MRI</td>
<td>samples of spatial freq. in freq. domain</td>
<td>FT⁻¹ → spatial object (like a time-domain signal)</td>
</tr>
</tbody>
</table>

- the original analogy only 'works' because $\text{FT} \approx \text{FT}^{-1}$ (except sign change)
FREQUENCY ENCODING (1)

- Frequency encode gradient ($G_x$) causes precession rates to vary linearly in $x$-direction

- Different frequency signals are mixed together and recorded as a 1-D signal over time

- A Fourier transform, which can convert back and forth between $x$-position (cf. time) and spatial frequency (cf. temporal freq.) is done on signal

- Spatial frequencies get confused/conflicted with precession frequencies

**N.B.:** Gradient freq. ramp does not record exactly the same as during recording!!

- Therefore, the Fourier transform is used to convert position-dependent precession frequencies into spatial position

- The spatial frequency increases for each time point in the readout

- The precession freq. ramp is constant each timestep
FREQUENCY ENCODING (2) - connect intuition - why phase critical

- “Frequency”-encode gradient ($G_x$) turned on during during echo causes precession rates to immediately vary with x-position.

- At beginning of gradient on, the phase of signal coming from each x-position is the same. Summed phase angle is what we measure.

- Early after gradient on, phase advances (because of faster precession frequency) arise with greatest phase advance at largest x-position.

- Later during gradient on, phase advances cause multiple wraparounds of phase angle across space.

- In practice, the lowest spatial frequency ($\delta = 0$) occurs in the middle of the gradient on time because the phase is “wrapped” negatively by a preparatory gradient (to the highest negative spatial frequency) before data collection occurs. Valid RF data samples (after demodulation).
FREQUENCY ENCODING (3) why each datapoint is 1 spatial freq

Standard Fourier transform: (Temporal freq $\leftrightarrow$ time)

$$ H(f) = \int_{-\infty}^{\infty} h(t) \cdot e^{-i 2\pi f t} \, dt $$

"k" is often used instead of "f" for the frequency variable

Imaging equation: (Spatial freq $\leftrightarrow$ space)

$$ S(f) = \int_{-\infty}^{\infty} I(x) \cdot e^{-i 2\pi f x} \, dx $$

Signal strength at one x-position (brightness of image point)

Spin density (spectral density)

Sum across x of object

This is done by RF coil recording sum

Oscillations come from readout phase wrapping, where f is single spatial freq (e.g. 5) and x goes across object

F = Gx(t-TE), that is, spatial freq depend on amount of time gradient was on (this f increases with time!)

Don't confuse with instantaneously changed precession freqs which are constant across entire readout time (for each x position)

One data point (i.e., one spatial freq) during readout (2 components)

G_x

RF

takes

get this single readout point by summing signal across x-position (RF coil records sum)

even though variable is f, it represents one time period during readout

To make image, do inverse Fourier transform of recorded signal S(f)
- Oscillators at $w = \gamma B$ at each position (just $x$ for now)

$$S(t) = \int x M(x) e^{-i \phi(x)} dx$$

- By definition, freq. $w$ is rate of change of phase, $\phi$

$$\frac{d\phi(x,t)}{dt} = w(x,t) = \gamma B(x,t) \quad \text{and} \quad \phi(x,t) = \int_0^t w(x,t) dt = \gamma \int_0^t B(x,t) dt$$

- Assuming phase initially 0, $B$ affected by gradients

$$B(x,t) = B_0 + G_x(t) x$$

$$\phi(x,t) = \gamma \int_0^t B_0 dt + \left[ \gamma \int_0^t G_x(t) dt \right] x$$

$$= \omega_0 t + 2\pi k_x(t) x$$

- Demodulation removes the $B_0$-caused carrier frequency $e^{-i \omega_0 t}$ from the first equation

$$S(t) = \int x M(x) e^{-i 2\pi k_x(t)} dx$$

- Amplitude of each oscillator gradient-controlled phase
Phase-encode Gradient $G_y$  

- Turn on gradient after excitation but before readout

- Different levels of $G_y$
  
- Higher levels of $G_y$ (slope of $B_z$ in $y$-direction!)
  
- Higher spatial freq. (more phase wraps) in $y$-direction

- Phase wraps persist after phase-encode gradient off

- Read-out gradient ($G_x$) phase wraps then add to phase-encode phase

**2D Imaging Equation**

$$S(k_x, k_y) = \iiint_{x \times y} I(x, y) \cdot e^{-i 2\pi (k_x x + k_y y)} \, dx \, dy$$

- Signal recorded at single time point (one readout point)
  
- Complex signal (from phase-sensitive detection)
  
- Done by RF coil
  
- Scalar (what we try to reconstruct)
  
- Fixed strength
  
- Image (strength of magnetization at each $x,y$ point)
  
- Phase (vector of unit length and particular angle which is function of $G_x$ and $G_y$)
  
- Phase angle (of scalar magnetization!) in rotating frame, set by gradients

- Ignoring relaxation, spatial frequency $k_x$ and $k_y$ have no "inertia" — they stay wherever the gradients last left them
3-D IMAGING - two phase-encode gradients

- use z-gradient for 2nd phase-encoding instead of slice selection
- excitation of whole slab (slice-select is whole brain)
- simple spin echo example (in real life, usu. done with echo trains [FSE] or small flip angle to allow short TR [3PGR])

\[ S(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(x, y, z) e^{-i2\pi (k_x x + k_y y + k_z z)} dx dy dz \]

- i.e., freq-encode phase, first phase-encode phase, and second phase-encode phase all just add (= 3D rotation of phase stripes)
- S/N much better than 2-D because each entire excited volume contributes to signal from each pulse instead of just slice
  \[ \text{phase stripes created throughout volume vs. slice:} \]

N.B. this ignores relaxation effects for now
PHASE & FREQ, 2D & 3D

Since the phase-encode gradient and the freq-encode gradient both affect phase, the result is a rotation of phase "stripes" when the two add.

Stripes here represent complex value.

Phase of whole image summed to one (complex) number by RF coils.

- Successive readout steps:
  - More rotation
  - Higher spatial freq.

Small phase encode $G_y$  
Large phase encode $G_y$

TE

Read out $G_x$

$t_1, t_2, t_3$

- 3D phase encode w/ $G_y$ and $G_z$ starts rotated in y-z plane

Large phase encode $G_z$

Small phase encode $G_y$
GRADIENTS MOVE K-SPACE LOCATION OF DATA POINT

- k-space (spatial-frequency space) location is set by integral of gradient over time up to recording point

\[ k = \int_0^t G(t) \, dt \]

- all of the following gradients end up at the same point in k-space:

Frequency-encode FID

\[ G_x \]

\( \rightarrow \)

\[ k_x \quad k_y \]

Frequency-encode gradient echo

\[ G_x \]

\[ \rightarrow \]

\[ k_x \quad k_y \]

N.B. 180° moves to conjugate point

Frequency-encode spin-echo (plus gradient echo!!)

\[ G_x, G_y \]

\[ \rightarrow \]

\[ k_x \quad k_y \]

Phase-encode then frequency encode gradient echo

\[ G_x, G_y \]

\[ \rightarrow \]

\[ k_x + G_y \]
**IMAGE RECONSTRUCTION**

\[ S(k_x, k_y) = \sqrt{ I(x, y) e^{i 2\pi (k_x x + k_y y)} \, dx \, dy} \]

\[ I(x, y) = \sum_{k_y} \sum_{k_x} S(k_x, k_y) e^{i 2\pi (x k_x + y k_y)} \, dk_x \, dk_y \]

- In practice, finite number of samples, \( N \) and \( M \), are collected.
- \( k_x \) and \( k_y \) directions of \( k \)-space (integral \( \Rightarrow \) discrete sum).

\[ I(x, y) = \sum_{m=-M/2}^{M/2} \left[ \sum_{n=-N/2}^{N/2} S(n, m) e^{i 2\pi n \Delta k_x \Delta k_y} \right] e^{i 2\pi m \Delta k_x x \, \Delta k_y y} \]
- Aliasing occurs in the spatial domain.
- Replicas overlap, causing high-frequency ringing.
- As above, but with spatial blurring.
- Aliasing occurs in the frequency domain.
- Finite frequency range is sampled, but not at the Nyquist rate.
- Limited in range of frequencies sampled.
- NB: aliasing less familiar when result of limited frequency domain sampling is sampled in space.
- Limited in rate of sampling (ΔK).
- Must consider effects of sampling limited points in K-space.
- Insufficient sampling in K-space.
- Thus, finer sampling of same range of frequency gives poorer results.
- Inverse Fourier transform.
- Sampling in space.
**UNDER/OVER SAMPLE**

**More Examples**

\[
\text{FOV}_x = \frac{1}{\Delta k_x}
\]

\[
\delta_x = \frac{\text{FOV}_x}{N} = \frac{1}{N \Delta k_x}
\]

- **FOV (distance to repeat)** is reciprocal of spatial frequency sampling interval.
- **Pixel size** is FOV divided by *K*-space sample count.

**3 More Examples** (not incl. less samples to same spat. freq [bottom last page])

- **Basic Image**
- **Same num samp. to 2x spat. freq.**
  - (i.e. gradients stronger or time ON longer)
  - \( N = 10 \)
  - \( k_x = 5 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV} = 2 \)
  - \( \delta_x = 0.1 \)

- **2x num. samples to same spat. freq.**
  - (i.e. gradients weaker or time ON shorter)
  - \( N = 10 \)
  - \( k_x = 10 \)
  - \( \Delta k_x = 0.5 \)
  - \( \text{FOV} = 2 \)
  - \( \delta_x = 0.05 \)

- **2x number samples to 2x spat. freq.**
  - (i.e. gradients stronger or time ON longer)
  - \( N = 20 \)
  - \( k_x = 5 \)
  - \( \Delta k_x = 1 \)
  - \( \text{FOV} = 1 \)
  - \( \delta_x = 0.05 \)

**Space**

- **Basic image**
- **Square pix**
  - \( x \)-pix half width
  - \( x \)-pix half width
  - \( x \)-pix half width
  - \( x \)-pix half width

- **Square pix**
  - \( x \)-pix half width
  - \( x \)-pix half width
  - \( x \)-pix half width

- **Twice x-pix count**
  - twice x-pix count
  - twice x-pix count
  - twice x-pix count

- **Same FOV**
  - same FOV
  - same FOV
  - same FOV

- **Phase oversamp**
  - this is "phase oversamp"
  - this is "phase oversamp"

- **Scanner makes square image "wrap" occurs**
  - scanner makes square image "wrap" occurs
  - scanner makes square image "wrap" occurs

- **Scanner crops to square replicas move out**
  - scanner crops to square replicas move out
  - scanner crops to square replicas move out

- **This is decrease pixel size w/o change FOV**
  - this is decrease pixel size w/o change FOV
  - this is decrease pixel size w/o change FOV
Fourier Transform Solution to Replicas

1. Image/brain space
   \[ \mathcal{F} \]
   \[ \ast \] convolve
   \[ \Rightarrow \]

2. Sampled data
   \[ \mathrm{Spatial} \text{ frequency} \]

\[ \times \] multiply

\[ \Rightarrow \]

Limit approach to Fourier transform of conv

\[ \uparrow \uparrow \]

\[ \Delta k \]

\[ \text{Fov} \]

\[ \Delta k = \frac{1}{\text{Fov}} \]

Useful FT's

\[ \text{Rect} \]

\[ \text{Rect}(\frac{x}{w}) \xrightarrow{\mathcal{F}} W \cdot \text{sinc}(\pi Wk) \]

\[ \text{Gaussian} \text{ (special case)} \]

\[ e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi k^2} \]

\[ \text{Gaussian} \text{ (adj width)} \]

\[ e^{-\alpha x^2} \xrightarrow{\mathcal{F}} \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\pi k^2}{\alpha}} \]

\[ \text{Comb} \]

\[ \sum_{n=-\infty}^{\infty} \delta(x - \frac{n}{\Delta k}) \xrightarrow{\mathcal{F}} \Delta k \sum_{p=-\infty}^{\infty} \delta(k - p \Delta k) \]
**Point Spread Function**

\[ \hat{f}(x) = \Delta k \sum_{n} S(n \Delta k) e^{i 2\pi n \Delta k x} \]

- Set true image to S-function, then measured signal is:
  \[ S(m \Delta k) = 1 \]

- Substitute into recon to get PSF:
  \[ h(x) = \Delta k \sum_{n} e^{i 2\pi n \Delta k x} \]

- Simplify:
  \[ h(x) = \Delta k \frac{\sin (\pi n \Delta k x)}{\sin (\pi \Delta k x)} \Rightarrow \text{periodic} \]

- That is, image is reconstructed from a sum of sinc's, because the FT of a boxcar pixel in k-space is an image sinc

---

**Image**

- How PSF modifies ideal (infinite k) image
  - Convolving
  - Ringing

**FT**

- Under rect, narrower sinc
- FT
- Multiply

**Spatial freq. data**

- Acquisition window (truncates hi spat. f)
GENERAL LINEAR INVERSE RECON FOR MRI

\[ S(k_x) = \int \frac{I(x) e^{-i \frac{2\pi}{\text{perfect spin phase}} k_x x}}{dx} \]
\[ I(x) = \int_{k_x} S(k_x) e^{+i \frac{2\pi}{\text{imperfect spin phase}} k_x x} \text{dk}_x \]

Recon eq. \( \to \) inv. problem

\[ s = F i \]

[Matrix] vectors have complex entries

Linear "forward solution"

can build in any measurable prior

could insert x-y-dependent gradient non-lin.

\[ F_{x,y,t} = g(x,y) e^{-i f(x,y)} e^{-(nT \pm m \Delta t \pm TE)/2} e^{-i \gamma \Delta B(x,y) \Delta t} \]

T2 decay

EP Error
(xy dep.)

Freq + phase

multi-coil

\[ s = \begin{bmatrix} f_{x,y} \end{bmatrix} \]

\[ i = F^+ s \overline{\text{over-determined}} \]

More
Powerful
Inverse

\[ F^+ = (F^T F)^{-1} F^T \]

(x,y) \( \rightarrow \) "small"

(x,y, coils) \( \rightarrow \) 10x bigger

For 4 coils

\[ i = \left[(F^T)^{-1} F^T\right] s \]

Slice-by-slice

assume slice select swamps others
FAST SPIN ECHO (FSE)  RARE, FSE, 3DFSE

- one 90° pulse followed by multiple 180° pulses (e.g., 8)
  each with a different phase-encode gradient

- each phase “winder” is “unwound” because leftover phase would
  be re-focused away by 180° (vs. EPI where it persists between blips)

- the “effective TE” is the TE when center of k-space
  is collected (largest effect on contrast, largest echo)

- each subsequent echo has more T2 decay: \( E_n = e^{-nTE/T2} \) \( n = 1, 2, ..., M \)

- by arranging to collect \( ky = 0 \) early, PD-weighted instead of T2-weighted

- possible to correct different T2-weighting of echoes by
  estimating T2 curve from \( G_y = 0 \) echo train

- 3DFSE — like 2D except
  wind/unwind added to thick
  Slice select (w/currents on 180°)

N.B.: only one read-reflex
subsequent 180° reset
(meaning low dephasing)

N.B. all those 180°
pulse deposit a lot
of RF power:
90° + 180° = 45x power 30°
MULTI-SLAB 3DFSE, PROBLEMS

- echoes die out quickly by $e^{-t/T_2}$
- since echoes after 90° limited to <30, can't fill 3-D k-space in a reasonable time
- SAR constraint $\text{SAR} \propto B_0^2 T_2^2 \Delta f$
  $\rightarrow$ 180° pulses deposit 4-6x power of 90°
- "multi-slab" is halfway between slices and single-slab

- problem at slice boundaries — esp. movement
- multislab requires slice selective RF pulses $\rightarrow$ longer than non-selective 'hard' pulses

4 ms RO

hard to get under 8 msec inter-echo spacing

limits speed of covering k-space
**SINGLE-SLAB 3D FSE**

- **Regular FSE (180° pulse train)**
- Sub 180° pulses cause each successive pulse to also generate a stimulated echo (STE)
  - This "storage" in z-axis preserves magnetization for longer time
  - Smaller flip angles allow much longer echo trains
  - Enough to collect whole plane of 3-D k-space
  - Different than hyper echoes (not symmetric)
  - Contrast must consider STE

\[
\text{SE} = \sin \alpha, \sin^2 \alpha, e^{-2\gamma / \tau_2} \\
\text{STE} = \frac{1}{2} \sin \alpha_2 \sin \alpha_3 \sin \alpha_3 e^{-2\gamma / \tau_2} e^{-2\gamma / \tau_2}
\]

**Single-slab 3DFSE pulse seq:**
- Variable flip angle (≤1msec)
- Hard (non-selective) pulse not 180°

**RFm:**
- 90°
- 3.9 msec

**Gz, Gy, Gx:**
- Crushers
- Etc to 150 echoes (all SEs plus STEs)

**FID, spin echo, FID:**
- Spin echo plus stimulated echo

**Long TE eff (T2):**
- Don't collect
- Kx
- Ky
- Echo a
- Echo 2
- Echo 5
- Echo 15
- Don't collect
- Train 1
- Train 26

**Short TE eff (PD):**
- Don't collect
- Kx
- Ky
- Echo 20
- Echo 15
- Echo 5

**NB:** time to cent k-space is ≈5X apparent contrast time b/c of "storage"
- E.g. TEeff = 585 ms looks like FSE TE = 140 ms
FAST GRADIENT ECHO (GRASS, FLASH, FISP, SPGR, MPRAGE)

- Small tip so TR can be greatly reduced (e.g., 10 msec, less than $T_2$)
- 'Leakage' undecayed transverse magnetization "unwound" and re-used "spoiled" before next shot

**STEADY-STATE COHERENT (GRASS, FISP)**

- Unwind phase from phase-encode $M_r$ before next pulse (here because $TR < TE$)
- Unwind read gradient, too

$$S = k \sin \alpha \left[ \frac{1}{1 + \cos \alpha \cdot (1 - \cos \alpha)} \right] \exp \left[ - \frac{T_2}{T_1} \right]$$

- Brain 0.1, Fat 0.3, Soft 0.7

**STEADY-STATE SPOILED (SPGR, FLASH)**

- Spoil with random gradient (but this still allows some $\alpha$ refocusing)
- Spoil with gradient plus incremented phase of RF pulses (RF spoiling)
- Good gray-white contrast ($T_1$-weighted)

**NON-STEADY STATE, MAGNETIZATION-PRESERVING**

(Shown as 3D sequence — possible with ones above, too)

**MPRAGE**

- Longitudinal mag. not affected much by low angle pulses

- Preparation pulse $\rightarrow$ strong $T_1$-weighting
- Contrast varies in spatial $\times$ freq-dependent way

- Nominal inversion time
- Effective $T_1$ actually time to $G_z$
- $TR$ that records signal
- TR $\approx 10$ msec
- Preparatory pulse
- Strong $T_1$-weighting
- Contrast varies in spatial $\times$ freq-dependent way

- K-space filled for this $k_z$ (i.e., on $G_z$ phase encode)
- Record $k_y = 0$ here

- WM
- GM
Motivation

- Image values are arbitrary/relative (diff. seqs., manufacturers)
- Uncorrected coil fall-off (receive inhomogeneity) can result in 2-3x differences in voxel brightness
- Uncorrected variation in local $T_1$ field can cause contrast variation

\[ \text{at } 3T, T_1 \text{ can vary by 25\% across the brain} \]
\[ \text{this can invert contrast in a fast gradient echo} \]

Pre-scan normalise

- Collect low-res GE image, receive w/ body coil (no coil fall-off)
- Set parms. to get low GM/WM contrast
- Collect data scan (e.g. MPRAGE) w/surface coils, strong GM/WM
- Use ratio between scans to generate smooth correction field

$T_1$ divided by $T_2$

- MPRAGE $\Rightarrow$ strong $T_1$-Contrast
- SPACE $\Rightarrow$ $T_2$-weighted (no $T_1$ weighting)
- $T_1 / T_2$ removes coil fall-off
- Problems: Noise in regions of low signal

MPRAGE

1st volume $\rightarrow$ PD-weighted

2nd copy of volume $\rightarrow$ max $T_1$-weighted

N.B. SSFP-like in partition, phase-encode dir
- Convert to $-0.5 \text{ to } 0.5$ image: $S = \text{real}$
- Calc. PD & $T_1$ from above $\text{cf. 2 flip angles}$
QUANTITATIVE T1 - HELMS 2-FIIP ANGLE METHOD

- Start with gradient echo signal e.g., dropping T2-decay \( e^{-TE/2} \)

\[
S_{\text{Ernst}} = A \cdot \sin \alpha \cdot \frac{1 - e^{-TR/T1}}{1 - \cos \alpha \cdot e^{-TR/T1}}
\]

- Simplify/linearize/estimate

\[ TR \ll T1 \]
linear approx. of exponentials

Taylor expansion simplification of \( \sin, \cos \), drop small term

Helms et al. (2008)

\[
S \approx A \cdot \alpha \cdot \frac{TR/T1}{\alpha^{2/2} + TR/T1}
\]

- Solve for TD and

\( A \) (proton-density) given signals from 2 diff flip angles

\[
T1_{\text{est}} = 2TR \frac{S_1/\alpha_1 - S_2/\alpha_2}{S_2\alpha_2 - S_1\alpha_1}
\]

\[ A_{\text{est}} = \frac{S_1S_2(\alpha_2/\alpha_1 - \alpha_1/\alpha_2)}{S_2\alpha_2 - S_1\alpha_1} \]

- Tiny error for flip \( \leq 15\) deg.

- Problem: Flip angle varies a lot at 3T (e.g., 25%) from nominal (desired) (e.g., flip series)

- acq. spin-echo and stimulated echo (EVI)

\[
S = k \cdot \sin^3 \alpha \cdot e^{-TE/T2}
\]

\[ S_{\text{STE}} = k/2 \cdot \sin^3 \alpha \cdot \sin 2\alpha \cdot e^{-TF/T1} \]

\[ \alpha = \cos^{-1}\left( \frac{S_{\text{STE}} \cdot e^{-TM/T1}}{S_{\text{SE}}} \right) \]

- 3T map

\[
\text{RF}_{\text{in}} \rightarrow 90^\circ \text{TE}_{1/2} \rightarrow 180^\circ \text{TE}_{1/2} \rightarrow \alpha \text{ TE}_{1/2} \rightarrow \text{TM (x,y,z)} \rightarrow \text{partition (3D)}
\]

\[
\text{Slice} \rightarrow \text{phase} \rightarrow \text{read} \rightarrow \text{RF}_{\text{out}}
\]

- Add EPI-like echo train to each FLASH excite.

Jiru & Klose (2006)
ECHO PLANAR IMAGING, EPI (another fast gradient echo)

- Single shot EPI collects all k-space lines (e.g., 64) after a 90° RF pulse using a train of gradient echoes

RF pulse sequence

- Since there is only one RF pulse per slice, spins never get reset to all-the-same (= zero freq, center of k-space)

- Therefore, the recording point (Δt) in k-space (= spin phase stripe pattern) stays wherever the x and y gradients last left it

- That explains why successive y phase-encode steps are achieved without changing the size of the G_y "blips"

- Echoes are T2*-weighted (gradient echo)

- Contrast mainly determined by echoes near center of k-space, which are only recorded after about 32 echoes
**SPIN ECHO EPI**

why SE-BOLD may be selective for capillary bed

- Standard EPI is a gradient echo method, which results in $T_2^*$-weighting

- Deoxygenated hemoglobin is paramagnetic, which reduces signal in a $T_2^*$-weighted image due to greater dephasing

- The excess of deoxygenated hemoglobin (probably the result of the need to drive $O_2$ into tissue, which requires more $O_2$ in the blood than is actually used) leads to the positive BOLD effect

- Spin echo corrects (cancels) static $T_2^*$ ($T_2$) dephasing, incl. deoxygenated hemoglobin

- If all spins stayed in the same position, spin echo would eliminate the BOLD effect by eliminating dephasing

- Diffusion exposes spins to different fields (reducing gradient echo dephasing)

- Magnetic field gradients produced by large vessels are smoother across space than those produced by small vessels

- For TE = 100 ms, spins diffuse 10's of mm, which is larger than diameter of small capillary, meaning that spins will likely experience different fields over time

- Therefore, spin echo will be less successful at canceling BOLD effect near small vessels (BOLD effect will be reduced near large vessels where diffusion less likely to expose spin to different fields here)

- This argument only works for extracellular spins — intracellular signal in BOLD is large (despite being only 4% by volume) because large gradient produced around red blood cells

- Measure intra/extra v/with bipolar pulse which kills signal in faster moving blood in moderate and larger vessels

\[ \Rightarrow \text{over half of SE-BOLD at LST is venous...} \]
SPIN ECHO EPI

- EPI is a multi-gradient echo pulse sequence.

- "Spin-echo EPI" uses a 180° pulse to add a single spin echo to the contrast-controlling gradient echo through the center of k-space.

- "Asymmetric spin-echo EPI" arranges for the spin echo to occur T msec before the gradient echo, which gives more T2* weighting (for ky = 0 echo).

- The 180° pulse rephasing reduces the T2* signal, which is why the partially rephased asymmetric spin echo has been more commonly used.

- At higher fields, spin echo EPI is more promising.

  - Signal to noise is higher so we can take spin echo hit
  - Contribution from venous blood is reduced, since blood T2 is so short, we can let it decay away before recording...
- Coil fall-off intuitively contains info about location if same brain location imaged by different coils w/ diff. fall-offs

- but what does this look like in k-space?

- Slow variation in RF-field fall-off (e.g., 1-4 cycle/FOV) causes a blur in acquired data in k-space

  (N.B. not addition!)

- To see this, consider multiplication by coil fall-off function in image space, which equals convolution (w/ FT of that function) in k-space – at all spatial frequencies!!

- Simple example w/ "brain" consisting of one spatial freq.

  Image domain
  
  Image ("brain")
  
  × (multiply)
  
  Coil fall-off function
  
  = (equals)
  
  Acquired image

  Forward FT
  
  FT
  
  Spatial freq. domain
  
  Space is sharp image w/ fall-off (not blurred image)

- N.B. inverse FT of k-space data "smeared" in spatial freq.

- "Smeared" means normally orthogonal spatial freq's "leak" to adj. freqs.

  Grappa - construct k-space "kernel" to fill in missing k-space lines by training on fully-sampled data from near k-space center

  Sense - general linear inverse approach

- N.B.: neither would work unless normally ortho. spatial freqs. blurred!
Pulse-Se SMS/Multiband/Blipped CAIPI

$V b_z$

$A b_z$

RF$_{in}$

RF$_{out}$

$G_z$

$G_y$

$G_x$

-excite multiple slices at once

-function of $G_z$ blips is to shift slices in $G_y$ direction

-this occurs because for given slice, a phase pedestal is added to the entire slice

- problem w/ all up $G_z$ blips $\Rightarrow$ phase error builds up

-trick#1: start w/2 slices, one at $z=0$, other above

-trick#2: for multiple slices not all at $z=0$, phase no longer same for even/odd

-trick#3: for more than 2 slices:

1st even odd even odd

etc.
MULTI-BAND/BLIPPED CHIP (Cont.)

- Relation between leave-one-out aliasing and nominally fully-sampled SMS

- Leave alternate lines out wraps image
- SENSE/GRAPPA to fix blc coil view smear k-space data
- Nominally, w/ SMS we record every line of k-space
- But equivalent to leave alternate out blc our multi-slice
  FOV was not big enough

- Slice-GRAPPA
  - reg GRAPPA -> recon missing lines
  - slice-GRAPPA -> recon multiple k-spaces
  - For each overlapped slices by training on fully-sampled data at beginning of scan

- Inter-slice "leakage block"
  - when training GRAPPA kernel on fully-sampled data,
    also minimize inter-slice leakage (split-slice-GRAPPA)
  - can also do regular GRAPPA on top of this
  - reason: for diffusion, loss in SNR from undersample
  - cancelled by shorter TE readout
  - gain from reduced image distortion from shorter readout
**ECHO-VOLUME IMAGING EVI**

- multi-shot (like FLASH) but acquiring one plane of 3-D k-space per shot (can do spin echo too)

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**Diagram Description**

- RF_in
- G2
- Gy
- Gx
- RF_out

---

- partitions (2nd phase encode)
- phase-encode blips
- second partition
- center 6 kx, ky for this partition; TE off
- finished the first partition
- entire k-space must be filled before 3D image is reconstructed
- since entire volume is excited each shot, potentially higher S/N
- must use smaller flip angle to avoid killing M_L since entire volume excited every partition (e.g. every 80 msec)

---

- main issue is movement artifact since data assembled from many shots over several secs
- breathing-induced B0 problems in different partitions may cause blur
SPIRAL IMAGING

- by using smoothly changing gradients (sinusoids) less gradient power required than w/trapezoids (less eddy currents)

earlier EPI hardware like this: sinusoidal gradient waveform from resonant circuit w/non-uniform sampling to get constant $\Delta k_x$

- sinusoids in both $G_x$ and $G_y$ give spiral k-space trajectory

90°

RF

$G_x$

$G_y$

$G_x$

Sig

Sample all orientations of each spatial frequency while slowly increasing spatial freq.

- constant angular velocity goes too fast at large $k_x$, $k_y$

- constant linear velocity better but impractical near $k_x=0$, $k_y=0$

- compromise: start constant angular, end constant linear

**Constant angular velocity**

$\omega(t) = \omega_0 t$

$k(t) = A t e^{i \omega_0 t}$

$G(t) = \frac{1}{A} \frac{d}{dt} k(t) = A e^{i \omega_0 t} + i A \omega_0 e^{i \omega_0 t}$

$G_x(t) = A \cos \omega_0 t - A t \omega_0 \sin \omega_0 t$

$G_y(t) = A \sin \omega_0 t + A t \omega_0 \cos \omega_0 t$

**Constant linear velocity**

$\omega(t) = \omega_0 T_e$

$k(t) = A T_e e^{i \omega_0 T_e}$

$G(t) = \frac{1}{A} \frac{d}{dt} k(t) = A e^{i \omega_0 T_e} + \frac{A}{2} \omega_0 e^{i \omega_0 T_e}$

$G_x(t) = \frac{A}{2T_e} \cos \omega_0 T_e + \frac{A}{2} \omega_0 \cos \omega_0 T_e$

$G_y(t) = \frac{A}{2T_e} \sin \omega_0 T_e + \frac{A}{2} \omega_0 \sin \omega_0 T_e$
SPIRAL 3D IR FSE (from Eric Wong)

- 3D: block select vs. slice select
- FSE: multiple echoes from one 90°
- Spiral: multiple spirals vs. multiple lines
- Interleaved spirals (like FSE interleaves)
- True IR (vs. MPRAGE)
- All echoes after 90° derive from mag w/ same T1 contrast (vs. non-steady-state)
- Possible to present sign
- High, uniform contrast, but lots of waiting (T1), high BW

RF

\[ 180°(\text{pre}) \Rightarrow T1 \approx 700 \text{ msec} \]

\[ 90° \]

\[ 180° \]

\[ 180° \times 16 \]

\[ 180°(\text{pre2}) \]

\[ G_x \]

\[ G_y \]

\[ G_z \]

\[ \text{FID} \quad \rightarrow \quad \text{echo}_1 \quad \rightarrow \quad \text{echo}_2 \]

3D k-space ("stack of spirals")

Spiral interleaves

\( k_z \) interleaves

\( k_z \) echoes

Echos \( \rightarrow \) (after one 90°)
Phase Errors & Echo-Centering Errors

Anything that causes a deviation of the $B_0$ field strength from the expected value $(B_{0,z} + G_{x,z} x + G_{y,z} y + G_{z,z} z)$ changes precision frequency and therefore, expected phase angle.

- Incorrect phase of spatial frequency stripes results in a shift in space in the magnitude image after reconstruction.

Fourier shift theorem:

Phase shift in spatial freq. domain causes spatial shift in image domain.

$$I(x-x_0) = \int \frac{e^{-i2\pi k_x x}}{k_x} S(k_x) e^{i2\pi k_x x_0} dk_x$$

- Correct w/ shimming and $B_0$-mapping/phase unwrapping before reconstruction.

Echo Centering Error:

- If realignment of all spins ($k_x = k_y = 0$) doesn't occur at the middle of read gradient, echo is shifted.
- Since echo is in spatial frequency domain, this is frequency shift.

- Spatial frequency shift results in wrapping in phase image after reconstruction → magnitude image unchanged.

Fourier freq. shift theorem:
Freq. shift in freq. domain causes phase shift in spatial freq. space.

$$I(x) = \int e^{i2\pi k_x x} S(k_x-k_x_0) e^{i2\pi k_x x_0} dk_x$$
FAST SCAN ARTIFACTS  EPI vs. Spiral

Brain-induced field defects lead to phase errors

**EPI**
- $G_x$ readout gradient strong $\rightarrow$ field defects smaller percentage
  less deformation of $k_x$ (vertical stripe components)
- $G_y$ "blips" are weak and total $G_y$ record time
  much longer (5 times) than standard readout (50 ms vs. 10 ms)
- an extra gradient in the $x$-direction, for example, maps
  and unmaps phase as a function of $x$-position
- but $G_x$ big, so effect on freq.-encode direction is much
  less than on phase-encode direction, where error accumulates
- for a given $x$-position, the strength of the spurious gradient
  is constant, so the accumulation of phase error results
  shift in the $y$-direction ($k_x$-space spin-stripe displ.)
  the phase error causes a shift in the $y$-direction
  proportional to $x$-gradient strength (shear) but no blurring
  (N.B. Shift varies w/x-position) $y \rightarrow y$

**Spiral**
- with center-out spirals phase errors accumulate
  in a radial direction
- thus, higher spatial frequencies have more error
  (= more shearing)
- for spurious $x$-direction gradient as above, there is
  a radial blurring, rather than a vertical shift
  because higher frequency phase stripes misaligned
  relative to low spatial freq.$y \rightarrow y$

- For defects with more complex contours in the $y$-direction
  (than linear, as above) the vertical shifts (in EPI)
  will vary with $y$-position, and may result in signals from different
  $y$-positions being reconstructed on top of each other
**Image-Space View of Localized $\Delta \phi$ Defect, Effect on Recon**

- Localized $\Delta \phi$ defects often arise from air pockets embedded in tissue
  - Air in middle/outer ear $\rightarrow$ indentation in inferior temporal lobe
  - Air under olfactory epithelium $\rightarrow$ orbitofrontal d/v, ant, thal. compression

- Collect one data (k-space) point
  - 4 cycles of phase in y-dir (g-polarized) $+$ localized $\Delta \phi$ defect $\rightarrow$ complex multiply $= \text{correlate} \sin \phi \cos \theta$ with brain
  - Brain structure sampled with distorted stripes $\rightarrow$ one complex number

- Reconstruction from distorted data points
  - $\cdots + \begin{bmatrix} \text{amplitude and phase of this component} \end{bmatrix} + \begin{bmatrix} \text{same for 5 cycles} \end{bmatrix} + \cdots = \text{image}

- Local upward displacement image phase (phase encode dir)

- **N.B.:** Image shifts only occurs if shift spatially sampled w/successively later echoes (see next page)

- Same defect makes leftward dent in vertical phase stripes

- Spatial information can be lost when continuous changes in phase are flattened by $\Delta \phi$ defect

- Shifts can pile multiple pixels on top of each other into one bright pixel

- Local estimates of $\Delta \phi$ needed to correct images
  1) Fieldmap method: $<\text{multiple TE's} >$ to est. local $\Delta \phi$ from $\Delta T E$ slope
  2) Point-spread-function: $<\text{extra phase encode to estimate PSF (should be } \delta-\text{function)} >$
     - Deconvolve distorted image in phase-encode direction
LOCALIZED Bφ DEFECT, EFFECT ON RECON

- When local Bφ defect disturbs image space phase stripes during signal acquisition, estimates of local spatial freq. are affected (compressed stripes = higher spatial freq.).

- If each successive ky line recorded w/ same echo time (e.g., w/ single line phase encoding) this will correspond to constant spat. freq. offset in k-space.

- A k-space freq. offset only results in image space phase shift (Fourier freq. shift theorem), which is invisible in amplitude image (cf. echo cont. error).

- However, with w/EPI, static Bφ defect causes more and more local displacement of image phase stripes for each additional ky line.

  - That is, later lines have greater spat. freq. offset.
  - Effectively stretches k-space in ky direction.
  - Same num samples to higher spatial freq. shrinks FOV (squishes voxels — see FOV page).

- When image is reconstructed, region with local Bφ defect shifted oppositely.

- Thus, local shift effect due to combination of 3 things:
  
  1) Static local ΔBφ defect
  2) Successive increases in phase error for successive spat. freq. measurements during long EPI readout
  3) Small size of ky phase encode blips relative to Bφ defect (much less of this effect in freq. encode direction)

- Respiration (which affects Bφ) in 3D FLASH might cause similar effect within k2 partition (if successive spat. freqs.)
GRADIENT NON-LINEARITIES

- Ideally, the $G_x$, $G_y$, and $G_z$ gradient coils attempt to impose a linear variation onto the $z$-component of the $B$ field — $B_z$ — in the $x$, $y$, and $z$-directions.

- In practice, gradient coils are non-linear (e.g., printed-circuit-like).

- Non-linearities are worse in smaller coils, but also in higher performance coils, designed for post-processing correction of distortions.

- Non-linearities result in phase errors, which result in 3-D image distortion.

  - A non-linear slice-select gradient will excite a curved slice.
  - Non-linear phase and frequency encode gradients will distort in-plane features.

- Some scanners correct these differently:
  - For 3-D scans (all directions), 2-D scans (just in plane), and EPI scans (no corrections!).

- This can result in errors approaching 1 cm in functional overlays.

- Different coil designs have different directions of distortion (!).

- The assumption of non-Maxwellian gradients results in additional phase errors.

- These can also be corrected since the $B_x$ and $B_y$ components are known.

- There are Fourier shift theorems.

- These effects do not build up over time in phase-encode directions, since they only occur when gradients are turned on.

- These distortions are predictable and can be corrected.

- That is, the assumption that gradients cause no field in the $B_x$ and $B_y$ direction.
SHimming AND Bo-MAPPING

- Passive iron shims inserted to flatten Bo field
- Additional coils (usu. in the gradient coils) can be statically energized in an attempt to flatten the Bo field (a few ppm good)
- Primary use is to compensate for defects in flatness present without a sample in the magnet (geometric imperfections, impurities in metal, etc) (= several hundred ppm)

[Linear shim coils impose gradients in x, y, and z]

[Higher order shims impose higher order spherical harmonic field components (e.g., z^2)]

Secondary use is to compensate for inhomogeneities caused by introducing the sample into the Bo field

Local resonance offsets caused by Bo defects estimated from images

- e.g., sample phase at multiple echo times

- Fit defective field using combination of fields generated by shim coils, then add these corrections to base shim currents

- This only corrects spatially gradual field defects

- Local defects due to air in sinuses much higher order than shims

- After shimming, field map measured again

- Image voxel displacements calculated from resonance offset map are used to unwarp the reconstructed magnitude image

- For EPI images, assume displacements all in phase-encode direction (since freq encode gradient is still strong relative to defects)
**1D navigator**

- Bϕ drift problem
  - slow up/down drifts in Bϕ continuously occur
  - a pedestal in Bϕ is pedestal in phase (not gradient)
  - which causes spatial shift (Fourier shift theorem)
  - in EPI, mainly affects phase-encode dir b/c of small slab readout
  - result is successive volumes drift in phase encode dir

---

**Gradient balance problem**

- unequal L/R readout gradients cause L/R shift in position of even/odd lines in k-space
- causing N/2 (Nyquist) ghosting -> another phase error

---

**3D navigator:** collect 3D sphere in k-space

- rotated object = rotation of k-space amplitude pattern
- translation of object = phase shift of k-space phase (Fourier shift)
- sample at sufficient radius to pick up high spatial freq features
- N.B.: excite whole volume
- do N,S hemispheres separately (less $T_2^*$, cancel EPI-like error accumulation)

Welch et al. (2002) MRM

- $\text{equator} \rightarrow \text{up, equator} \rightarrow \text{down}$

---

**RF, Gz, Gy, Gx**

- can be used for prospective motion correction (rotate, translate w/ gradients)
- better estimate, because of speed, than full TR & EPI images (27 ms vs. 2.4 sec)
- may need to smooth rot,trans estimates across time (e.g. Kalman filter)
RF FIELD INHOMOGENEITIES \( \mathbf{B}_0 \) inhomogeneities

- receive coil inhomogeneities alter the amplitude of the received signal, altering the reconstructed proton density in a spatially varying way
  - variations can be used (cf. GRAPPA, SENSE) and/or corrected

- transmit coil inhomogeneities affect the flip angle in a spatially varying way (can affect contrast: FLASH)
  - potentially worse (why local transmit is still in progress)
  - usu. fixed by using a large transmit coil (e.g. body coil)

- RF penetration at higher fields (\( \geq \) higher RF frequencies)
  - is less uniform:
    1) decreased RF wavelength (closer to size of head) at higher freq.
    2) increased permittivity \( (\varepsilon) \) and conductivity \( (\sigma) \) at higher field

- 2nd advantage of the falloff in signal recorded with a small, receive-only RF coil is better signal-to-noise (less noise received from other parts of brain)

- different sensitivity functions from different coils can be used to scan less lines in k-space (GRAPPA/SENSE/SPACE-RIP)

normalization ("pre-scan normalize")

- record lo-res volume (b/c coil fall-off is smooth) through both body coil and small coil(s)

- divide small coil body coil at each voxel to determine receive field

- use receive field to normalize main image(s)

[ see also: \( \text{ETL, MP2RAGE, T1/T2} \) ]
**DIFFUSION - WEIGHTED IMAGING**

Simple diffusion weighting

- RF
- $G_z$: select
- $G_y$: (not a practical sequence)
- $G_x$: (prepare, readout)

"Apparent diffusion coefficient" map

- to get large $b$, need $G_y$, $G_z$ (need big $G$'s)
- Long $t$ gives spurious T2-weighting
- can use stimulated echoes: 90° RF → $G_y$ → 90° RF → $G_z$ → diffusion echo 1 → back to transverse

1) Anisotropic Diffusion (Gaussian)

- measure $D$ along multiple axes
- have to measure tensor, not scalar
- even for determining one primary direction

$D = \begin{bmatrix} U_x & U_y & U_z \\ D_{xy} & D_{yz} & D_{xz} \\ D_{xz} & D_{yz} & D_{yy} \end{bmatrix}$

$D = u^T \cdot D \cdot u$

- scalar diffusion
- diffusion tensor measurement direction

Diffusion Surface (Non-Gaussian)

- need to measure diffusion in many directions (e.g., >6) to properly characterize even 2 main directions

2) Length Scale by multiple b-values

- fit line to semilog signal as function of $b$
- check if straight line: multi-exponential, e.g., $S = A_1 e^{-bD_1} + A_2 e^{-bD_2}$

- e.g., assume $A_1 = A_2 = 0.5$
PRACTICAL DIFFUSION-WEIGHTED PULSE SEQ

- Spin-echo 'Stejskal-Tanner' EPI seq. (standard on scanners)

  \[ 90^\circ \rightarrow 180^\circ \rightarrow \text{flips } M_z \text{ so rephase gradient, same sign as dephase} \]

  RF<sub>in</sub> • slice select
  G<sub>1</sub>, G<sub>y</sub>
  TE<sub>eff</sub> • center k-space
  RF<sub>out</sub> • TE<sub>eff</sub>/2

- Eddy-currents are long time-constant currents in metal of scanner that distort B field → spatial image distortion

- "doubly-refocused" spin echo sequence (DSE) can cancel the effects of eddy current (w/ partic. time constants)

  \[ 90^\circ \rightarrow 180^\circ \rightarrow 180^\circ \]

  RF<sub>in</sub> • phase dispersion (6 echo)
  G<sub>1</sub>, G<sub>y</sub>, G<sub>x</sub>
  RF<sub>out</sub> • multiple

  Nav<sup>y</sup>, etc.

  \[ \psi_{TRSE} = 0 = \psi_1 + \psi_2 + \psi_3 + \psi_4 \]

  \[ \text{twice-refocused Spin-echo (for center k-space)} \]
PERFUSION - ARTERIAL SPIN LABEL

- basic idea: tag blood below area of interest, collect control & tagged image, assume directional input flow, tag is 180° pulse, sign not problem when delay long enough (see below)

- continuous ASL (CASL) - continuously tag a plane, greatest on, blood gets adiabatically inverted as it passes through location w/convected resonant tag

- pseudo-contin. ASL (pCASL) - see next

- pulsed ASL (PASL) - e.g., EPSTAR, FAIR, PICORE, QUIPSS II

- small diff between control and tag (~1%) requires accurate balancing of control & tag images, control mag. transfer

- contrast problems: transit delays biggest confounding factor, motion artifact, other clearances (vs. microphones, which get stuck!)

- QUIPSS II - quantitative perfusion

  1) pre-saturate spins in target slices

  2) tag - 180° pulse below slices (to control off-resonance)

  3) saturate tagged block to end tag (TI), 4) EPI or spiral images of target slices (T2)

  both tag and control can use train of thin slices, pulses at top of tag band

  image most distal slice last to cancel delays, fast between slice so imaging excitation don't get interpreted as flow

  \[ \Delta M \approx \text{flow} \times \left[ 2M_0(TI, e^{-T2/T1a}) \right] \]

- solutions for quantitative perfusion

  - in sort delay so all spins arrive into low-velocity capillaries

  - kill end of tag to reduce spatial variation of tag

- QUIPSS II - alternate tag and control, GRE TE=30 ms

  1) control-tag \Rightarrow \text{flow}

  2) dual echo spiral

  \[ k=0 \text{ early } \Rightarrow \text{Hi S/N flow} \]

  \[ \text{TE=30 ms } \Rightarrow \text{BOLD} \]
PERFUSION - pCASL

- Original CASL (continuous arterial spin labeling) requires
  RF on continuously to adiabatically invert blood flowing
  through one plane
  - can only image one slice (bc dephasing from gradient)
  - hard to keep RF on continuously on modern scanner (esp. EC)
  - can use special purpose RF transmit (separate xmt channel)

A) Original CASL

<table>
<thead>
<tr>
<th>RF</th>
<th>[G_2]</th>
</tr>
</thead>
</table>
|     | image formation module ("readout") | \[\text{multiple possibilities}\]

B) pCASL - pseudo continuous arterial spin labeling  Dai, Alsop (2008)

<table>
<thead>
<tr>
<th>RF</th>
<th>[G_2]</th>
</tr>
</thead>
</table>
|     | \[\text{begin readout}\]

- Problem: multiple pulsers create aliased slice planes
- \(RF(t) = \frac{1}{\Delta t} \text{comb}(\frac{t}{\Delta t}) \otimes \text{rect}(\frac{t}{\delta})\)
- [use: convolution of 2 funct equals multiplying their FTs]
  \(F[RF(t)] = \text{comb}(6\Delta t) \cdot 8 \text{sinc}(\pi \delta \Delta t)\)

- Aliased labeling planes at: \(6 = n/\Delta t\) in frequency space, modulated by broad sinc()
- Use Hann or hyperbolic secant to reduce replicas

C) pCASL w/ shaped gradients

- Tag pulses have phase offset respecting gradient
- Control identical except every other has \(+\pi\) phase
- No net flip
**OFF RESONANCE EXCITATION**

- main idea: examine evolution of $\mathbf{\hat{M}}$ vector in rotating coord syst set to "off-resonance" $\mathbf{B}_1$ field freq ($\omega_{rf}$), not Larmor freq of $\mathbf{\hat{M}}$ ($\omega_0$)

- normally, if rotating coord syst freq set to Larmor freq ($\omega_{rf}=\omega_0$), an actually precessing $\mathbf{\hat{M}}$ will be stationary (ignoring decay) → implies effective $B_z=0$ in rotating

- now, move $\mathbf{\hat{M}}$ to rotating coord syst at $\mathbf{B}_1$ freq lower than $\omega_0$ (assume $\mathbf{B}_1=0$): existing $\mathbf{\hat{M}}$ will now appear to precess around z-axis:

  N.B.: this is precession in already rotating coordinate system!

  (slow relative to $\omega_0$)

  $\Delta \omega_0 = \omega_0 - \omega_{rf}$

  freq of precession in rotating coordinate syst of $\mathbf{\hat{M}}$

  Larmor rotation freq of $\mathbf{\hat{M}}$ = "incorrectly set" rotating coord syst freq.

- thus, viewing $\mathbf{\hat{M}}$ vector in off-resonance rotating coord syst makes it look like additional $\mathbf{B}_z$ field is causing "extra" precession

  "extra" $\mathbf{B}_z$ component is proportional to $\Delta \omega_0$ offset

  $\Rightarrow$ can be pos or neg: not coord syst freq too low → pos $\mathbf{B}_z$

  not coord syst freq too high → neg $\mathbf{B}_z$

- extra $\mathbf{B}_z$ adds to $\mathbf{B}_1$ resulting in slow precession around tipped axis: $\mathbf{B}_{\text{eff}}$ (effective)

- extra $z$-gradient can have same effect on $\Delta \omega_0$ (changes $\omega_0$ instead of changing $\omega_{rf}$)

  $\mathbf{B}_{\text{eff}} = \left(\frac{\Delta \omega_0}{\gamma}\right)\mathbf{k} + B_z\mathbf{k} + B_1\mathbf{i}$

  effective $\mathbf{B}$ in rotating frame set to $\mathbf{B}_1$ freq ($\omega_{rf}$) from Larmor-$\mathbf{B}_1$

  freq mismatch ($\omega_0$ vs $\omega_{rf}$)

  (if on-res -> 0)

  apparent "extn" of $\mathbf{B}_z$

  transverse RF stim (here, around x-axis)

  optional 2-gradient (pos or neg)

  (additional source of $\mathbf{B}_{\text{eff}}$ tilt)

  RF: sweep freq

  W0: constant

  RF: const freq

  W0: sweeps because spins flow along gradient direction

adiabatic RF pulse $\approx$ flow-driven CASL tag
SPECTROSCOPY + IMAGE

- chemical shift: small displacement resonant freq due to shielding of target nucleus (e.g. 1H) by surrounding electron orbitals

- e.g., acetic acid:
  - oxygen attracts electron so less shielding of target nucleus
  - 3 of these H's (more shielded)
  - 1 of these H's (less shielded)

- how we get chemical shift spectrum:

  RF stim → Larmor oscillations are multiplied (PSD) by center freq to obtain Δf (not MHz high freq)

- data before FT is a series of time-domain samples of the mix of shifted-freq offsets

- FT turns data into "shift spectrum"

Pulse Sequence

- since we are already using phase (freq) encoding for space, we need an "extra dimension" w/ all gradients OFF!

- use spin-echo to undo built-up chemical shifts, then record chemical-NMR-like signal (if) and FT it like chemists do!
PRESS, MEGA-PRESS

usu. single voxel by using 3 orthog. slice selects
(tho can add PE gradients & more excitations to get multiple vox)

PRESS — 3 orthog. slice select

MEGA-PRESS — add "editing" RFs to suppress solvent (water)

G2, G3 — asymmetric spoilers to dephase spins in bandwidth of selective MEGA pulses
**PHASE-ENCODED STIMULUS & ANALYSIS**

- **map polar angle**
- **map frequency**
- **map eccentricity**
- **map prox/distal axis, Road maps**

Periodic stimuli (Phase-encoded) - e.g., 8 cycles at 64 sec/cycle

**Calculate significance**
- Ratio between amplitude at stimulus frequency (= signal) and average of amplitudes at other frequencies (= noise)
- Ignore harmonics, low freq (= movement)

**Smooth**
- Vector average of complex significance \( A, \phi \) with that at nearest neighbor surface points

**Display**
- Plot phase using hue and saturation to indicate significance

**Delay correction**
- Record responses to opposite directions of stimulus (ccw/lcw, in/out, up/down)
- Vector average after reversing angle of one
- Penalizes inconsistent more than just avg of angles

Typically 0.5-5% amplitude

Strongly periodically activated single voxel time course

Remove constant (avg) and linear trend

\[
\text{real} \quad \begin{array}{c}
\cdots \\
\text{imaginary} \\
\end{array}
\quad t \rightarrow \\
\quad t = \text{total TRs}
\]

FFT, convert to \( A, \phi \)

\[
\begin{array}{c}
A \\
\phi \\
\end{array}
\quad t \rightarrow \\
\quad t = \text{total TRs}/2
\]

Reversed CCW vector average CW significance

CCW significance (complex)
CONVOLUTION

\[ h(x) = f(x) \ast g(x) = \int_{-\infty}^{\infty} f(z) \cdot g(x-z) \, dz \]

- **linear time/space invariant system**
- **definition of convolution** \((f \ast g)(x)\)
- **commutative**

**Why reverse makes sense**

blc commutative, this is thinking like: \( \int g(x) \cdot f(x-z) \, dz \)

**intuitive non-reversed view of convolution output**

**impulse response function (HDR)**

**impulses (exp. design)**

**N.B. cross-corr same as convolution except no reversed**

**N.B. auto-corr same, except no-reversal**
**General Linear Model**

\[ \hat{y} = \hat{X} \hat{h} + \hat{S} \hat{b} + \hat{n} \]

- **Data** = design · HDR + drifts · weights + noise

\[ \begin{bmatrix} \hat{y} \\ \text{data} \end{bmatrix} = \begin{bmatrix} \hat{X} \\ \text{homo} \end{bmatrix} \begin{bmatrix} \hat{h} \\ \text{unknowns} \end{bmatrix} + \begin{bmatrix} \hat{S} \\ \text{fixed} \end{bmatrix} \begin{bmatrix} \hat{b} \\ \text{poly} \end{bmatrix} + \begin{bmatrix} n \end{bmatrix} \]

- **Goal** is to solve for the hemodynamic response functions, \( \hat{h} \)

- **Simpler:** preconvolve
  1. conv. \( \hat{X} \) with \( \hat{h} \)
  2. solve for \( \hat{\beta} \)

\[ \hat{y} = \hat{X} \hat{\beta} + \hat{n} \]

- **Scalar** \( \hat{x} \)

**Maximum Likelihood Estimate**

1. Assume white noise, solve for \( \hat{h} \)

2. \[
\hat{h} = (X^T P_s^+ X)^{-1} X^T P_s^+ y \quad \text{where} \quad P_s^+ = I - S (s^T S)^{-1} s^T
\]

   \( \rightarrow \) projection matrix that removes part of vector that lies in \( S \) space

   or

\[
= (X_\perp X_\perp)^{-1} X_\perp y \quad \text{where} \quad X_\perp = P_s^\perp X
\]

3. **Significance** (how to construct F-ratio)

\[
F = \frac{N - K - \ell}{k} \left[ \frac{y^T (P_{xs} - P_s) y}{y^T (I - P_{xs}) y} \right]
\]

- \( P_{xs} \) = projection data on \( X \) + nuisance subspace
- \( P_s \) = projection data onto nuisance subspace

**See diagram next page for geometric interpretation.**
GEOMETRIC INTERPRETATION OF GENERAL LINEAR MODEL

- With no nuisance functions (S), we could look at orthogonal projection of data onto experimental design and compare that to error to determine significance.

\[ \hat{y} = X\hat{h} + \hat{e} \]

Projection matrix, \( P_x \), operates on \( \hat{y} \) to give projection of data into experiment space, \( X \).

- When nuisance functions, \( S \), are considered, problem: \( S \) may not be orthogonal to \( X \)

For example: linear trend not orthogonal to std. block design.

[Remember: "orthogonal" means \( \text{dot prod} = 0 \) \( \text{corr} = 0 \)]

[Geometric Picture]

- Orthogonal projection
- Oblique projection onto nuisance (\( P_{yS} \))
- Orthogonal projection onto reference (\( P_{y} \))

\( P_{xS} \) y

Unit S

How much more of data you can explain by adding reference functions (\( F \) numerator):

\[ [(I - P_{yS}) y] \]

Same as projection onto reference only in special case where \( S \perp X \)
1) MNI auto-Talairach \( \rightarrow \) generates 4x4 matrix
- make average brain target (blurry)
- blur target (further), blur single brain (a lot), gradient descent on \( x \times \text{corr} \)
- repeat w/ less blurring of avg target and current brain
- problems: variable neck cut off
  but much better than standard! \( \leq \) fit to bounding box
  only 2 points near center of brain!
- 5-10 times

2) Intensity Normalization (output: "T1")
- histogram of pixel values in 10 mm thick HTR slices
- smooth histogram
- peak find to get initial estimate of white matter
- discard outlier peaks across slices
- fit splines to peaks across slices
  \( \Rightarrow \) interpolated scaling factor \( 1 \) to HTR
- scale each pixel so WM peak is 110
- refine estimate to interpolate in 3D
- find points in 5x5x5 within 10% of WM, get new scale for them
  build Voronoi to interpolate scales unset above
  soap-bubble-smooth Voronoi boundaries (3 iterations)
  re-scale each voxel

3) Skull Stripping (output: "brain")
- "shrink-wrap" algorithm
- start with ellipsoid template
- minimize brain penetration and curvature
  \( \text{curvature} : \text{spring force} \)
  \( \begin{align*}
  &\text{from center-to-neighbor vect sum} \\
  \longrightarrow & \text{decompose into } 1 \text{ and tangential (local norm of summed, normed cross products)}
\end{align*} \)
- brain penetration
  apply force along surface normal that prevents surface from entering gray matter
SEGMENTATION & SURFACE RECON

- implementing a "force" is like directly constructing the operator that minimizes something (without first defining the 'something')
- more formally, we would define cost function, then take its derivative (gradient) to minimize it

Shrinkwrap update e.g. (skull strip, original Dale & Sereno surface refinement)

\[ \mathbf{r}_{\text{center}}(t+1) = \mathbf{r}_{\text{center}}(t) + F_{\text{smooth}}(t) + F_{\text{MRI}}(t) \]

**F** _smooth_ = \[\lambda_{\text{tang}} \sum_{\text{neigh}} (I - n_{\text{center}} n_{\text{center}}^T) \cdot (r_{\text{neigh}} - r_{\text{center}}) \]

**F** _MRI_ = \[\lambda_{\text{MRI}} n_{\text{center}} \max_{\mathbf{d}} \left[ 0, \tanh \left[ I(\mathbf{r}_{\text{center}} - d n_{\text{center}}) - I_{\text{thresh}} \right] \right] \]

**HOW TO**

\[ \mathbf{n}^{n} = \mathbf{x}^t \times \mathbf{y}^t \times \mathbf{z}^t \]

\[ \mathbf{n}^{n} = \begin{bmatrix} x^t \\ y^t \\ z^t \end{bmatrix} \]

\[ \mathbf{d} = \text{project} \]

\[ \text{average normal component} \]

\[ \text{vector subtract off} \]

\[ \text{projection of a neighbor vertex vector onto normal in the direction of the normal (n is squared (as above) so we get a vector out (not a scalar))} \]

\[ \text{vector to neighbor vertex} \]

\[ \text{expand by distribution to neighbor vector minus projection of neighbor onto normal = tangential!} \]

\[ \text{stronger than normal} \]

\[ \text{(0.5)} \]

\[ \text{identity 3x3} \]

\[ \text{weaker than tangential} \]

\[ \text{(0.1)} \]
4) Non-isotropic filtering (output: "win")
   - preliminary hard threshold: output GM2 WM
   - find ambiguous/boundary voxels
     \[ \geq 20\% \text{ of more of } 26 \text{ immediate neighbors different} \]
   - find plane of least variance
     \[ \text{for each direction (from icosahedral supertessellation)} \]
     \[ \text{consider } 5 \times 5 \times 5 \text{ volume around 1 voxel} \]
     \[ \text{find plane of least variance in this hemisphere} \]
     \[ \text{median filter w/ hysteresis} \]
     \[ \text{if } 60\% \text{ of within-slab differ, reverse classification} \]
   - "flosses" sulci without blurring

5) Find cutting planes
   - midbrain, to avoid fill into cerebellum (HOR)
   - callosum, to separate hemispheres (SAH)
   - Talairach to start; fill WM in SAH or HOR till min area

6) Region-growing to define connected parts (output: "filled")
   - inside-out, outside-in, inside-out - for each hemisphere
   - up/down cycles within each plane
   - plane-by-plane
   - "wormhole filter" (3x3x3 = center + 26)
     \[ \text{fill (unfilled) voxel if } 66\% \text{ neighbors differ} \]
     \[ \text{eliminates structures within 1-D structure} \]
7) Surface Tessellation (output: rh.orig, lh.orig)

- variable num neighbors possible!
- quads to triangles

- find filled voxels bordering unfilled
- make ordered list of neighboring vertices
  \[ \Rightarrow \text{so cross-products oriented properly} \]

- long list of values associated with each numbered vertex
  e.g., position (orig, morphed)
  area (orig, morphed)
  curvature (intrinsic, Gaussian)
  "Sulcushess" (summed 1 movement during unfolding)
  cortical thickness
  FMRI data \[ \Rightarrow \]
  EEG/MEG dipole strength

- separate FMRI data set must be aligned, sampled

\[ \text{FMRI voxels larger} \]
\[ \text{Sample at each surface vertex} \]
\[ \text{nearest-neighbor "soap bubble" smoothing to interpolate data onto hi-res mesh} \]

- some quantities only well-defined on surface
  \( \Rightarrow \) gradient of magnitude of cortical map measure (e.g., eccentricity)
SEGMENTATION & SURFACE RECON

- smoothing/inflation/WM, pial done as derivative of energy functional

\[ \mathcal{J} = \mathcal{J}_{\text{tangential}} + \lambda_{\text{normal}} \mathcal{J}_{\text{normal}} + \lambda_{\text{image}} \mathcal{J}_{\text{image}} \]

- total scalar error to minimize
- scalar tangential error (fixed by redistributing vertices)
- scalar normal error (fixed by reducing curvature)
- scalar image error (fixed by moving toward target image value)

\[ \mathcal{J}_{\text{normal}} = \frac{1}{2} \#\text{vert} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 \]

- across all vertices
- \( \frac{1}{2} \) so no coefficient on derivative
- across all vertices of one vertex
- vertex unit normal
- vector from current center to one neighbor (position vector diff.)
- \( \mathbf{t}^x \), \( \mathbf{t}^y \) are first 2 eigen.vector of neighbor vector cloud (\( \mathbf{n} \) is third)

\[ \mathcal{J}_{\text{tangential}} = \frac{1}{2} \#\text{vert} \sum_{\text{centers}} \sum_{\text{neighbors}} \left[ \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 + \left[ \mathbf{t}^y_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \right]^2 \]

- "squishing" or "wash"
- x-direction in tangent plane
- y-direction in tangent plane
- project vector to neighbor onto x & y
- \( I_{\text{targ}} \) for WM: mean of voxels labeled WM in 5 mm neighborhood
- \( I_{\text{targ}} \) for pia: global - small num for CSF-like

\[ \mathcal{J}_{\text{image}} = \frac{1}{2} \#\text{vert} \sum_{\text{centers}} \left[ I_{\text{targ}} - I(\mathbf{r}_{\text{center}}) \right]^2 \]

- take directional derivative of energy functional (to find steepest uphill)
- move each vertex in the opposite (negative) direction w/ self-interest test

\[ \frac{\partial \mathcal{J}}{\partial \mathbf{r}_{\text{center}}} = \lambda_{\text{image}} \left[ I_{\text{targ}} - I(\mathbf{r}_{\text{center}}) \right] \nabla I(\mathbf{r}_{\text{center}}) \]

- go the direction (vector) of largest scalar error for one vertex
- \( \mathbf{n} \) goes 0/const
- \( \sum \) across neighbors
- \( \lambda_{\text{normal}} \) scalar
- \( \sum \) neighbors
- \( \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \) x-component of tangential
- \( \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \) y-component of tangential

N.B.: eq. 9 in Lata, Fisch & Silverman different - and incorrect!

HOW TO derivative:
- vector - calculate gradient on image (first blur w/Gaussian)
- scalar - \( \sum \) across neighbors
- \( \mathbf{n}_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \) normal
- \( \mathbf{t}^x_{\text{center}} \cdot (\mathbf{r}_{\text{center}} - \mathbf{r}_{\text{neighbor}}) \) tangential
SULCUS-BASED CROSS-SUB. ALIGN

- use summed perpendicular vertex movement during inflation
  as per-vertex measure of "sulcus-ness"
- add term to energy function: "sulcus-ness" error: \((S_{\text{sent}} - S_{\text{targ}})^2\)
- bootstrap
  - morph to one brain
  - make avg target
  - remorph to avg target

Smooth wm \(\rightarrow\) inflated \(\rightarrow\) sphere \(\rightarrow\) registered

\(\text{Sub}_1\) \(\rightarrow\) inflated \(\rightarrow\) sphere \(\rightarrow\) morph

\(\text{Sub}_2\)

\(\cdots\)

\(\text{Sub}_n\) \(\rightarrow\) inflated \(\rightarrow\) sphere \(\rightarrow\) morph

- each sub's native surf has diff # vertices
- interpolate values (coords, surface measures, stats) to each icosahedral vertex from neighboring vertices of native mesh (dashed lines)
- average surface made from folded/inflated avg coords
  - folded: loses area from sulcal crinkles (\(\text{fsaverage "inflated"}\))
  - inflated: retains orig area, correct sulc/gyrus ratio ("inflated-avg")
- can use sampled-to-icos individual subj coords to draw icosahedral surface in shape of an indiv. brain

⇒ N.B.: morph will have changed local vertex density compared to more uniform native mesh (use native for sing. subj.)
**Source of EEG/MEG**

- PSPs
  - anisotropic cables + aligned spatially + coherent/biased stim one end
  - no distant signal from axon spike

- Head
  1. Local dipole
  2. EEG through skull, skin
  3. Swimming because skull 1/80 conductivity of brain

**MEG**

- Radial dipoles lost
- Tangential dipole generates Gabor-like scalp distrib. of \( \mathbf{B} \) field
INTRACORTICAL CIRCUITS & ORIGIN OF EEG

Cell types
- excitatory (spiny)
  - pyramidal
  - spiny stellate (e.g. V1 layer 4C)
- inhibitory (smooth)
  - basket
  - double bouquet
  - chandelier
  - clutch

Circuits
- huge complexity
- first principal components: input $\rightarrow$ layer 4 $\rightarrow$ layer 2/3 $\rightarrow$ feedback layer 5/6 $\rightarrow$ output feedback
- microelectrode recording (e.g. 10 $\mu$m tip):
  - high pass $\rightarrow$ spikes
  - low pass $\rightarrow$ local field potentials
- spikes only recordable in gray matter
- white matter spikes only recordable with pipette w/ very fine tip b/c inward & outward currents so spatially close in axon/spike ($>1$ $\mu$m)

Intra/inter cortical connections cartoon

"Lower" (e.g. V1)
- 2/3 feedforward
- 4 input
- 5 motor output
- 6 feedback

"Higher" (e.g. V2)
- 2/3
- 4
- 5
- 6

Ascending input (e.g. dLGN) $\rightarrow$ output to sup. collic. $\rightarrow$ feedback avoids layer 4 $\rightarrow$ motor striatum $\rightarrow$ to higher areas
GRADIENT, DIVERGENCE, CURL

**Gradient** \( \nabla \) (generalized derivative)

\[ \nabla f(r) = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} \]

- Turns scalar field into vector field at each \( x, y, z \) point \( r \)
- Unit vector in \( x \)-dir

**Divergence** \( \nabla \cdot \mathbf{v}(r) \) (div, dot prod)

\[ \nabla \cdot \mathbf{v}(r) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \]

- Turns vector field into scalar field at each \( x, y, z \) point \( r \)
- Change of \( \mathbf{v} \) in \( x \)-direction at point \( r \)

**Curl** \( \nabla \times \mathbf{v}(r) \) (curl, "cross product")

\[ \nabla \times \mathbf{v}(r) = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{i} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{j} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{k} \]

- Change of \( \mathbf{v} \) in \( y \)-direction at point \( r \)

**Vector identities**

\[ \nabla \times \nabla f = 0 \] \( \text{curl of the gradient of any scalar field is zero} \)

\[ \nabla \cdot (\nabla \times \mathbf{A}) = 0 \] \( \text{divergence of the curl of any vector field is zero} \)

\[ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \]
**Potential ($\Phi$), Electric Field ($\nabla \Phi$) & CSD ($\nabla \cdot (-\nabla \Phi) = \nabla^2 \Phi$)**

**Low-frequency field approximation**
- Electric fields uncoupled from magnetic (vs. electromagnetic radiation)
- Pre-Maxwellian approx. (EEG freq's $\ll 1$ MHz)
- Calculate electric fields as if magnetic fields don't exist
- Calculate magnetic fields strictly from distribution of currents
- Ignore capacitative effects, too

**Scalar potential, $\Phi$** (what we measure with electrode)

\[
\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}
\]

Scalar potential at one position

(1) $\vec{E}$ defined as force (vector) acting on unit charge at a given point in space (as result of arbitrary distribution of other charges)

(2) Current density, $\vec{J}$ (not current source density!), is proportional to $\vec{E}$ (still a vector!)

\[
\vec{J} = \sigma \vec{E}
\]

(3) Two defs of $\vec{A}$:
- $\vec{A} = \frac{\phi}{4\pi} \int \frac{\vec{J}}{r^2} \, d\text{vol}$

3D CSD gold standard (rat RAER paper)

**Data**

- (360 points) $\nabla \Phi$

- Electric vector field

- Scalar field source/sink movie as function of $t$
1D and 2D Current Source Density Expts.

1D CSD
- Raw event-related signal relative to ground, 1/f (e.g. skull)
- Low-pass spike
- LFP (local field potential)
- Both types of data can be recorded from same electrode

Rationale: CSD changes much more slowly parallel to cortex than perpendicular to cortical sheet
⇒ assume approx constant (≈ 0) parallel to cortex

2D CSD
- 2D array of electrodes on pial surface or on scalp

Rationale: All electrodes record along same surface so assume depth profiles are constant
INTRACORTICAL C.S.D.

- e.g. click evoked rat A-I
  (Sukov & Barth, 1998)

CSD - (10)

Layer 4

- Source
  Sink

50 msec

50 msec

P1

N1

P2

- phase-locked CSD p
  Gamma shifts with each cycle
MAXWELL EQUATIONS

Electrodynamics, Magnetodynamics

\[ \nabla \cdot (\sigma \nabla \Phi) = \nabla \cdot \vec{J} \]

Ohm's law: \( \sigma \vec{E} \)

Conductivity constant (a tensor constant if inhomogeneous in different direction)

\[ \nabla \times \vec{B} = \mu_0 (\vec{J} - \sigma \nabla \Phi) \]

curl magnetic field

Impressed currents

currents due to ionic flow that "appear out of nowhere"

N.B. These are all defined at a (every) point in space

\[ \nabla \cdot \vec{B} = 0 \]

divergence

Incidentally, this Maxwell equation violated by a linear gradient in \( B_z \) in \( x \)

\[ \Phi = \text{potential} \]

\[ \vec{B} = \text{magnetic fields} \]

\[ \vec{J} = \text{currents} \]

- propagation of potentials, magnetic fields instantaneous (no capacitance)
- simultaneous can't solve \( \vec{J} \& \text{sources} \), \( \Phi, \vec{B} \) are data
- linear

Potential (\( \Phi \)) and magnetic fields (\( \vec{B} \)) produced by a weighted sum of two current source distributions are equal to weighted sum of fields produced by each current source distribution by itself.
WHY WE CAN IGNORE MAGNETIC INDUCTION

\[ \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t} \]

(from Nunez, 1981)

\[ \vec{B} = \nabla \times \vec{A} \]

"vector potential"

\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \]

magnetic field
field component due to charge distribution
field component due to coupling between electric & magnetic fields

\( \nabla \cdot \vec{D} = \sigma \vec{E} \)

permittivity \( \varepsilon \)
permeability \( \mu \)
conductivity \( \sigma \)

\( \nabla \cdot \vec{B} = 0 \)

characteristic of substance linear in all three

\[ \nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} \left( \vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \]

\( \nabla \times \nabla \times \vec{E} = -2\pi i \mu (\sigma + 2\pi i \varepsilon) \vec{E} \)

to neglect:

\[ \frac{2\pi i \mu (\sigma + 2\pi i \varepsilon) |\vec{E}|}{|\nabla \times \nabla \times \vec{E}|} \ll 1 \]

1) \( |\nabla \times \nabla \times \vec{E}| \propto |\vec{E}|/L^2 \) where \( L \) is dim over which \( \vec{E} \) varies significantly
2) \( \varepsilon \) in tissue similar to empty space
3) Assume conductor (large) \( \sigma \), dielectric unit, and EEG freq

\( \lambda \) number is about \( 10^{-6} \) \( \rightarrow \) small
**MONPOLE, DIPOLE FORWARD SOL’N**

\[
\Phi_1 = \frac{S}{4\pi \sigma r}
\]

the potential recorded for a source monopole

distance source to measuring point (=|| \vec{r} ||)

\[
\Phi_2 = \frac{S}{4\pi \sigma} \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right)
\]

potential recorded for a source-sink pair

"near field"

\[
\Phi_2 \approx \left( \frac{1}{4\pi \sigma} \right) \frac{\vec{S} \cdot \vec{r}}{r^2}
\]

scalar potential

approximations for "far enough away" measurements
(subtracting two \(1/r\)'s gives inverse square)

\[
\vec{B}_2 = \left( \frac{\mu_0}{4\pi} \right) \frac{\vec{S} \times \vec{r}}{r^3}
\]

vector potential

3.4x

**N.B.:** both assume inside infinite isotropic conductor

\[
\Phi_i(t) = e_i s_i(t)
\]

\[
\vec{b}_i(z) = \vec{m}_i s_i(t) \rightarrow \text{Squid’s measure component of } \vec{B}
\]

Linear superposition with fixed electrodes and sensors

\[
\vec{X}(t) = \sum_j g_{ij} s_j(t) \rightarrow \vec{X}(t) = \mathbf{G} \mathbf{s}(t)
\]
Forward Solution

- well-posed (one answer)
- linear: \( 6(A) + 6(B) = 6(A+B) \)
- approximations due to unknown electrical properties of head

3-shell spherical analytic

- skull conductivity & brain
- "smearing" (cf. cable theory)

3-shell boundary element

- arbitrary shape
- homogeneous conductivity
- solution = infinite homogeneous + correction factors

Finite element

- most general
- computational intensive w/ small grid
- many unknown parameters to estimate

remember, we only need one shell b/c currents thru skin/skull too small to make sig. \( \textbf{B} \)
**FORWARD SOLN**

\[ \sum_j V_i = \sum_j E_{ij} S_j + \eta_i \]

**matrix form**

\[ \begin{bmatrix} \mathbf{v} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{s} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{n} \end{bmatrix} \]

\[ \mathbf{v} = \mathbf{E} \mathbf{s} + \mathbf{n} \]

**note:**
- lower case bold \( \rightarrow \) vector
- upper case bold \( \rightarrow \) matrix

**electric recordings**

\[ \begin{bmatrix} \mathbf{V} \\ \mathbf{m} \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{n} \end{bmatrix} \]

\[ \hat{\mathbf{x}} = \hat{\mathbf{A}} \hat{\mathbf{s}} + \hat{\mathbf{n}} \]

**magnetic recordings**

\[ \text{current Source dipole amplitudes (same length as above)} \]
WHY LOCALIZE?

most of ERP literature based (instead) on temporal "components"

1) undely local cortical generators (from micro electrode LFP)
   - extended in time (400 msec)
   - multiphasic in every cortical area
   - temporally non-static, depending on stimulus
     e.g. simple contrast-brightness drifts can modulate retinal delay by 50 msec!

2) thus, any "component" consists of sum of activity
   from multiple cortical areas at different hierarchical levels

3) stimulus manipulations will change temporal overlap
   - may cause "component" peak to disappear without changing cortical areas being activated

4) verified by intracortical LFP/CSD (Schroeder et al, 1998)

macaque monkey intracortical data

these areas span the visual system from bottom to top, accounting for roughly 50% of the entire macaque monkey CTZ

V1  MT  V4  LIP  AIP  IT
LFP's from approx. layer 4 in cortex ("input layer")

psychologists now identify a few temporal "component" peaks...

...but each one comes from every one of these cortical areas!!

- by contrast, the spatial signature of the signal from one cortical area is static -- a better area-based component

- origin of "components"
  - easier to record many temporal points (EEG started w/ few electrodes, many time points)
  - easier to "paste" high level psychological functions onto a few waveform deflections
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Derivation of Ill-posed Inverse
(from Dale & Sereno, 1993)

\[ x = As + n \]

\( A = \) forward solution matrix (E+B)

\( s = \) source vector

\( n = \) sensor noise vector

\[ \text{solve for inverse operator} \]

\[ \text{inverse operator} \]

\[ \text{expected value: } = \sum_k P_k k \]

\[ \text{probability that rand. var. value is } k \]

assume \( n, s \) normal, zero-mean \( w \) corresponding var. matrices \( C, R \)

\[ \text{Err}_w = \left\langle \| Wx - s \|^2 \right\rangle \]

\[ = \left\langle \| (WA-I)s + Wn \|^2 \right\rangle \]

\[ = \left\langle \| Ms + Wn \|^2 \right\rangle \]

where \( M = WA-I \)

\[ = \left\langle \| Ms \|^2 + \| Wn \|^2 \right\rangle \]

\[ = \text{tr}(MRMT) + \text{tr}(WCW^T) \]

\[ = \text{trace is sum of diag elements} \]

\[ \text{re-expand: } = \text{tr} \left( WAR^TW^T - RAT^TW^T - WAR + R \right) + \text{tr} \left( WCW^T \right) \]

Explicitly minimize by taking derivative w.r.t. \( W \), set to zero, solve for \( W \)

\[ 0 = 2WARAT - 2RAT + 2WC \]

\[ WARAT + WC = RAT \]

\[ W(ARA^T + C) = RAT \]

\[ W = RAT (ARA^T + C)^{-1} \]

\[ W \] is inverse solution operator: \( \text{sensors} \rightarrow \text{sources} \)
Inverse \( \text{Sin}'N \) (2)

\[ W = RA^T(ARA^T + C)^{-1} \]

\( \rightarrow \) "minimum norm" solution
\( \text{find } \hat{\beta} \text{ w/ smallest norm } = ||\hat{\beta}|| \)

- the minimum norm solution appropriatelydownplays
  deeper (weaker scalp signal) sources since these
  are more likely to fall into the noise floor

- "problems" of minimum norm:
  - deeper sources get displaced
    to the surface
  - small superficial sources "win" because
    of approximate inverse square form of fund solution
    \( \Rightarrow \) smaller norm of distributed superficial soln.
  - can't fix by increasing priors of deep sources!!
    \( \Rightarrow \) that will give deep sources given noise as input!!
**Inverse Solutions to Ill-Posed Compared**

$$s = Wx$$  

**sensor data**  

how to use the inverse solution, $W$  

same $W$ for all time points  

"minimum norm" solution  

i.e., norm $\|W\|$ of solution is smallest of infinitely many alternate solutions

**Linear inverse operator**

$$W = RA^T(ARA^T + C)^{-1},$$

from error minimization derivation

$$[W] = [R][A^T][A]\left([A^T][R][A^T] + [C]\right)^{-1}$$

$[ARA^T] \Rightarrow$ square in # of sensors (small)

alternate, algebraically equivalent Bayesian derivation (w/ bigger inverses!)

$$W = (A^TC^{-1}A + R^{-1})^{-1}A^TC$$

$$[W] = \left([A^T][C][A] + \left[R [A^T]C\right]\right)$$

$\Rightarrow$ both square in # of sources (large)
PROBLEMS W/ SURFACE NORMAL

- Since nearby points on surface often have different orientation, surface normal constraint can help (since fwd soln A, B very different)

- But, some point spread functions typically extends across sulci, artificial sign reversals occur

- Solutions

1) Ignore sign → saves useful orientation info!

2) Solve onto 3 orthogonal dipoles at each critical point instead of a single oriented dipole

⇒ more appropriate when averaging across subjects, since detailed variations vary a lot

⇒ also, fills in bottom of sulci (else unsigned stripes)
use inverse-2

FMRI Constrained Inverse

- insert FMRI values for Rii's
- but still allow other sites to have non-zero Rii's
- pathologies occur if solution restricted completely to FMRI points by setting non FMRI Rii's to zero
  set to small number instead!

- this allows extracting time course from sources visible in EEG/MEG and FMRI

- N.B.: sources that are only visible in EEG/MEG will be dispensed to small distributed values at a large number of vertices

visible in both EEG/MEG and FMRI

visible only in EEG/MEG and not FMRI — distributed at small amplitude across many vertices
**NOISE SENSITIVITY NORMALIZATION**

\[ W = R A (A R A^T + C)^{-1} \]

Forward: \( x = A s \) well-posed (Liu, Dale, and Ballard, 2002)
Inverse: \( s = W x \) ill-posed
Solve: \( x = A s + n \) for \( s \)

\[ W_{\text{norm}} = D W \]

\[ s_{i, \text{norm}} = \left( W_{\text{norm}} x \right)_i = \left( D W x \right)_i = \frac{W_i x}{\sqrt{(W x)^T W}} = \frac{\sqrt{(W x x^T W)^T}}{C_W} \]

If assume Gaussian white noise, noise covariance \( C_W \) is multiple of \( I \), so

\[ W_{\text{norm}} = \frac{W_i \text{orig}}{\| W_i \text{orig} \|} \]

\[ s_{i, \text{norm}} = \frac{W_i \text{orig} x}{\| W_i \text{orig} \|} \]

i.e., scale each row of \( W \) by single value — the norm of that row row of \( W \) is:

\[ a_{ij} \rightarrow \frac{a_{ij}}{\| W_{ij} \|} \]

\[ \text{Inverse solution coefficients for one source} \]

\[ W_{\text{norm}} \]

\[ s_{i, \text{norm}} \]

scale by norm of this row

i.e., if inverse soln for source is big (e.g., deep source), noise normed inverse for that source reduced by scaling
**Noise Sensitivity Normalization (2)**

Shallow source (unit strength)  
Deep source (unit strength)

- Shallow: small
- Deep: even smaller

\[ S_i = \frac{\mathbf{W}_i^{\text{org}} \cdot \mathbf{x}}{\| \mathbf{W}_i^{\text{org}} \|} \]

- Effect on inverse solution: more like significance than actual power
- Effect on point-spread function: to equalize shallow & deep
  - Shallow spread out more than min norm
  - Deep shrunk to same as shallow

Point-spread functions:

- Noise normalized

Point spread data spread in
**Conclusions**

- More EEG or more MEG better
- EEG better than MEG (cf. radial)(EEG far less currently less accurate)
- Biggest gain from adding small # EEG (a MEG) (e.g. 30) to many MEG (a EEG) (e.g. 150)
- Easier to add many MEG, so: optimal < 30 EEG < 300 MEG
- EEG/MEG forward-solution-scaling-factor error causes more cross talk
**Music**

(from Dale & Sereno, 1993) (cf. Mosher & Leahy)

- using **sensor covariance**

\[
D = \langle xx^T \rangle = \sigma^2 I + \sum_i \sum_j \sigma_i \sigma_j \text{Corr}(i,j) A_i A_i^T
\]

\[
\approx \frac{[x_1 \ldots x_n]}{n} [x_1 \ldots x_n]^T
\]

\[
D = U \Lambda U^T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}
\]

- find most significant **spatial patterns**
- in **sensors** over **time**

**Project forward solutions onto these spatial patterns**

(“project” = dot prod = similarity) for each point in brain

\[
\xi_i = A_i^T U \Lambda U^T A_i^T
\]

- big single number if forward soln looks like Us
how to weight the minimum norm inverse

\[ R_{ii} \approx \frac{A_i^T A_i}{A_i^T U \Lambda U^T A_i} \]

\[ W = R A^T (A R A^T + C)^{-1} \]

cf.

- like parallel resistance

\[ R_{\text{parallel}} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \ldots} \]

- any low resistance (\( \xi \)) decreases overall resistance (small \( R_{ii} \))

- i.e., if forward soln has appearance like any low eigenvalue spatial pattern, it gets devalued
- How it works

- How it fixes min norm problem

N.B., if two widely separated sources (i.e., diff fwd solns) are highly correlated, MUSIC will eliminate both since no single fwd soln will look like that 2-separated dipole pattern (e.g., L/R A-I)

La various "dual MUSIC" hacks possible